# COFINITELY GENERALIZED WEAK SUPPLEMENTED MODULES

## FIGEN ERYILMAZ[1], SENOL EREN[2]

**Abstract.** *Let $R$ be a ring and $M$ be a left $R-$module. $M$ is called cofinitely generalized weak supplemented (or briefly CGWS-module) if every cofinite submodule of $M$ has a generalized weak supplement in $M$. In this paper, we give various properties of that kind of modules. It is shown that (1) A module $M$ is CGWS-module if and only if every maximal submodule has a generalized weak supplement in $M$. (2) The class of cofinitely generalized weak supplemented modules are closed under taking homomorphic images, arbitrary sums, generalized covers and short exact sequences. (3) A ring $R$ is semilocal if and only if every left $R-$module is a CGWS-module.(4) A commutative domain $R$ is h-semilocal if and only if every torsion $R-$module is cofinitely generalized weak supplemented.*

*__Keywords:__ Cofinite Submodule, Cofinitely Generalized Weak Supplemented Module, Generalized Cover, Semilocal Ring.*

## 1. INTRODUCTION AND PRELIMINARIES

Supplemented modules have been discussed by several authors in the literature. Bass calls a ring $R$ is *semi-perfect* if every factor module of $R$ modulo a right ideal has a projective cover. He proved in [4] that $R$ is *semi-perfect* if and only if $\dfrac{R}{J(R)}$ is semi-simple and decompositions of $\dfrac{R}{J(R)}$ can be raised to $R$ where $J(R)$ is Jacobson radical of $R$. Mares generalized this concept to modules such as a module $M$ is called *semi-perfect* if $M$ is projective and every factor module of $M$ has a projective cover. She proved in [11] that $M$ is *semi-perfect* if and only if

$*\dfrac{M}{J(M)}$ is semi-simple,

*decompositions of $\dfrac{M}{J(M)}$ can be raised to $M$, and

$* J(M)$ is small in $M$.

Throughout the paper, $R$ will be an associative ring with identity, $M$ be an $R-$module and all modules are unital left $R-$modules unless otherwise specified. By $N \leq M$, we mean that $N$ is a submodule of $M$. Recall that a submodule $L$ of $M$ is *small* in $M$, denoted as $L \ll M$, if $L + K \neq M$ for every proper submodule $K$ of $M$. $Rad(M)$ will indicate Jacobson radical of $M$. $M$ is called *supplemented*, if every submodule $N$ of $M$ has a *supplement* in $M$, i.e. a submodule $K$ minimal with respect to $N + K = M$. $K$ is

[1] Ondokuz Mayıs University, Department of Math. Edu., 55200, Samsun, Turkey. E-mail: fyuzbasi@omu.edu.tr.
[2] Ondokuz Mayıs University, Department of Mathematics, 55200, Samsun, Turkey. E-mail: seren@omu.edu.tr.

supplement of $N$ in $M$ if and only if $N + K = M$ and $N \cap K = K$ [15]. Kasch and Mares proved that a module $M$ is *semi-perfect* if and only if $M$ is supplemented. In [8], it is proved that the local Dedekind domains are Dedekind domains for which there exists a nonzero torsion-free supplemented module. It is also proved in [9] that a ring $R$ is (semi-) perfect if and only if every (finitely generated) $R-$ module is supplemented.

If $N + K = M$ and $N \cap K = M$, then $K$ is called a *weak supplement* of $N$ in $M$, and clearly in this situation $N$ is also weak supplement of $K$. $M$ is a *weakly supplemented* module if every submodule of $M$ has a weak supplement in $M$. By using this definition, Büyükas k and Lomp showed that a ring $R$ is left perfect if and only if every left $R-$ module is weakly supplemented, if and only if $R$ is semilocal and the radical of the countably infinite free left $R-$ module has a weak supplement in [7]. Furthermore Alizade and Büyüka s k showed that a ring $R$ is semilocal if and only if every direct product of simple modules is weakly supplemented in [3].

Let $N$ and $K$ be any submodules of $M$ with $M = N + K$. If $N \cap K \leq Rad(K)$ ( $N \cap K \leq Rad(M)$ ) then $K$ is called generalized (weak) supplement of $N$ in $M$. The notion of generalized supplemented modules was introduced by Xue in [16]. $M$ is called *generalized supplemented* module (or briefly *GS*-module) if every submodule $N$ of $M$ has a generalized supplemented $K$ in $M$. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is a generalized supplement. In [14], an $R-$ module $M$ is called *generalized weakly supplemented* (or briefly *GWS*-module) ( *WGS*-module in [14]) if every submodule $K$ of $M$ has a generalized weak supplement $N$ in $M$. For characterizations of generalized supplemented and generalized weakly supplemented modules we refer to [14] and [16].

Alizade et al. studied certain modules whose maximal submodules have supplements, and introduced cofinitely supplemented modules in [1]. A submodule $N$ of a module $M$ is said to be *cofinite* if $\dfrac{M}{N}$ is finitely generated. $M$ is called a *cofinitely (weak) supplemented* module if every cofinite submodule of $M$ has a (weak) supplement in $M$ (see [1], [2], respectively). Nevertheless, it is known by [1, Theorem 2.8] and [2, Theorem 2.11], an $R-$ module $M$ is cofinitely (weak) supplemented if and only if every maximal submodule of $M$ has a (weak) supplement in $M$. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented. $M$ is called a *cofinitely generalized supplemented* if every cofinite submodule of $M$ has a generalized supplement [5]. Since every submodule of a finitely generated module is cofinite, a finitely generated module is generalized supplemented if and only if it is cofinitely generalized supplemented.

A module $M$ is called *locally noetherian* if every finitely generated submodule of $M$ is noetherian. It is known that every module over a noetherian ring is locally noetherian.

In this paper, we will call a module $M$ is *cofinitely generalized weak supplemented* (or briefly *CGWS*-module) if every cofinite submodule of $M$ has a generalized weak supplement. It is easy to see that generalized weakly supplemented modules and cofinitely weak supplemented modules are CGWS-modules. CGWS-modules are defined as analogous of cofinitely weak supplemented (CWS) modules. We will study closure properties of cofinitely generalized weak supplemented modules in the next section. Also, we will show that $M$ is a CGWS-module if and only if every maximal submodule of $M$ has a generalized weak supplement in $M$. Using CGWS-modules and their properties, we will characterize semilocal rings in analogous to [14].

In Section 3, for a commutative domain $R$, we will prove that $R$ is h-semilocal if and only if every torsion $R-$ module is a CGWS-module. As a consequence we will obtain some

conditions equivalent to be CGWS-module for some certain modules over a Dedekind domain.


## 2. COFINITELY GENERALIZED WEAK SUPPLEMENTED MODULES


The following lemma shows that without loss of generality generalized weak supplements of cofinite submodules can be regarded as finitely generated.

**Lemma 2.1** Let $M$ be a module and $U$ be a cofinite (maximal) submodule of $M$. If $V$ is a generalized weak supplement of $U$ in $M$, then $U$ has a finitely generated (cyclic) generalized weak supplement in $M$ contained in $V$.

*Proof.* If $U$ is cofinite, then $\dfrac{M}{U} \cong \dfrac{V}{(V \cap U)}$ is finitely generated. Let $\dfrac{V}{(V \cap U)}$ be generated by elements $x_1 + V \cap U, \; x_2 + V \cap U, .......,x_n + V \cap U$. Then for the finitely generated submodule $W = Rx_1 + Rx_2 + .... + Rx_n$ of $V$, we have $W + U = W + V \cap U + U = V + U = M$ and $W \cap U \leq V \cap U \leq Rad(M)$. Therefore $W$ is a finitely generated generalized weak supplement of $U$ in $M$ contained in $V$. If $U$ is maximal, then $\dfrac{V}{(V \cap U)}$ is a cyclic module generated by some element $x + (V \cap U)$ and $W = Rx$ is a generalized weak supplement of $U$.

The following example shows that a generalized weak supplement of a cofinite submodule need not to be finitely generated. Firstly we need the following lemma.

**Lemma 2.2** The $\mathsf{Z}-$submodule $M = \displaystyle\sum_{q \; prime} \mathsf{Z}\dfrac{1}{q}$, consisting of all rational numbers with square-free denominators, is a small submodule of the $\mathsf{Z}-$module $\mathsf{Q}$ of all rational numbers: $_\mathsf{Z}M = {}_\mathsf{Z}\mathsf{Q}$.

*Proof.* See [2, Lemma 2.2].

**Example 2.1** Consider $\mathsf{Q} \oplus \mathsf{Z}_p$ as a $\mathsf{Z}-$module where $p$ is a prime. Since $\mathsf{Q} \oplus 0$ is a maximal submodule of $\mathsf{Q} \oplus \mathsf{Z}_p$, it is a cofinite submodule of $\mathsf{Q} \oplus \mathsf{Z}_p$. Let $M$ be as in Lemma 2.2. Then $M \oplus \mathsf{Z}_p$ is a generalized weak supplement of $\mathsf{Q} \oplus 0$ since we have $(M \oplus \mathsf{Z}_p) \cap (\mathsf{Q} \oplus 0) = M \oplus 0 = \mathsf{Q} \oplus 0 = Rad(\mathsf{Q} \oplus \mathsf{Z}_p)$ by Lemma 2.2. Since $\dfrac{M}{\mathsf{Z}}$ is a direct sum of the cyclic groups $\langle \dfrac{1}{q} + \mathsf{Z} \rangle$, $\dfrac{M}{\mathsf{Z}}$ is not finitely generated. Hence $M$ is not finitely generated. So generalized weak supplements of cofinite submodules need not be finitely generated.

**Lemma 2.3** Let $M$ be a module. If, for every cofinite submodule $U$ of $M$, there exists a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V$ has a weak supplement in $V$, then $M$ is a CGWS-module.

*Proof.* Let $U$ be a cofinite submodule of $M$. By assumption, there is a submodule $V$ in $M$ such that $M = U + V$ and $U \cap V$ has a weak supplement $X$ in $V$. Then $U \cap V + X = V$ and $(U \cap V) \cap X = U \cap X \ll V$.    Note    that    $M = U + V = U + U \cap V + X = U + X$    and $U \cap X \ll M$. Hence $X$ is a generalized weak supplement of $U$ in $M$. It follows that $M$ is a CGWS-module.

**Lemma 2.4** Let $M$ be a module and $U$ be a cofinite submodule of $M$. If $U$ has a generalized weak supplement $V$ in $M$ and $Rad(K) = K \cap Rad(M)$ for every finitely generated submodule $K$ of $V$, then $U$ has a finitely generated generalized supplement in $M$.

*Proof.* Let $V$ be a generalized weak supplement of $U$ in $M$, i.e. $U + V = M$ and $U \cap V \leq Rad(M)$. Since $\dfrac{M}{U}$ is finitely generated, $U$ has a finitely generated generalized weak supplement $K \leq V$ in $M$, i.e. $M = U + K$ and $U \cap K \leq Rad(M)$ by Lemma 2.1. Then $U \cap K \leq K \cap Rad M) = Rad(K)$. So $K$ is a generalized supplement of $U$ in $M$.

**Theorem 2.1** Let $M$ be a module such that for every finitely generated submodule $K$ of $M$, $Rad(K) = K \cap Rad(M)$. Then $M$ is cofinitely generalized weak supplemented if and only if $M$ is cofinitely generalized supplemented.

*Proof.* Let $U$ be a cofinite submodule of $M$. Since $M$ is a CGWS-module, $U$ has a generalized weak supplement $V$ in $M$ and by Lemma 2.4, $U$ has a generalized supplement. Hence $M$ is cofinitely generalized supplemented.
The converse is obvious.

**Corollary 2.1** Let $M$ be a finitely generated such that for every (finitely generated) submodule $K$ of $M$, $Rad(K) = K \cap Rad(M)$. Then $M$ is generalized weakly supplemented if and only if $M$ is generalized supplemented.

*Proof.* Follows from Theorem 2.1 as in a finitely generated module, every submodule is cofinite.

**Proposition 2.1** Any factor module of a CGWS-module is a CGWS-module.

*Proof.* Let $M$ be a CGWS-module and $L$ be a submodule of $M$. Suppose that $\dfrac{U}{L}$ is a cofinite submodule of $\dfrac{M}{L}$. Note that $\dfrac{(\dfrac{M}{L})}{(\dfrac{U}{L})} \cong \dfrac{M}{U}$. Then $U$ is a cofinite submodule of $M$. Since $M$ is a CGWS-module, $U$ has a generalized weak supplement $V$ in $M$, i.e. $U + V = M$, $U \cap V \leq Rad(M)$. Thus $\dfrac{M}{L} = \dfrac{U}{L} + \dfrac{(V + L)}{L}$. Let $f : M \to \dfrac{M}{L}$ be a canonical

epimorphism. Since $U \cap V \leq Rad(M)$, we get

$$\left(\frac{U}{L}\right) \cap \left(\frac{(V+L)}{L}\right) = \left(\frac{U \cap (V+L)}{L}\right) = \frac{(L+(U \cap V))}{L} = f(U \cap V) \leq f(Rad(M)) \leq Rad\left(\frac{M}{L}\right),$$

which completes the proof.

**Corollary 2.2** Any homomorphic image of a CGWS-module is a CGWS-module.

To prove that an arbitrary sum of CGWS-modules is a CGWS-module, we use the following standard lemma.

**Lemma 2.5** Let $M$ be a module, $N$ and $U$ be submodules of $M$ with cofinitely generalized weak suplemented $N$ and cofinite $U$. If $N+U$ has a generalized weak supplement in $M$, then $U$ also has a generalized weak supplement in $M$.

*Proof.* Let $X$ be a generalized weak supplement of $N+U$ in $M$. Then we have

$$\frac{N}{\left[N \cap (X+U)\right]} \cong \frac{N+(X+U)}{X+U} = \frac{M}{X+U} \cong \frac{\left(\dfrac{M}{U}\right)}{\left(\dfrac{(X+U)}{U}\right)}.$$

Since $U$ is a cofinite submodule, $\dfrac{M}{U}$ is a finitely generated module. The last module is a finitely generated module. Hence $N \cap (X+U)$ has a generalized weak supplement $Y$ in $N$, i.e. $Y + \left[N \cap (X+U)\right] = N$ and $Y \cap \left[N \cap (X+U)\right] = Y \cap (X+U) \leq Rad(N) \leq Rad(M)$. Since $M = U + X + N = U + X + Y + \left[N \cap (X+U)\right] = X + U + Y$, $Y$ is a generalized weak supplement of $X+U$ in $M$. Therefore

$$U \cap (X+Y) \leq \left[X \cap (Y+U)\right] + \left[Y \cap (X+U)\right] \leq Rad(M).$$

This means that $X+Y$ is a generalized weak supplement of $U$ in $M$.

**Proposition 2.2** Any arbitrary sum of CGWS-modules is a CGWS-module.

*Proof.* Let $M = \sum_{i \in I} M_i$ where each module $M_i$ is a cofinitely generalized weak supplemented and $N$ be a cofinite submodule of $M$. Then $\dfrac{M}{N}$ is generated by some finite set $\{x_1 + N, x_2 + N, ...., x_n + N\}$ and therefore $M = Rx_1 + Rx_2 + ... + Rx_n + N$. Since each $x_i$ is contained in the sum $\sum_{j \in J} M_j$ for some finite subset $J = \{1_1, ..., 1_{s(1)}, ..., n_{s(n)}\}$ of $I$, $M = M_{1_1} + \sum_{j \in J - \{1_1\}} M_j + N$ has a trivial generalized weak supplement 0 in $M$ and since $M_{1_1}$ is

a CGWS-module, $N + \sum\limits_{j \in J} M_j$ has a generalized weak supplement by Lemma 2.5. Continuing in this way, we will obtain (after we have used Lemma 2.5 $\sum\limits_{i=1}^{n} s(i)$ times) $N$ has a generalized weak supplement in $M$.

Let $M$ and $N$ be $R-$modules. If there is an epimorphism $f : M^{(\Lambda)} \rightarrow N$ for some set $\Lambda$, $N$ is called an $M - generated\ module$. The following corollary follows from Corollary 2.2 and Proposition 2.2.

**Corollary 2.3** If $M$ is a CGWS-module, then any $M - $generated module is a CGWS-module.

Now we are going to prove that a module is cofinitely generalized weak supplemented if and only if every maximal submodule has a generalized weak supplement in $M$. Firstly we need the following lemma.

**Lemma 2.6** Let $M$ be a module and $K \leq M$ be a generalized weak supplement of a maximal submodule $N$ of $M$. If $K + U$ has a generalized weak supplement in $M$, then $U$ has a generalized weak supplement in $M$ for each submodule $U$ of the module $M$.

*Proof.* Let $X$ be a generalized weak supplement of $K + U$ in $M$, i.e. $X + K + U = M$ and $X \cap (K + U) \leq Rad(M)$. If $K \cap (X + U) \leq Rad(M)$ then

$$(K + X) \cap U \leq \left[ K \cap (X + U) \right] + \left[ X \cap (K + U) \right] \leq Rad(M).$$

So, in this case $K + X$ is a generalized weak supplement of $U$ in $M$. Suppose that $K \cap (X + U) \acute{\cup} Rad(M) \subseteq N$, i.e. $K \cap (X + U) \acute{\cup} K \cap N$. Since $\dfrac{K}{(K \cap N)} \cong \left( \dfrac{(K + N)}{N} \right) = \dfrac{M}{N}$ and $N$ is a maximal submodule of $M$, $K \cap N$ is a maximal submodule of $K$. Therefore $(K \cap N) + \left[ K \cap (X + U) \right] = K$. Also, we get

$$M = U + K + X = U + (K \cap N) + \left[ K \cap (X + U) \right] + X = U + (K \cap N) + X$$

and

$$\left( U \cap \left[ (K \cap N) + X \right] \right) \leq \left[ (K \cap N) \cap (U + X) \right] + \left[ \left( (K \cap N) + U \right) \cap X \right]$$
$$\leq (K \cap N) + \left[ (K + U) \cap X \right] \leq Rad(M).$$

So $\left( (K \cap N) + X \right)$ is a generalized weak supplement of $U$ in $M$.

For a module $M$, let $E$ be the set of all submodules $K$ such that $K$ is a generalized weak supplement for some maximal submodule of $M$ and CGWS($M$) denote the sum of all submodules from $E$.

**Theorem 2.2** Let $M$ be a module. Then the following statements are equivalent:
(i) $M$ is a CGWS-module,
(ii) Every maximal submodule of $M$ has a generalized weak supplement,
(iii) $\dfrac{M}{CGWS(M)}$ has no maximal submodules.

*Proof.* (i) $\Rightarrow$ (ii) is obvious since every maximal submodule is cofinite.

(ii) $\Rightarrow$ (iii) Suppose that there is a maximal submodule of $\dfrac{N}{CGWS(M)}$ of $\dfrac{M}{CGWS(M)}$. Then $N$ is a maximal submodule of $M$. By (ii), there is a generalized weak supplement $K$ of $N$. Then $K \in E$ and so $K \leq CGWS(M) \leq N \leq M$. Hence $N = M$. This contradiction shows that $\dfrac{M}{CGWS(M)}$ has no maximal submodules.

(iii) $\Rightarrow$ (i) Let $U$ be a cofinite submodule of $M$. Since

$$\dfrac{\left(\dfrac{M}{U}\right)}{\left(\dfrac{\left(U + CGWS(M)\right)}{U}\right)} \cong \dfrac{M}{\left(U + CGWS(M)\right)},$$

we have $U + CGWS(M)$ is a cofinite submodule of $M$. If $\dfrac{M}{\left[U + CGWS(M)\right]} \neq 0$ i.e. $U + CGWS(M) \neq M$, then there is a maximal submodule $\dfrac{N}{\left[U + CGWS(M)\right]}$ of $\dfrac{M}{\left[U + CGWS(M)\right]}$. It follows that $N$ is a maximal submodule of $M$ and $\dfrac{N}{CGWS(M)}$ is a maximal submodule of $\dfrac{M}{CGWS(M)}$. This contradicts (iii). So $U + CGWS(M) = M$. Since $\dfrac{M}{U}$ is finitely generated, say some elements $x_1 + U, x_2 + U, ..., x_m + U$, we obtain $M = Rx_1 + Rx_2 + ... + Rx_m + U$. Each element $x_i \ (i = 1, 2, .., m)$ can be written as $x_i = u_i + c_i$, where $u_i \in U, c_i \in CGWS(M)$. Since each $c_i$ is contained in the sum of finite number of submodules from $E$, $M = U + K_1 + K_2 + .... + K_n$ for some submodules $K_1, K_2, ..., K_n$ of $M$ from $E$. Now $M = (U + K_1 + .... + K_{n-1}) + K_n$ has a generalized weak supplement, namely 0. By Lemma 2.6, $U + K_1 + K_2 + .... + K_{n-1}$ has a generalized weak supplement. Continuing in this way (applying Lemma 2.6 $n$ times) we obtain that $U$ has a generalized weak supplement in $M$.

**Theorem 2.3** Let $M$ be CGWS-module and $Rad(M) \leq N$. Then every cofinite submodule of $\dfrac{M}{N}$ is a direct summand.

*Proof.* Let $\dfrac{U}{N}$ be a cofinite submodule of $\dfrac{M}{N}$. Since $\dfrac{\left(\dfrac{M}{N}\right)}{\left(\dfrac{U}{N}\right)} \cong \dfrac{M}{U}$, $U$ is a cofinite submodule

of $M$. Since $M$ is a CGWS- module, $U$ has a generalized weak supplement $V$, i.e.

$U + V = M$ and $U \cap V \le Rad(M)$. Let $f : M \to \dfrac{M}{N}$ be a canonical epimorphism. Then

$\left(\dfrac{(V+N)}{N}\right)$ is a generalized weak supplement of $\dfrac{U}{N}$ in $\dfrac{M}{N}$ by Proposition 3.2 in [14]. Note

that $V \cap U \le Rad(M) \le N$. So $\dfrac{U}{N}$ is a direct summand of $\dfrac{M}{N}$.

**Corollary 2.4** Let $M$ be a CGWS-module. Then every cofinite submodule of $\dfrac{M}{Rad(M)}$ is a
direct summand.

**Theorem 2.4** Let $M$ be a module and $N$ be a submodule with $N \le Rad(M)$. If $\dfrac{M}{N}$ is a
CGWS-module, then $M$ is a CGWS-module.

*Proof.* Let $U$ be any cofinite submodule of $M$. Note that $\left(\dfrac{M}{(U+N)}\right) \cong \left(\dfrac{\left(\dfrac{M}{U}\right)}{\dfrac{(U+N)}{N}}\right)$. Then

$U + N$ is a cofinite submodule of $M$. Since $\dfrac{U+N}{N}$ is a cofinite submodule of $\dfrac{M}{N}$, there is

a submodule $\dfrac{X}{N}$ of $\dfrac{M}{N}$ such that $\left(\dfrac{(U+N)}{N}\right) + \dfrac{X}{N} = \dfrac{M}{N}$ and

$$\left(\dfrac{(U+N)}{N}\right) \cap \left(\dfrac{X}{N}\right) = \left(\dfrac{(U \cap X + N)}{N}\right) \le Rad\left(\dfrac{M}{N}\right).$$

Therefore $N \le Rad(M)$, $Rad\left(\dfrac{M}{N}\right) = \dfrac{(RadM)}{N}$ and so $U \cap X \le Rad(M)$. Lastly,

$U + X = M$ implies that $X$ is a generalized weak supplement of $U$ in $M$.

Let $M$ and $N$ be $R-$modules. An epimorphism $f : M \to N$ is called a *small cover* if
$Ker f = M$. Recall that an epimorphism $f : M \to N$ is called a *generalized cover* if
$Ker f \le Rad(M)$ and $M$ is called a generalized cover of $N$ with an epimorphism
$f : M \to N$.

**Corollary 2.5** A generalized cover of a CGWS-module is a CGWS-module.

**Theorem 2.5** Let $0 \to L \to M \to N \to 0$ be a short exact sequence. If $L$ and $N$ are CGWS-
modules and $L$ has a weak supplement in $M$, then $M$ is a CGWS-module.

*Proof.* Without restriction of generality, we will assume that $L \le M$. Let $S$ be weak supplement of $L$ in $M$, i.e. $L + S = M$ and $L \cap S = M$. Then we have, $\frac{M}{L \cap S} \cong \frac{L}{L \cap S} \oplus \frac{S}{L \cap S}$. $\frac{L}{L \cap S}$ is cofinitely generalized weak supplemented as a factor module of $L$ which is cofinitely generalized weak supplemented. On the other hand, $\frac{S}{L \cap S} \cong \frac{M}{L} \cong N$ is cofinitely generalized weak supplemented. Then $\frac{M}{L \cap S}$ is cofinitely generalized weak supplemented as a sum of cofinitely generalized weak supplemented. Therefore $M$ is a CGWS-module by Corollary 2.5.

**Theorem 2.6** Suppose that $M$ is a module with $Rad(M) = M$. Then $M$ is a cofinitely generalized weak supplemented if and only if $M$ is a cofinitely weak supplemented.

*Proof.* Since $Rad(M) = \left( \sum_{K = M} K \right) = M$, the proof is obvious.

Let $M$ and $N$ be $R -$ modules. If every proper submodule of $M$ is contained in a maximal submodule of $M$, then $M$ is called *coatomic*. Note that every coatomic module has small radical.

**Corollary 2.6** Let $M$ be a coatomic module. Then $M$ is a cofinitely generalized weak supplemented if and only if $M$ is a cofinitely weak supplemented.

**Corollary 2.7** For a finitely generated module $M$, the following statements are equivalent:
 (i) $M$ is generalized weakly supplemented,
 (ii) $M$ is cofinitely generalized weak supplemented,
 (iii) $M$ is cofinitely weak supplemented,
 (iv) $M$ is weakly supplemented.

*Proof.* (i) $\Rightarrow$ (ii) is obvious.
 (ii) $\Rightarrow$ (iii) Since $M$ is finitely generated, the result follows from Theorem 2.6.
 (iii) $\Rightarrow$ (iv) Let $U$ be a submodule of $M$. Since $M$ is finitely generated, $U$ is a cofinite submodule of $M$. By the assumption, $M$ is weakly supplemented.
 (iv) $\Rightarrow$ (i) Since $Rad(M) = \sum_{K = M} K$, the proof is obvious.

Theorem 3.9 in [14] says that a ring $R$ is semilocal if and only if every cyclic module is a generalized weakly supplemented module. Since every left $R -$ module is $_R R -$ generated, we have the following corollary by Corollary 2.3.

**Corollary 2.8** Let $R$ be a ring. Then $R$ is semilocal if and only if every $R -$ module is a CGWS-module.

# 3. COFINITELY GENERALIZED WEAK SUPPLEMENTED MODULES OVER COMMUTATIVE DOMAIN

In [12], Matlis called a commutative domain $S$ h-local if every non-zero ideal of $S$ is contained in only finitely many maximal ideals and $\dfrac{S}{P}$ is a local ring for every non-zero prime ideal $P$ of $S$. $S$ is h-semilocal if every non-zero ideal $I$ of $S$ is contained in only finitely many maximal ideals of $S$ i.e. $\dfrac{S}{I}$ is a semilocal ring. This concept was introduced in there as a simultaneous generalization of both a Dedekind domain and a local domain. Matlis claims that an h- local domain is "a domain in which localizations with respect to maximal ideals behave properly." in the same paper. Among examples of h-local domains Matlis mentions
     (a) local domains
     (b) Dedekind domains and, more generally, one dimensional domains.
In [1], it is proved that a commutative domain $R$ is h-local if and only if every torsion $R-$module is cofinitely supplemented. A relation between h-semilocal domains and CGWS-modules is given in the following theorem.

**Theorem 3.1** The following statements are equivalent for a commutative domain $R$.
     (1) $R$ is h-semilocal.
     (2) Every cyclic torsion $R-$module is generalized weakly supplemented.
     (3) Every torsion $R-$module with $Rad(M) = M$ is generalized weakly supplemented.
     (4) For every torsion $R-$module $M$, every maximal submodule of $M$ has a generalized weak supplement in $M$.
     (5) Every torsion $R-$module is a CGWS-module.

*Proof.* $(1) \Rightarrow (2)$ Let $M \cong \dfrac{R}{I}$ for some non-zero ideal $I$ of $R$. Then $\dfrac{R}{I}$ is a semilocal $\dfrac{R}{I}-$module and so $\dfrac{R}{I}$ is a semilocal $R-$module. Therefore $M$ is generalized weakly supplemented.

$(2) \Rightarrow (3)$ Let $M$ be a torsion $R-$module. By hypothesis $Rm$ is generalized weakly supplemented, then semilocal for every $m \in M$. Since $M = \sum\limits_{m \in M} Rm$, $M$ is generalized weakly supplemented by Proposition 2.12 in [6].

$(3) \Rightarrow (5)$ Let $M$ be a torsion $R-$module. Then $Rm$ is torsion and by hypothesis $Rm$ is generalized weakly supplemented. Therefore $M = \sum\limits_{m \in M} Rm$ is a CGWS-module by Proposition 2.2.

$(4) \Leftrightarrow (5)$ Follows from Theorem 2.2.

$(5) \Rightarrow (1)$ Let $I$ be a non-zero ideal of $R$. Then $\dfrac{R}{I}$ is a torsion $R-$module and a generalized weakly supplemented $R-$module. This means that $\dfrac{R}{I}$ is generalized weakly

supplemented $\frac{R}{I}$ – module. Therefore by [14, Theorem 3.9], $\frac{R}{I}$ is a semilocal and so $R$ is h-semilocal.

**Corollary 3.1** Let $R$ be a h-semilocal domain, $M$ be an $R$ – module and $T(M)$ be the torsion submodule of $M$. Suppose $T(M)$ has a generalized weak supplement in $M$ and $\frac{M}{T(M)}$ is a CGWS-module. Then $M$ is a CGWS-module.

*Proof.* By Theorem 3.1, $T(M)$ is a CGWS-module. Then by Theorem 2.5, $M$ is a CGWS-module.

Let $R$ be a Dedekind domain and $M$ be an $R$ – module. Note that $R$ is noetherian. Denote by $D(M)$ the divisible part of $M$. Then $D(M)$ is injective by [13, Lemma 4.4], hence $M = D(M) \oplus N$ for some submodule $N$ of $M$. In this case $N$ is called *reduced part* of $M$. By [1, Lemma 4.4], $D(M)$ is cofinite submodule of $D(M)$. Thus $D(M)$ is a CGWS-module. In this case, we get $M$ is a CGWS-module if and only if the reduced part of $M$ is a CGWS-module. Hence using Corollary 2.4 we have the following Corollary.

**Corollary 3.2** Let $R$ be a Dedekind domain and $M$ be an $R$ – module and $N$ be the reduced part of $M$. Then the following are equivalent.
    (1) $M$ is a CGWS-module.
    (2) $N$ is a CGWS-module.
    (3) Every maximal submodule of $\frac{N}{Rad(N)}$ is a direct summand.

**Corollary 3.3** Let $R$ be a Dedekind domain and $M$ be an $R$ – module. Suppose the reduced part $N$ of $M$ is torsion. Then $M$ is a CGWS-module.

*Proof.* Since $R$ is h-semilocal, $N$ is a CGWS-module by Theorem 3.1. Hence $M$ is a CGWS-module by Corollary 3.2.

**Proposition 3.1** Let $R$ be a Dedekind domain, $K(R)$ be a quotient field of $R$ and $\{P_i\}_{i \in I}$ be an infinite collection of distinct maximal ideals of $R$. Let $M = \prod_{i \in I}(R / P_i)$ be the direct product of the simple $R$ – modules $R / P_i$ and $T(M)$ be the torsion submodule of $M$. Then the following hold,
    (i) $M / T$ is divisible, i.e. $M / T \cong K(R)^{(J)}$ for some index set $J$,
    (ii) $Rad(M) = \{0\}$.
    Proof. See [3, Lemma 2.9].

The following example shows that a CGWS-modules need not be a GWS-module.

**Example 3.1** Consider $M$ left $R-$module for $R = \mathbb{Z}$ as in Proposition 3.1. Then we can write $\dfrac{M}{T(M)} = \mathbb{Q}^{(J)}$. Due to this isomorphism, there is a submodule $N$ of $M$ such that $\dfrac{N}{T(M)} \cong \mathbb{Q}$. Since $Rad(\mathbb{Q}) = \mathbb{Q}$, there is no maximal submodule of $N$ which includes torsion submodule $T(M)$. Let $U$ be an arbitrary cofinite submodule of $N$. Then there is a maximal submodule of $M$ namely $P$, such that $U \leq P \tilde{} N$. Therefore we obtain $P + T(M) = N$ since $P$ doesn't include torsion submodule $T(M)$. On the other hand, since $T(M)$ is semisimple, there is a submodule $X$ of $T(M)$ such that $(P \cap T(M)) \oplus X = T(M)$. Then we get

$$P \cap X = P \cap ((T(M) \cap X) = (P \cap T(M)) \cap X = \{0\}$$

and

$$N = P + T(M) = P + (P \cap T(M)) \oplus X = P \oplus X.$$

Therefore $N$ is a CGWS-module. Suppose that $N$ is a GWS-module. Since $Rad(N) \leq Rad(M) = \{0\}$, we have $Rad(N) = \{0\}$. So $N$ is semisimple. Thus we can say that $\mathbb{Q}$ left $R-$module is semisimple. This is a contradiction. Consequently, $N$ is a CGWS-module but not a GWS-module.

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