ORIGINAL PAPER

ON NEW TYPES OF WEAKLY NANO CLOSED FUNCTIONS

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Abstract. *In this paper, we should utilize the concepts of $N\alpha$-closed and $Ns\alpha$-closed sets to define some new types of weakly nano closed functions such as; $N\alpha$-closed functions, $N\alpha$*-closed functions, $N\alpha$**-closed functions, $Ns\alpha$-closed functions, $Ns\alpha$*-closed functions and $Ns\alpha$**-closed functions. Also, we should explain the relationships between these types of weakly nano closed functions and the concepts of nano closed functions. Furthermore, we should prove some theorems, properties and remarks.*

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Keywords: *$Ns\alpha$-closed sets, $N\alpha$-closed functions, $N\alpha$*-closed functions, $N\alpha$**-closed functions, $Ns\alpha$-closed functions, $Ns\alpha$*-closed functions and $Ns\alpha$**-closed functions.*

1. INTRODUCTION

Thivagar and Richard [1] present nano topological space on a subset $\mathcal{C}$ of a universe which is characterized with respect to lower and upper approximations of $\mathcal{C}$. He studied about the weak forms of nano open sets. Imran [2] presented the concept of Nsα-open sets in nano topological spaces. Abdulkadhim [3] presented new types of weakly nano open functions. The purpose of this paper is to present new types of weakly nano closed functions such as; Nα-closed functions, Nα*-closed functions, Nα**-closed functions, Nsα-closed functions, Nsα*-closed functions and Nsα**-closed functions. Also, we should explain the relationships between these types of weakly nano closed functions and the concepts of nano closed functions. Furthermore, we should prove some theorems, properties and remarks.

2. PRELIMINARIES

Throughout this paper, $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C})), (\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ and $(\mathcal{W},\rho_{\mathcal{R}}(\mathcal{E}))$ (or frugally $\mathcal{U},\mathcal{V}$ and $\mathcal{W}$) always mean nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a set $\mathcal{M}$ in a nano topological space $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))$, $Ncl(\mathcal{M})$, $Nint(\mathcal{M})$ and $\mathcal{M}^c=\mathcal{U}-\mathcal{M}$ denote the nano closure of $\mathcal{M}$, the nano interior of $\mathcal{M}$ and the nano complement of $\mathcal{M}$ respectively.

Definition 2.1: A subset $\mathcal{M}$ of a nano topological space $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))$ is said to be:

1. A nano α-closed set (briefly Nα-closed set) [1] if $Ncl(Nint(Ncl(\mathcal{M})))\subseteq\mathcal{M}$. The complement of a Nα-closed set is called a nano α-open set (briefly Nα-open set) in

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$(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C}))$. The family of all Nα-closed sets (resp. Nα-open sets) of $\mathcal{U}$ is denoted by NαC$(\mathcal{U}, \mathcal{C})$ (resp. $\tau_{\mathcal{R}}\alpha(\mathcal{C})$).

2. A nano semi-α-closed set (briefly Nsα-closed set) [2] if there exists a Nα-closed set $\mathcal{F}$ in $\mathcal{U}$ such that $Nint(\mathcal{F}) \subseteq \mathcal{M} \subseteq \mathcal{F}$ or equivalently if $Nint(Ncl(Nint(Ncl(\mathcal{M})))) \subseteq \mathcal{M}$. The complement of a Nsα-closed set is called a nano sα-open set (briefly Nsα-open set) in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C}))$. The family of all Nsα-closed sets (resp. Nsα-open sets) of $\mathcal{U}$ is denoted by NsαC$(\mathcal{U}, \mathcal{C})$ (resp. $\tau_{\mathcal{R}}$sα$(\mathcal{C})$).

Remark 2.2 [2]: In a nano topological space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C}))$, the following statements hold and the inverse of each statement is not true:

1. Every N-closed set is a Nα-closed and Nsα-closed.

2. Every Nα-closed set is a Nsα-closed.

Example 2.3: Let $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$ with $\mathcal{U}/\mathcal{R} = \{\{p_1\}, \{p_3\}, \{p_2, p_4\}\}$ and $\mathcal{C} = \{p_1, p_2\}$. Then $\tau_{\mathcal{R}}(\mathcal{C}) = \{\phi, \{p_1\}, \{p_2, p_4\}, \{p_1, p_2, p_4\}, \mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: NC$(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{p_2, p_3, p_4\}, \{p_1, p_3\}, \{p_3\}, \phi\}$. The family of all Nα-closed sets of $\mathcal{U}$ is: NαC$(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{p_2, p_3, p_4\}, \{p_1, p_3\}, \{p_3\}, \phi\}$. The family of all Nsα-closed sets of $\mathcal{U}$ is: NsαC$(\mathcal{U}, \mathcal{C})$ = NαC$(\mathcal{U}, \mathcal{C}) \cup \{\{p_2, p_4\}, \{p_1\}\}$.

Theorem 2.4 [2]: For any subset $\mathcal{M}$ of a nano topological space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C}))$, $\mathcal{M} \in$ NαC$(\mathcal{U}, \mathcal{C})$ iff there exists a N-closed set $\mathcal{F}$ such that $Ncl(Nint(\mathcal{F})) \subseteq \mathcal{M} \subseteq \mathcal{F}$.

Definition 2.5: Let $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ be a function, then $\hbar$ is said to be:

1. Nano closed (briefly N-closed) [4] iff for each $\mathcal{M}$ N-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a N-closed set in $\mathcal{V}$.

2. Nano α-closed (briefly Nα-closed) [5] iff for each $\mathcal{M}$ N-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$.

Theorem 2.6 [4]: A function $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ is N-closed iff $Ncl(\hbar(\mathcal{M})) \subseteq \hbar(Ncl(\mathcal{M}))$, for every $\mathcal{M} \subseteq \mathcal{U}$.

Definition 2.7 [4]: Let $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ be a function, then $\hbar$ is said to be nano continuous (briefly N-continuous) iff for each $\mathcal{M}$ N-closed set in $\mathcal{V}$, then $\hbar^{-1}(\mathcal{M})$ is a N-closed set in $\mathcal{U}$.

Theorem 2.8 [4]: A function $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ is N-continuous iff $Nint(\hbar(\mathcal{M})) \subseteq \hbar(Nint(\mathcal{M}))$, for every $\mathcal{M} \subseteq \mathcal{U}$.

3. WEAKLY NANO CLOSED FUNCTIONS

Definition 3.1: Let $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ be a function, then $\hbar$ is said to be:

1. Nano α*-closed (briefly Nα*-closed) iff for each $\mathcal{M}$ Nα-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$.

2. Nano α**-closed (briefly Nα**-closed) iff for each $\mathcal{M}$ Nα-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a N-closed set in $\mathcal{V}$.

Definition 3.2: Let $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})\right)$ be a function, then $\hbar$ is said to be:

1. Nano semi-α-closed (briefly Nsα-closed) iff for each $\mathcal{M}$ N-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a Nsα-closed set in $\mathcal{V}$.
2. Nano semi-α*-closed (briefly Nsα*-closed) iff for each $\mathcal{M}$ Nsα-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a Nsα-closed set in $\mathcal{V}$.
3. Nano semi-α**-closed (briefly Nsα**-closed) iff for each $\mathcal{M}$ Nsα-closed set in $\mathcal{U}$, then $\hbar(\mathcal{M})$ is a N-closed set in $\mathcal{V}$.

Theorem 3.3: A function $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})\right)$ is Nsα-closed if and only if for every $\mathcal{M} \in \mathcal{U}$, $Nint(Ncl(Nint(Ncl(\hbar(\mathcal{M}))))) \subseteq \hbar(Ncl(\mathcal{M}))$.

Proof: Necessity: For any $\mathcal{M} \in \mathcal{U}$, $\hbar(Ncl(\mathcal{M}))$ is Nsα-closed set in $\mathcal{V}$ this implies that $Nint(Ncl(Nint(\hbar(Ncl(\mathcal{M}))))) \subseteq \hbar(Ncl(\mathcal{M}))$. Hence $Nint(Ncl(Nint(Ncl(\hbar(\mathcal{M}))))) \subseteq \hbar(Ncl(\mathcal{M}))$.
Sufficiency: For any $\mathcal{M} \in \mathcal{U}$, we have by hypothesis $Nint(Ncl(Nint(Ncl(\hbar(\mathcal{M}))))) \subseteq \hbar(Ncl(\mathcal{M}))$. So $\hbar(\mathcal{M})$ is Nsα-closed set in $\mathcal{V}$ and hence $\hbar$ is a Nsα-closed function.

Theorem 3.4:

1. Every N-closed function is a Nα-closed, so it is Nsα-closed, but the inverse is not true in general.
2. Every Nα-closed function is a Nsα-closed, but the inverse is not true in general.

Proof:

1. Let $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})\right)$ be a N-closed function and $\mathcal{M}$ be a N-closed set in $\mathcal{U}$. Then $\hbar(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since any N-closed set is Nα-closed (Nsα-closed), $\hbar(\mathcal{M})$ is a Nα-closed (Nsα-closed) set in $\mathcal{V}$. Hence $\hbar$ is a Nα-closed (Nsα-closed) function.
2. Let $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})\right)$ be a Nα-closed function and $\mathcal{M}$ be a N-closed set in $\mathcal{U}$. Then $\hbar(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since any Nα-closed set is Nsα-closed, $\hbar(\mathcal{M})$ is a Nsα-closed set in $\mathcal{V}$. Hence $\hbar$ is a Nsα-closed function.

Example 3.5: Let $\mathcal{U} = \{1,2,3,4\}$ with $\mathcal{U}/\mathcal{R} = \left\{\{2\}, \{4\}, \{1,3\}\right\}$ and $\mathcal{C} = \{1,2\}$.
Then $\tau_{\mathcal{R}}(\mathcal{C}) = \{\phi, \{3\}, \{1,3\}, \{1,2,3\}, \mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\text{NC}(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{1,2,4\}, \{2,4\}, \{4\}, \phi\}$. The family of all Nα-closed (Nsα-closed) sets of $\mathcal{U}$ is: $\text{N}\alpha\text{C}(\mathcal{U}, \mathcal{C}) = \text{Ns}\alpha\text{C}(\mathcal{U}, \mathcal{C}) = \text{NC}(\mathcal{U}, \mathcal{C}) \cup \left\{\{1,4\}, \{1,2\}, \{2\}, \{1\}\right\}$.
Define a function $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right)$ as $\hbar(1) = 1, \hbar(2) = 4, \hbar(3) = 3$ and $\hbar(4) = 2$. Then $\hbar$ is a Nα-closed, so it is Nsα-closed but not N-closed function.

Example 3.6: Let $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$ with $\mathcal{U}/\mathcal{R} = \{\{p_1\}, \{p_3\}, \{p_2, p_4\}\}$ and $\mathcal{C} = \{p_1, p_2\}$. Then $\tau_{\mathcal{R}}(\mathcal{C}) = \{\phi, \{p_1\}, \{p_2, p_4\}, \{p_1, p_2, p_4\}, \mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\text{NC}(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{p_2, p_3, p_4\}, \{p_1, p_3\}, \{p_3\}, \phi\}$. Let $\mathcal{V} = \{q_1, q_2, q_3, q_4\}$ with $\mathcal{V}/\mathcal{R} = \{\{q_2\}, \{q_4\}, \{q_1, q_3\}\}$ and $\mathcal{D} = \{q_1, q_2\}$.

Then $\sigma_{\mathcal{R}}(\mathcal{D}) = \{\phi, \{q_2\}, \{q_1, q_3\}, \{q_1, q_2, q_3\}, \mathcal{V}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{V}$ is: $\text{NC}(\mathcal{V}, \mathcal{D}) = \{\mathcal{V}, \{q_1, q_3, q_4\}, \{q_2, q_4\}, \{q_4\}, \phi\}$.

Define a function $\hbar: \left(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})\right) \longrightarrow \left(\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})\right)$ as $\hbar(p_1) = \hbar(p_2) = q_2, \hbar(p_3) = \hbar(p_4) = q_4$. Then $\hbar$ is a Nsα-closed function but it is not Nα-closed function.

Remark 3.7: The concepts of N-closed function and Nα*-closed function are independent, as the following examples show:

Example 3.8: In example (3.5), the function $\hbar$ is a Nα*-closed but it is not N-closed.

Example 3.9: Let $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$ with $\mathcal{U}/\mathcal{R} = \{\{p_1\}, \{p_4\}, \{p_2, p_3\}\}$ and $\mathcal{C} = \{p_1, p_4\}$. Then $\tau_{\mathcal{R}}(\mathcal{C}) = \{\phi, \{p_1, p_4\}, \mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\mathrm{NC}(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{p_2, p_3\}, \phi\}$.
Let $\mathcal{V} = \{q_1, q_2, q_3, q_4\}$ with $\mathcal{V}/\mathcal{R} = \{\{q_1\}, \{q_3\}, \{q_2, q_4\}\}$ and $\mathcal{D} = \{q_1, q_2\}$. Then $\sigma_{\mathcal{R}}(\mathcal{D}) = \{\phi, \{q_1\}, \{q_2, q_4\}, \{q_1, q_2, q_4\}, \mathcal{V}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{V}$ is: $\mathrm{NC}(\mathcal{V}, \mathcal{D}) = \{\mathcal{V}, \{q_2, q_3, q_4\}, \{q_1, q_3\}, \{q_3\}, \phi\}$.
Define a function $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ as $\hbar(p_2) = q_1, \hbar(p_1) = q_2, \hbar(p_3) = q_3$ and $\hbar(p_4) = q_4$. Then $\hbar$ is a N-closed function but it is not Nα*-closed.

Proposition 3.10:
1. If $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ is a N-closed, N-continuous function, then $\hbar$ is a Nα*-closed function.
2. $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ is a Nα*-closed function iff $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}\alpha(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}\alpha(\mathcal{D}))$ is a N-closed.

Proof:
1. Let $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ is a N-closed, N-continuous function. To prove $\hbar$ is a Nα*-closed function. Let $\mathcal{M} \in \mathrm{N\alpha C}(\mathcal{U}, \mathcal{C})$, then there exists a N-closed set $\mathcal{N}$ such that $Ncl(Nint(\mathcal{N})) \subseteq \mathcal{M} \subseteq \mathcal{N}$ (by theorem (2.4)). Hence $\hbar(Ncl(Nint(\mathcal{N}))) \subseteq \hbar(\mathcal{M}) \subseteq \hbar(\mathcal{N})$ but $Ncl(\hbar(Nint(\mathcal{N}))) \subseteq \hbar(Ncl(Nint(\mathcal{N})))$ (since $\hbar$ is a N-closed function). Then $Ncl(\hbar(Nint(\mathcal{N}))) \subseteq \hbar(Ncl(Nint(\mathcal{N}))) \subseteq \hbar(\mathcal{M}) \subseteq \hbar(\mathcal{N})$.
   But $Ncl(Nint(\hbar(\mathcal{N}))) \subseteq Ncl(\hbar(Nint(\mathcal{N})))$ (since $\hbar$ is a N-continuous function). Therefore we get $Ncl(Nint(\hbar(\mathcal{N}))) \subseteq \hbar(\mathcal{M}) \subseteq \hbar(\mathcal{N})$. But $\hbar(\mathcal{N})$ is a N-closed set in $\mathcal{V}$ (since $\hbar$ is a N-closed function). Hence $\hbar(\mathcal{M}) \in \mathrm{N\alpha C}(\mathcal{V}, \mathcal{D})$ (by theorem (2.4)). Thus is a Nα*-closed function.
2. The proof of a part (2) is easily.

Remark 3.11: Every Nα*-closed function is a Nα-closed and Nsα-closed but the inverse is not true in general as the following example show:

Example 3.12: Let $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$ with $\mathcal{U}/\mathcal{R} = \{\{p_1\}, \{p_2\}, \{p_3\}, \{p_4\}\}$ and $\mathcal{C} = \{p_1, p_4\}$. Then $\tau_{\mathcal{R}}(\mathcal{C}) = \{\phi, \{p_1, p_4\}, \mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\mathrm{NC}(\mathcal{U}, \mathcal{C}) = \{\mathcal{U}, \{p_2, p_3\}, \phi\}$. Let $\mathcal{V} = \{q_1, q_2, q_3, q_4\}$ with $\mathcal{V}/\mathcal{R} = \{\{q_2\}, \{q_3\}, \{q_1, q_4\}\}$ and $\mathcal{D} = \{q_1, q_3\}$. Then $\sigma_{\mathcal{R}}(\mathcal{D}) = \{\phi, \{q_3\}, \{q_1, q_4\}, \{q_1, q_3, q_4\}, \mathcal{V}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{V}$ is: $\mathrm{NC}(\mathcal{V}, \mathcal{D}) = \{\mathcal{V}, \{q_1, q_2, q_4\}, \{q_2, q_3\}, \{q_2\}, \phi\}$.

Define a function $\hbar: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ as $\hbar(p_1) = q_1, \hbar(p_2) = q_2, \hbar(p_3) = q_3$ and $\hbar(p_4) = q_4$. Then $\hbar$ is a Nα-closed function and Nsα-closed function but not Nα*-closed.

Remark 3.13: The concepts of N-closed function and Nsα*-closed function are independent, for examples:

Example 3.14: In example (3.6), the function $\hbar$ is a Nsα*-closed but it is not N-closed.

Example 3.15: Let $\mathcal{U}=\{p_1,p_2,p_3,p_4\}$ with $\mathcal{U}/\mathcal{R}=\{\{p_1\},\{p_3\},\{p_2,p_4\}\}$ and $\mathcal{C}=\{p_2,p_4\}$. Then $\tau_{\mathcal{R}}(\mathcal{C})=\{\phi,\{p_2,p_4\},\mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\mathrm{NC}(\mathcal{U},\mathcal{C})=\{\mathcal{U},\{p_1,p_3\},\phi\}$.

Let $\mathcal{V}=\{q_1,q_2,q_3,q_4\}$ with $\mathcal{V}/\mathcal{R}=\{\{q_1\},\{q_3\},\{q_2,q_4\}\}$ and $\mathcal{D}=\{q_1,q_2\}$. Then $\sigma_{\mathcal{R}}(\mathcal{D})=\{\phi,\{q_1\},\{q_2,q_4\},\{q_1,q_2,q_4\},\mathcal{V}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{V}$ is: $\mathrm{NC}(\mathcal{V},\mathcal{D})=\{\mathcal{V},\{q_2,q_3,q_4\},\{q_1,q_3\},\{q_3\},\phi\}$.

Define a function $\hbar:(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ as $\hbar(p_1)=q_2,\hbar(p_2)=q_1,\hbar(p_3)=q_4$ and $\hbar(p_4)=q_1$. Then $\hbar$ is a N-closed function but it is not Nsα*-closed.

Proposition 3.16: A function $\hbar:(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ is a Nsα*-closed iff $\hbar:(\mathcal{U},\tau_{\mathcal{R}}\mathrm{s\alpha}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}\mathrm{s\alpha}(\mathcal{D}))$ is a N-closed function.

Proof: Obvious.

Remark 3.17: The concepts of Nα*-closed function and Nsα*-closed function are independent as the following examples show:

Example 3.18: In example (3.12), the function $\hbar$ is a Nsα*-closed but it is not Nα*-closed.

Example 3.19: Let $\mathcal{U}=\{p_1,p_2,p_3,p_4\}$ with $\mathcal{U}/\mathcal{R}=\{\{p_1\},\{p_3\},\{p_2,p_4\}\}$ and $\mathcal{C}=\{p_1,p_2\}$. Then $\tau_{\mathcal{R}}(\mathcal{C})=\{\phi,\{p_1\},\{p_2,p_4\},\{p_1,p_2,p_4\},\mathcal{U}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{U}$ is: $\mathrm{NC}(\mathcal{U},\mathcal{C})=\{\mathcal{U},\{p_2,p_3,p_4\},\{p_1,p_3\},\{p_3\},\phi\}$. Let $\mathcal{V}=\{q_1,q_2,q_3,q_4\}$ with $\mathcal{V}/\mathcal{R}=\{\{q_1\},\{q_3\},\{q_2,q_4\}\}$ and $\mathcal{D}=\{q_1,q_2\}$.

Then $\sigma_{\mathcal{R}}(\mathcal{D})=\{\phi,\{q_1\},\{q_2,q_4\},\{q_1,q_2,q_4\},\mathcal{V}\}$ is a nano topological space. The family of all N-closed sets of $\mathcal{V}$ is: $\mathrm{NC}(\mathcal{V},\mathcal{D})=\{\mathcal{V},\{q_2,q_3,q_4\},\{q_1,q_3\},\{q_3\},\phi\}$.

Define a function $\hbar:(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ as $\hbar(p_1)=q_1,\hbar(p_2)=q_2,\hbar(p_3)=q_2$ and $\hbar(p_4)=q_4$. Then $\hbar$ is a Nα*-closed function but it is not Nsα*-closed.

Theorem 3.20: If a function $\hbar:(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ is Nα*-closed and N-continuous, then it is Nsα*-closed.

Proof: Let $\hbar:(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{C}))\longrightarrow(\mathcal{V},\sigma_{\mathcal{R}}(\mathcal{D}))$ be a Nα*-closed and N-continuous function. Let $\mathcal{M}$ be a Nsα-closed set in $\mathcal{U}$. Then there exists a Nα-closed set say $\mathcal{F}$ such that $Nint(\mathcal{F})\subseteq\mathcal{M}\subseteq\mathcal{F}$. Therefore $Nint(\hbar(\mathcal{F}))\subseteq\hbar(Nint(\mathcal{F}))\subseteq\hbar(\mathcal{M})\subseteq\hbar(\mathcal{F})$ (since $\hbar$ is a N-continuous), but $\hbar(\mathcal{F})\in\mathrm{N\alpha C}(\mathcal{V},\mathcal{D})$ (since $\hbar$ is a Nα*-closed function). Hence $Nint(\hbar(\mathcal{F}))\subseteq\hbar(\mathcal{M})\subseteq\hbar(\mathcal{F})$. Thus, $\hbar(\mathcal{M})\in\mathrm{Ns\alpha C}(\mathcal{V},\mathcal{D})$. Therefore, $\hbar$ is a Nsα*-closed function.

Remark 3.21: The following diagram explains the relationship between weakly nano closed functions.

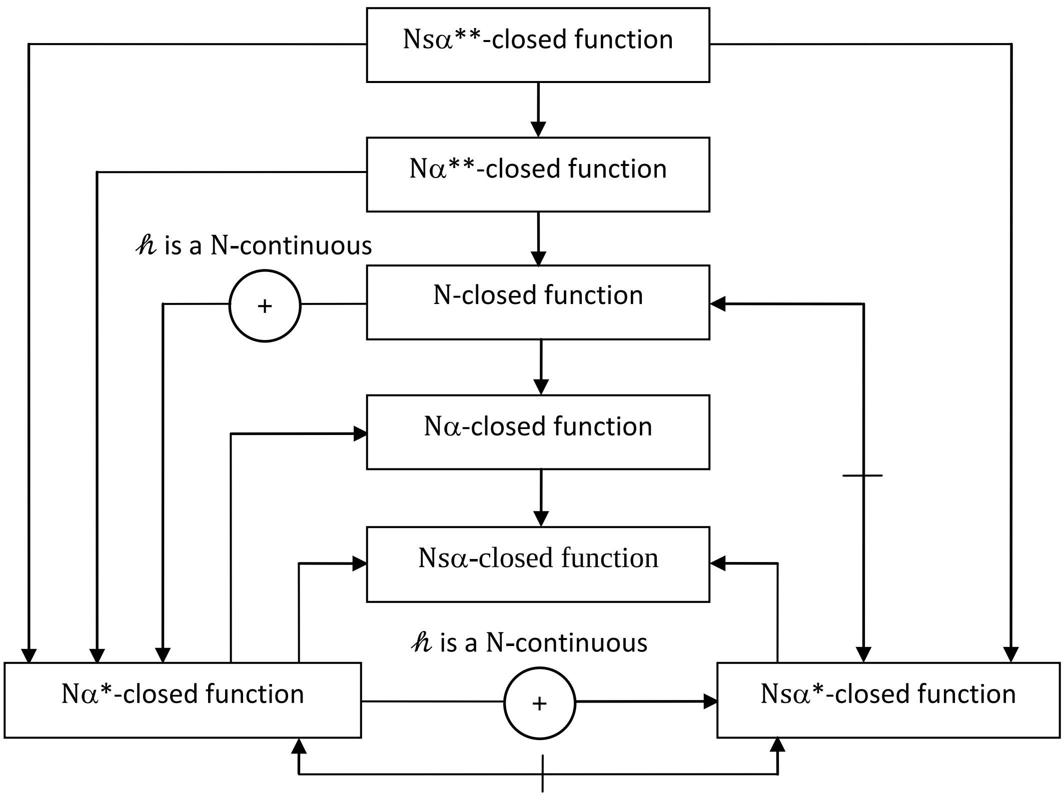

Diagram (3.1)

Theorem 3.22: Let $\hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D}))$ and $\hbar_2: (\mathcal{V}, \sigma_{\mathcal{R}}(\mathcal{D})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ be two functions, then:

1. If $\hbar_1$ is N-closed function and $\hbar_2$ is Nα-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα-closed function.

2. If $\hbar_1$ is Nα-closed function and $\hbar_2$ is Nα*-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα-closed function.

3. If $\hbar_1$ and $\hbar_2$ are Nα*-closed functions, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.

4. If $\hbar_1$ and $\hbar_2$ are Nsα*-closed functions, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nsα*-closed function.

5. If $\hbar_1$ and $\hbar_2$ are Nα**-closed functions, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα**-closed function.

6. If $\hbar_1$ and $\hbar_2$ are Nsα**-closed functions, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nsα**-closed function.

7. If $\hbar_1$ is Nα**-closed function and $\hbar_2$ is Nα*-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.

8. If $\hbar_1$ is Nα-closed function and $\hbar_2$ is Nα**-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a N-closed function.

9. If $\hbar_1$ is Nα**-closed function and $\hbar_2$ is Nα-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.

10. If $\hbar_1$ is Nα**-closed function and $\hbar_2$ is N-closed function, then $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα**-closed function.

Proof:

1. Let $\mathcal{M}$ be a N-closed set in $\mathcal{U}$. Since $\hbar_1$ is a N-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα-closed function.
2. Let $\mathcal{M}$ be a N-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα-closed function, $\hbar_1(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα*-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα-closed function.
3. Let $\mathcal{M}$ be a Nα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα*-closed function, $\hbar_1(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα*-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.
4. Let $\mathcal{M}$ be a Nsα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nsα*-closed function, $\hbar_1(\mathcal{M})$ is a Nsα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nsα*-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nsα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nsα*-closed function.
5. Let $\mathcal{M}$ be a Nα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα**-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since any N-closed set is Nα-closed, $\hbar_1(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα**-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a N-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα**-closed function.
6. Let $\mathcal{M}$ be a Nsα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nsα**-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since any N-closed set is Nsα-closed, $\hbar_1(\mathcal{M})$ is a Nsα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nsα**-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a N-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nsα**-closed function.
7. Let $\mathcal{M}$ be a Nα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα**-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since any N-closed set is Nα-closed, $\hbar_1(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα*-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.
8. Let $\mathcal{M}$ be a N-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα-closed function, $\hbar_1(\mathcal{M})$ is a Nα-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα**-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a N-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a N-closed function.
9. Let $\mathcal{M}$ be a Nα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα**-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since $\hbar_2$ is a Nα-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a Nα-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα*-closed function.
10. Let $\mathcal{M}$ be a Nα-closed set in $\mathcal{U}$. Since $\hbar_1$ is a Nα**-closed function, $\hbar_1(\mathcal{M})$ is a N-closed set in $\mathcal{V}$. Since $\hbar_2$ is a N-closed function, $\hbar_2 \circ \hbar_1(\mathcal{M}) = \hbar_2(\hbar_1(\mathcal{M}))$ is a N-closed set in $\mathcal{W}$. Thus, $\hbar_2 \circ \hbar_1: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{C})) \longrightarrow (\mathcal{W}, \rho_{\mathcal{R}}(\mathcal{E}))$ is a Nα**-closed function.

4. CONCLUSION

We should utilize the concepts of Nα-closed and Nsα-closed sets to define some new types of weakly nano closed functions such as; Nα-closed, Nα*-closed, Nα**-closed, Nsα-closed, Nsα*-closed and Nsα**-closed functions. The Nα-closed and Nsα-closed sets can be used to derive some nano separation axioms.

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