ORIGINAL PAPER

THE DECOMPOSITION OF SOME ANNIHILATOR POLYNOMIALS FOR LINEAR MAPS IN COPRIME POLYNOMIALS

VASILE POP[1]

Manuscript received: 12.08.2017; Accepted paper: 22.09.2017; Published online: 30.12.2017.

Abstract. *The decomposition of annihilator polynomials for linear maps or for the matrices of product of two coprime polynomials gives a decomposition of the spectrum as a direct sum of two subspaces with annihilator polynomials of smaller degree.*

Keywords: *Linear Map, Range and Null space, Rank, Idempotent and Involuntary matrices.*

MSC2010: *15A03, 15A04, 15A24.*

1. INTRODUCTION

Let V be a vector space over a field K and let $T:V\to V$ be a linear map for which there is an annihilator polynomial $P\in K[X]$, i.e. $P(T)=0$. If the annihilator polynomial P is irreductible then there are two coprime, non-constant polynomials such that $P=P_1\circ P_2$. We give some relations in which the null spaces $\ker(P_1(T))$, $\ker(P_2(T))$ and the numbers $\operatorname{rank}(P_1(T))$, $\operatorname{rank}(P_2(T))$ appear.

2. COMPLEMENTARY SUBSPACES $\ker(P_1(T))$ *AND* $\ker(P_2(T))$

Theorem 2.1 *If* V *is a vector space over the field* K *with* $T:V\to V$ *an endomorphism and* $P_1,P_2\in K[X]$ *two coprime polynomials then the following statements are equivalent:*

a) $P_1(T)\circ P_2(T)=0$ or $(P_1\cdot P_2)(T)=0$

b) $V=\ker(P_1(T))\oplus\ker(P_2)(T)$.

Proof: Since the polynomials are coprime there are two polynomials $Q_1,Q_2\in K[X]$ so that

$$P_1Q_1+P_2Q_2=1 \tag{1}$$

a)$\to$b): In the hypothesis $P_1(T)\circ P_2(T)=0$. We show that any vector $x\in V$ can be decomposed uniquely as $x=x_1+x_2$ with $x_1\in\ker(P_1(T))$ and $x_2\in\ker(P_2(T))$. Since $(P\cdot Q)(T)=P(T)\circ Q(T)=Q(T)\circ P(T)$, from (1) we obtain $P_1(T)\circ Q_1(T)+P_2(T)\circ Q_2(T)=I$, where $I:V\to V$ is the identity map $I(x)=x, x\in V$ .We select $x_1=P_2(T)\circ Q_2(T)(x)$ and

[1] Technical University of Cluj-Napoca, Department of Mathematics, Postal Code, Cluj-Napoca, Romania. E-mail: Vasile.Pop@math.utcluj.ro.

$x_2 = P_1(T)\circ Q_1(T)(x)$ and we have $P_1(T)(x_1) = (P_1(T)\circ P_2(T))(Q_2(T)(x)) = 0$, hence $x_1 \in \ker(P_1(T))$ and $P_2(T)(x_2) = (P_2(T)\circ P_1(T))(Q_1(T)(x)) = 0$ so that $x_2 \in \ker(P_2(T))$. Moreover, if

$y \in \ker(P_1(T)) \cap \ker(P_2(T))$ then $P_1(T)(y) = 0$, $P_2(T)(y) = 0$

and

$$y = (P_1Q_1 + P_2Q_2)(T)(y) = Q_1(T)(P_1(T)(y)) + Q_2(T)(P_2(T)(y)) = 0$$

hence

$$\ker(P_1(T)) \cap \ker(P_2(T)) = \{0\} \text{ and then } V = \ker(P_1(T)) \oplus \ker(P_2)(T).$$

b)$\to$a): Any vector $x \in V$ can be written uniquely as $x = x_1 + x_2$ with $x_1 \in \ker(P_1(T))$, $x_2 \in \ker(P_2(T))$. We have,

$$\begin{aligned} P_1(T)\circ P_2(T)(x) &= P_2(T)\circ P_1(T)(x_1) + P_1(T)\circ P_2(T)(x_2) = \\ &= P_2(T)(P_1(T)(x_1)) + P_1(T)(P_2(T)(x_2)) = P_2(T)(0) + P_1(T)(0) = 0 \end{aligned}$$

hence $P_1(T)\circ P_2(T) = 0$.

Corollary 2.1 *The linear map $T : V \to V$ is a linear projection on vector space V if and only if $V = \ker(T) \oplus \ker(I - T)$.*

Proof: The map T is a linear projection if and only if the polynomial $P(x) = x - x^2$ is the annihilator polynomial for T. We consider the polynomials $P_1(x) = x$, $P_2(x) = 1 - x$ which are coprime and $P(x) = P_1(x)\cdot P_2(x)$.

We have $P(T) = 0 \Leftrightarrow (P_1 \cdot P_2)(T) = 0 \Leftrightarrow P_1(T)\circ P_2(T) = 0$ and by Theorem 2.1 this is equivalent to $V = \ker(P_1(T)) \oplus \ker(P_2(T)) = \ker(T) \oplus \ker(I - T)$.

Corollary 2.2 *If V is a vector space over the field K of characteristic different than 2 then the linear map $S : V \to V$ is a symmetry if and only if $V = \ker(I - S) \oplus \ker(I + S)$.*

Proof: The map S is a symmetry if and only if the polynomial $P(x) = 1 - x^2$ is the annihilator polynomial of S $(P(S) = 0)$. We consider the polynomials $P_1(x) = 1 - x$ and $P_2(x) = 1 + x$, which in the ring $K[X]$ are coprime if the characteristic of the field K is different than 2. We have $P(x) = P_1(x)P_2(x)$ so that $0 = P(S) = P_1(S)\circ P_2(S)$ and by Theorem 2.1 this is equivalent with $V = \ker(P_1(S)) \oplus \ker(P_2(S)) = \ker(I - S) \oplus \ker(I + S)$.

3. THE SUM OF THE RANKS $\operatorname{rank}(P_1(T)) + \operatorname{rank}(P_2(T))$

Theorem 3.1 *Let V be a vector space of finite dimension over the field* K, $T : V \to V$ *an endomorphism and* $P_1, P_2 \in K[X]$ *two coprime polynomials. The following statements are equivalent:*
a) $P_1(T) \circ P_2(T) = 0$
b) $\operatorname{rank}(P_1(T)) + \operatorname{rank}(P_2(T)) = \dim(V)$.

Proof: a)$\to$b): By Theorem 2.1 it follows:

$$V = \ker(P_1(T)) \oplus \ker(P_2(T)) = \dim(V) - \operatorname{rank}(P_1(T)) + \dim(V) - \operatorname{rank}(P_2(T)),$$

which implies that $\operatorname{rank}(P_1(T)) + \operatorname{rank}(P_2(T)) = \dim(V)$.

b)$\to$a): From the rank-nullity theorem (see [1]) it follows that

$$\dim(V) = \dim(\ker(P_1(T))) + \dim(\ker(P_2(T))).$$

On the other hand, $\ker(P_1(T)) \cap \ker(P_2(T)) = \{0\}$; since $P_1(T)(y) = P_2(T)(y) = 0$ it follows that $y = (Q_1 P_1)(T)(y) + (Q_2 P_2)(T)(y) = 0$ hence $V = \ker(P_1(T)) \oplus \ker(P_2(T))$ and by Theorem 2.1 we have that $P_1(T) \circ P_2(T) = 0$.

Corollary 3.2 [6] *The linear map* $T : V \to V$ *is a projection of the vector space* V *if and only if* $\operatorname{rank}(T) + \operatorname{rank}(I - T) = \dim(V)$.

Corollary 3.2 [5] *If the field* K *has the characteristic different than 2 then the linear map* $S : V \to V$ *is a symmetry if and only if* $\operatorname{rank}(I - S) + \operatorname{rank}(I + S) = \dim(V)$.

4. THE DECOMPOSITION OF ANNIHILATORS POLYNOMIALS FOR MATRICES

Theorem 4.1 *If* $A \in \mathrm{M}_n(K)$ *is a matrix and* $P_1, P_2 \in K[X]$ *are two coprime polynomials the following statements are equivalent:*
a) $P_1(A) \cdot P_2(A) = 0$
b) $\operatorname{rank}(P_1(A)) + \operatorname{rank}(P_2(A)) = n$.

Proof: We consider the linear map $T : K^n \to K^n$, $T(X) = AX$, where

$$X = [x_1 \quad x_2 \quad \cdots \quad x_n]^t \in K^n$$

and we have $P_1(T)(X) = P_1(A)X$, $P_2(T)(X) = P_2(A)X$, $\operatorname{rank}(P_1(T)) = \operatorname{rank}(P_1(A))$, $\operatorname{rank}(P_2(T)) = \operatorname{rank}(P_2(A))$ and then Theorem 4.1 is a consequence of Theorem 3.1.

Corollary 4.1 [2, 5] *The matrix* $A \in \mathsf{M}_n(K)$ *is an idempotent matrix* $(A^2 = A)$ *if and only if* $\operatorname{rank}(A) + \operatorname{rank}(I_n - A) = n$.

Corollary 4.2 [2, 4] *If the field* K *has* $\operatorname{char}(K) \neq 2$ *then the matrix* $A \in \mathsf{M}_n(K)$ *is an involutory matrix* $(A^2 = I_n)$ *if and only if* $\operatorname{rank}(I_n - A) + \operatorname{rank}(I_n + A) = n$.

We mention that a recent book which contains a novel presentation of idempotent and involutory matrices of order two as well as properties and applications of symmetries and projections is [3].

REFERENCES

[1] Bernstein, D.S., *Matrix Mathematics. Theory, Fact and Formulas*, Princeton Univ. Press, 2009.
[2] Pop, V., *J. Sci. Arts*, **4**(37), 295, 2016.
[3] Pop. V, Furdui, O., *Square Matrices of Order Two. Theory Applications and Problems*, Springer, Cham, 2017.
[4] Tyan, I., Styan, G. *Linear Algebra and its Applications*, **335,** 101, 2001.
[5] Tyan, I., Styan, G. *Journal of Computational and Applied Mathematics*, **191(**1), 77, 2006.
[6] Zhang, F., *Matrix Theory – Basics Results and Techniques*, Springer New York, 1999.