ORIGINAL PAPER

THE DISCRIMINANT OF THE SECOND FUNDAMENTAL FORMS UNDER CONNECTION PRESERVING MAP

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Abstract. *Let $f: E^n \to E^n$ be an isometric immersion provided $f(M) = \overline{M}$ where M and $\overline{M}$ are $(n-1)-$ dimensional Riemannian manifolds .We study the discriminant of the second fundamental forms and also being $\lambda-$ isotropic of the Riemannian manifolds if f is a connection preserving map.*

Keywords: *Discriminant of the second fundamental form, Connection preserving map, $\lambda-$ isotropy*

1. INTRODUCTION

It is well known that modern geometry was born when Riemann first separated the concept of geometry from the concept of space. The geometry was redesigned according to the topological concepts by Riemann. In this geometry, the geometric structures which are named as manifold by Riemann play an important role. To do geometry on manifolds there are need some geometric concepts, such as metric and connection etc. One of the geometric structures added in this new geometry is the concept of the linear connection. Classically, a connection was defined by Christoffel (1869) to be a set of symbols $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ or Γ_{ij}^k. The modern view point was finally formulated about 1950 by Koszul. Another important geometric structure is the concept of the Riemannian metric. The notion of a Riemannian metric dates back to Riemann's 1854 lecture. The association of a connection is due to Levi-Civita, [11]. A property of a Riemannian manifold is intrinsic if it depends only on the metric. Otherwise the property is extrinsic. From here, we can say that the first fundamental form (or the second fundamental form) of a Riemannian manifold is intrinsic (or extrinsic). The first fundamental form completely describes the metric properties of a manifold and the second fundamental form gives the shape of a manifold. Therefore, the geometry of the second fundamental form has been studied by many scientists ([4],[9],[10],[16],[17] and [18]).

Properties of connection preserving and conformal maps in $n-$ dimensional $C^\infty-$ manifolds were given by N. J. Hicks in 1963, [6]. He proved that a conformal map f which defined between $C^\infty-$ class differentiable manifolds M and M' is connection preserving if and only if f is a homothety, [6]. Furthemore, N. J. Hicks studied the connection preserving spray maps, [8]. He investigated finding the necessary and sufficient conditions for a spray

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map to be connection preserving. Then, F. Erkekoğlu studied the differential geometry of the connection preserving maps, [3]. On the other hand, C. Tezer showed that for $n=3,7$ a conformal diffeomorphism of S^n into itself admits no invariant connection except the trivial case where it admits an invariant Riemannian metric, [15]. Besides, S. T. Pamuk proved that the results in [15] without any restiriction on the dimension of spheres, [14]. F. Bayar & A. Sarıoğlugil studied the geometry of Riemannian manifolds under the connection preserving maps, [1].

In this paper we investigate the relations between discrimnants of second fundamental forms of the manifolds M and $f(M)=\overline{M}$ where $f:E^{n+1}\to E^{n+1}$ is a connection preserving map.

2. PRELIMINARIES

In this section we will give the basic concepts of the theory of manifolds.

Let M be a $n-$ dimensional Riemannian submanifold of M^m with the metric $g=\langle\,,\rangle$. Let ∇ and $\overset{\square}{\nabla}$ be the Levi-Civita (or Riemannian) connections of M and M^m, respectively, and let $f:M\to M^m$ be a $C^\infty-$ map. The differential of the map is defined by

$$\begin{aligned} f_*:T_M(P) &\to T_{M^n}(f(P)) \\ X_P &\to f_*(\mathrm{X}_P)=X_P[f] \end{aligned} . \tag{1}$$

If there exists a C^∞ real valued function G on M such that for any P in M , then

$$\langle f_*X, f_*Y\rangle = G(P)\langle X,Y\rangle \tag{2}$$

for all X,Y in M^m; and if $G>0$ on M, then f is conformal. Furthermore, f_* has no kernel if f is conformal. We call the map G the scale function. If G is a constant function, we say f is homothetic. If $G=1$, we say f is an isometry, and if f is both an isometry and a diffeomorphism, we say M is isometric to M^m. A $C^\infty-$ *map* $f:M\to M^m$ is called the connection preserving map if

$$f_*(\nabla_X \mathrm{Y}) = \overset{\square}{\nabla}_{f_*(\mathrm{X})} f_*(\mathrm{Y}) \tag{3}$$

where $X,Y\in T_M(P)$,$[8]$. *If* M is a hypersurface in E^{n+1} with the connection $\overset{\square}{\nabla}$, the formulas of Gauss and Weingarten are given by

$$\overset{\square}{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{4}$$

and

$$\overset{\square}{\nabla}_X Y = S_N X + \nabla_X{}^{\perp} X \tag{5}$$

where h, S and $\nabla^{\perp}$ denote the second fundamental form, the shape operator and the normal connection of M, respectively, $[19]$. In (4), $h(X,Y)=\langle S(X),Y\rangle N$ is a symmetric, vector-valued, 2-covariant C^{∞} tensor called the second fundamental tensor, $[19]$.

For all $X,Y,Z\in T_M(P)$ the curvature tensor R is defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \\ &= [\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z \end{aligned} \tag{6}$$

where $[,]$ is Lie bracket operator. Then the sectional curvature $K(\sigma)$ of σ is given by

$$K(\sigma)=K(X,Y)=\frac{\langle R(X,Y)Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2} \tag{7}$$

where $\sigma=Span\{X,Y\}\subset T_M(P)$,$[2]$.

From here, it can easily seen that

$$\langle h(X,Y),N\rangle=\langle S_N X,Y\rangle. \tag{8}$$

For the covariant derivatives of the second fundamental form h and the shape operator S o M we have

$$(\nabla h)(Z,X,Y)=\nabla_Z^{\perp}\left(h(X,Y)\right)-h(\nabla_Z X,Y)-h(X,\nabla_Z Y) \tag{9}$$

and

$$(\nabla S)_N(X,Y)=\nabla_Y\left(S_N(X)\right)-S_{\nabla^{\perp}{}_Y N}X-S_N(\nabla_Y X). \tag{10}$$

The covariant derivative of the k^{th} order of the second fundamental form h is

$$\begin{aligned} (\nabla^k h)(X_1,X_2,\ldots,X_{k+2}) &= \nabla_{X_1}\left((\nabla^{k-1}h)(X_2,X_3,\ldots,X_{k+2})\right) \\ &\quad -\sum_{i=2}^{k+2}(\nabla^{k-1}h)\left(X_2,X_3,\ldots,\nabla_{X_1}X_i,\ldots,X_{k+2}\right) \end{aligned} \tag{11}$$

where $k\geq 1$ and $\nabla^0 h=0$,$[5]$.

Definition 1. Let M be an $n-$dimensional Riemannian submanifold of $\overline{M}$ with the metric $g=\langle\ ,\ \rangle$. M is called the isotropic manifold at $P\in M$ with a constant normal curvature $\lambda(P)$ if the normal curvature vector $h(X,X)$ of f in the $X-$direction satisfies

$$\langle h(X,X),h(X,X)\rangle=\lambda^2\langle X,X\rangle. \tag{12}$$

Especially, if λ is constant on M then M is called constant isotropic, $[5]$. For all $X, Y, Z, W\in T_M(P)$ the above isotropic condition is equivalent with

$$C\left(\left\langle h\left(X,Y\right),h\left(Z,W\right)\right\rangle\right)=C\left(\lambda^2\left\langle X,Y\right\rangle\left\langle Z,W\right\rangle\right) \tag{13}$$

where C denotes the cylic sum with respect to X,Y,Z and W,[12].

Hence, the discriminant Δ at $P\in M$ of the second fundamental form h is given by

$$\Delta_{XY}=\frac{\left\langle h\left(X,X\right),h\left(Y,Y\right)\right\rangle-\left\|h\left(X,Y\right)\right\|^2}{\left\|X\right\|^2\left\|Y\right\|^2-\left\langle X,Y\right\rangle^2}. \tag{14}$$

where $T_M\left(P\right)=Span\left\{X,Y\right\}$. Furthermore, we can say that the second fundamental form h is $\lambda-$ isotropic provided that $\left\|h\left(X,X\right)\right\|=\lambda$ for all $X\in T_M\left(P\right)$, $\left[12\right]$. If the vector system $\left\{X,Y\right\}$ is an orthonormal basis of $T_M\left(P\right)$, then by $\left(14\right)$ we get

$$\Delta_{XY}=\left\langle h\left(X,X\right),h\left(Y,Y\right)\right\rangle-\left\|h\left(X,Y\right)\right\|^2. \tag{15}$$

Then we may give the following lemmas:

Lemma 1. Let M be an $n-$dimensional Riemannian submanifold of $\overline{M}$ and let $\left\{X,Y\right\}$ be an orthogonal vector system on M. Then h is $\lambda-$ isotropic if and only if $\left\langle h\left(X,X\right),h\left(Y,Y\right)\right\rangle=0.$

Lemma 2. Let M be an $n-$dimensional Riemannian submanifold of $\overline{M}$ and let $\left\{X,Y\right\}$ be an orthonormal vector system on M. If h is $\lambda-$ isotropic, then we have

$$\Delta_{XY}+3\left\|h\left(X,X\right)\right\|^2=\lambda^2 \tag{16}$$

and

$$2\Delta_{XY}+\lambda^2=3\left\langle h(\mathrm{X},\mathrm{X}),\mathrm{h}\left(Y,Y\right)\right\rangle. \tag{17}$$

Lemma 3. Let M be an $n-$dimensional Riemannian submanifold of $\overline{M}$ and let $\left\{X,Y\right\}$ be an orthonormal vector system on M. If h is $\lambda-$ isotropic, then we have

$$\left\langle h\left(X,X\right),h\left(Y,Y\right)\right\rangle+2\left\|h\left(X,X\right)\right\|^2=\lambda^2. \tag{18}$$

3. THE DISCRIMINANT OF THE SECOND FUNDAMENTAL FORM UNDER CONNECTION PRESERVING MAP

In this section, we will investigate relations between discriminants of second fundamental forms of the manifolds M and $\overline{M}=f\left(M\right)$ where $f:E^{n+1}\to E^{n+1}$ is a connection preserving map.

Theorem 1. Let $f : E^{n+1} \to E^{n+1}$ be an isometric immersion provided $\overline{M} = f\left(M\right)$. If f is a connection preserving map, then

$$\overline{h}\left(f_*X ,\, f_*Y\right) = \delta h\left(X, Y\right) \tag{19}$$

where h and $\overline{h}$ are second fundamental form tensors of M and $\overline{M}$, respectively**.**

*Proof***:** By (1) we may write $f_*X ,\, f_*Y \in T_{\overline{M}}(\mathrm{f(P)})$ for all $X ,Y \in T_M(P)$. By the definition of second fundamental tensor, we have

$$\overline{h}\left(f_*X ,\, f_*Y\right) = \left\langle \tilde{\nabla}_{f_*X}\, \tilde{N},\, f_*Y \right\rangle \tilde{N} \tag{20}$$

where $\tilde{N}$ is the unit normal vector field of $\overline{M}$. On the other hand, we may take as

$$\tilde{N} = \frac{1}{\| f_*N\|}\, f_*N\ . \tag{21}$$

Substituting (21) into (20), we obtain

$$\overline{h}\left(f_*X ,\, f_*Y\right) = \frac{1}{\| f_*N\|^2} \left\langle \tilde{\nabla}_{f_*X}\, f_*N,\, f_*Y \right\rangle f_*N\ . \tag{22}$$

Since f is a connection preserving map, we get

$$\overline{h}\left(f_*X ,\, f_*Y\right) = \frac{1}{\| f_*N\|^2} \left\langle f_*(\nabla_X \mathrm{N}),\, f_*Y \right\rangle f_*N\ . \tag{23}$$

On the other hand, since f is an isometric immersion, we obtain

$$\left\langle \overline{h}\left(f_*X ,\, f_*Y\right), \overline{h}\left(f_*X ,\, f_*Y\right)\right\rangle = \frac{1}{\| f_*N\|^2} \left\langle \nabla_X \mathrm{N}, Y \right\rangle^2 . \tag{24}$$

Substituting $\delta^2 = \dfrac{1}{\| f_*N\|^2}$ and $h(\mathrm{X},\mathrm{Y}) = \left\langle \nabla_X \mathrm{N}, Y\right\rangle$ into (24), we get

$$\overline{h}\left(f_*X ,\, f_*Y\right) = \delta h(X ,Y).$$

This completes the proof.

The principal significance of this theorem is that it allows one to get some relations about the discriminants of the second fundamental forms under the above assumptions.

Theorem 2. Under the hypotheses of the previous theorem we have

$$\overline{\Delta}\left(\overline{\sigma}\right)=\delta^2\Delta\left(\sigma\right) \tag{25}$$

where Δ and $\overline{\Delta}$ are discriminants of the second fundamental form tensors h and $\overline{h}$, respectively.

Proof: Let us consider the basis $\sigma = Span\{X, Y\}$ for all $X, Y \in T_M(P)$.Since f_* is a linear map we may write

$$\overline{\sigma} = Span\{f_*X, f_*Y\} \subset T_{\overline{\overline{M}}}\left(f_*\left(P\right)\right). \tag{26}$$

Substituting (26) into (14) we get

$$\overline{\Delta}\left(\overline{\sigma}\right)=\frac{\left\langle \overline{h}\left(f_*X, f_*X\right), \overline{h}\left(f_*Y, f_*Y\right)\right\rangle - \left\|\overline{h}\left(f_*X, f_*Y\right)\right\|^2}{\left\|f_*X\right\|^2\left\|f_*Y\right\|^2-\left\langle f_*X, f_*Y\right\rangle^2}. \tag{27}$$

Since f is a connection preserving map and an isometric immersion we have

$$\overline{\Delta}\left(\overline{\sigma}\right)=\delta^2\,\frac{\left\langle h\left(X, X\right), \mathrm{h}\left(Y, Y\right)\right\rangle - \left\|h\left(X, Y\right)\right\|^2}{\left\|X\right\|^2\left\|Y\right\|^2-\left\langle X, Y\right\rangle^2}. \tag{28}$$

From (14) and (28), we have

$$\overline{\Delta}\left(\overline{\sigma}\right)=\delta^2\Delta\left(\sigma\right).$$

This is completed proof.

Corollary 1 Under the hypotheses of the theorem 1 if h is $\lambda-$ isotropic, then $\overline{h}$ is $\overline{\lambda}-$ isotropic.

Proof: Let h be $\lambda-$ isotropic.From Lemma 1 we may write

$$\left\langle h\left(X, X\right), h\left(Y, Y\right)\right\rangle = 0. \tag{29}$$

On the other hand, since f is a connection preserving map and an isometric immersion we get

$$\left\langle \overline{h}\left(f_*X, f_*X\right), \overline{h}\left(f_*Y, f_*Y\right)\right\rangle = \delta^2\left\langle h\left(X, X\right), h\left(Y, Y\right)\right\rangle. \tag{30}$$

Substituting (29) into (30) we get

$$\left\langle \overline{h}\left(f_*X, f_*X\right), \overline{h}\left(f_*Y, f_*Y\right)\right\rangle = 0.$$

Then, $\overline{h}$ is $\overline{\lambda}-$ isotropic.

Lemma 4. Let $f : E^{n+1} \to E^{n+1}$ be an isometric immersion provided $\overline{M} = f(M)$ and let h is $\lambda -$ isotropic. If f is a connection preserving map,then we have

I. $$\overline{\Delta}\left(\overline{\sigma}\right) + 3\left\|\overline{h}\left(f_* X, f_* X\right)\right\|^2 = \overline{\lambda}^2 \quad (31)$$

II. $$2\overline{\Delta}\left(\overline{\sigma}\right) + \overline{\lambda}^2 = 3\left\langle \overline{h}\left(f_* X, f_* X\right), \overline{h}\left(f_* Y, f_* Y\right)\right\rangle. \quad (32)$$

Proof:

I. Since h is $\lambda -$ isotropic, we can write

$$\left\|h\left(X, X\right)\right\| = \lambda. \quad (33)$$

From (19) and (25) we get

$$\overline{\Delta}\left(\overline{\sigma}\right) + 3\left\|\overline{h}\left(f_* X, f_* X\right)\right\|^2 = \delta^2 \Delta(\sigma) + 3\left\|h\left(X, X\right)\right\|^2 \quad (34)$$

Substituting (16) and (33) into (34) we obtain

$$\overline{\Delta}\left(\overline{\sigma}\right) + 3\left\|\overline{h}\left(f_* X, f_* X\right)\right\|^2 = \left(\lambda\delta\right)^2 \quad (35)$$

Setting $\overline{\lambda} = \lambda\delta$ in (35) we get

$$\overline{\Delta}\left(\overline{\sigma}\right) + 3\left\|\overline{h}\left(f_* X, f_* X\right)\right\|^2 = \overline{\lambda}^2.$$

II. From (30) we obtain

$$3\left\langle \overline{h}\left(f_* X, f_* X\right), \overline{h}\left(f_* Y, f_* Y\right)\right\rangle = 3\delta^2 \left\langle h\left(X, X\right), h\left(Y, Y\right)\right\rangle. \quad (36)$$

Substituting (17) into (36) we obtain $2\Delta_{XY} + \lambda^2 = 3\left\langle h(\mathrm{X}, \mathrm{X}), \mathrm{h}\left(Y, Y\right)\right\rangle$.

$$3\left\langle \overline{h}\left(f_* X, f_* X\right), \overline{h}\left(f_* Y, f_* Y\right)\right\rangle = \delta^2\left(2\Delta\left(\sigma\right) + \lambda^2\right). \quad (37)$$

On the other hand, by $\overline{\lambda} = \lambda\delta$ and (25) we get

$$2\overline{\Delta}\left(\overline{\sigma}\right) + \overline{\lambda}^2 = \delta^2\left(2\Delta\left(\sigma\right) + \lambda^2\right). \quad (38)$$

From (37) and (38) we have

$$2\overline{\Delta}\left(\overline{\sigma}\right) + \overline{\lambda}^2 = 3\left\langle \overline{h}\left(f_* X, f_* X\right), \overline{h}\left(f_* Y, f_* Y\right)\right\rangle.$$

Lemma 5. Let $f : E^{n+1} \to E^{n+1}$ be an isometric immersion provided $\overline{M} = f(M)$ and let h is $\lambda -$ isotropic. If f is a connection preserving map,then we have

$$\left\langle \overline{h}(f_*X , f_*X), \overline{h}(f_*Y, f_*Y) \right\rangle . + 2\left\| \overline{h}(f_*X , f_*Y) \right\|^2 = \overline{\lambda}^2 . \quad (39)$$

Proof: Since f is a connection preserving map, we have

$$\left\langle \overline{h}(f_*X , f_*X), \overline{h}(f_*Y, f_*Y) \right\rangle + 2\left\| \overline{h}(f_*X , f_*Y) \right\|^2 = \delta^2 \left(\left\langle h(X,X), h(Y,Y) \right\rangle + 2\left\| h(X,Y) \right\|^2 \right).$$

From (18) we obtain

$$\left\langle \overline{h}(f_*X , f_*X), \overline{h}(f_*Y, f_*Y) \right\rangle + 2\left\| \overline{h}(f_*X , f_*Y) \right\|^2 = (\lambda\delta)^2 \quad (40)$$

Substituting $\overline{\lambda} = \lambda\delta$ into (40) we get

$$\left\langle \overline{h}(f_*X , f_*X), \overline{h}(f_*Y, f_*Y) \right\rangle + 2\left\| \overline{h}(f_*X , f_*Y) \right\|^2 = \overline{\lambda}^2 .$$

This completes the proof.

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