ORIGINAL PAPER

NON-INVARIANT HYPERSURFACES OF A NEARLY SASAKIAN MANIFOLD WITH SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. *The present paper focuses on the study of non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection equipped with -structure. Firstly, some properties of this structure are obtained. Further, the second fundamental forms of non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection has been traced under the condition when is parallel. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection with structure of nearly Sasakian manifold to be totally geodesic.*

Keywords: *Nearly Sasakian, Semi-symmetric non-metric connection, Totally umbilical, Totally geodesic.*

1. INTRODUCTION

Goldberg and Yano [14], in 1970's studied the notion of a non –invariant hypersurface of an almost contact manifold such that transform of a tangent vector of hypersurface by the (1,1) structure tensor field ϕ defining the almost contact structure is never tangent to the hypersurface. Yano studied induced structures on submanifolds [2]. Yano et al [3-5, 7], introduced $(\mathrm{f}, \mathrm{g}, \mathrm{u}, \mathrm{v}, \lambda)$-structure and termed it as a non –invariant hypersurface of an almost contact metric manifold and studied their properties. A hypersurface of an almost contact manifold always admits a $(\mathrm{f}, \mathrm{g}, \mathrm{u}, \mathrm{v}, \lambda)$-structure was studied by Blair and Yano in [1] and [6] respectively. Prasad [12] studied the non –invariant hypersurfaces of trans-Sasakian manifold. In 2011, Prasad and Kishore [13] studied non-invariant hypersurfaces of nearly Sasakian manifold. In the present paper, we study non-invariant hypersurfaces of nearly Sasakian manifold with semi-symmetric non-metric connection.

2. PRELIMINARIES

Let $\bar{M}$ be an almost contact metric manifold with the almost contact metric structure $(\emptyset, \xi, \eta, g)$, where a tensor $\emptyset$ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ and g is a compatible Reimannian metric such that

$$\emptyset^2 X = -X + \eta(X)\boldsymbol{\xi}, \qquad \eta(\xi) = 1, \qquad \emptyset(\xi) = 0, \qquad \eta o \emptyset = 0, \tag{1}$$

$$g(\emptyset X, \emptyset Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

$$g(X, \emptyset Y) = -g(\phi X, Y) - \eta(X)\eta(Y), \;\; g(X, \xi) = \eta(X) \tag{3}$$

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for all $X, Y \in T\bar{M}$.
An almost contact metric manifold is a nearly Sasakian manifold if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X,Y)\boldsymbol{\xi} - \eta(Y)X - \eta(X)Y. \tag{4}$$

Now, we define a semi- symmetric non-metric connection by[10], [11]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{5}$$

such that

$$\bar{\nabla}_X g(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(\phi X, Y).$$

Using (5) and (4), we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 2g(X,Y)\boldsymbol{\xi} - \eta(Y)X - \eta(X)Y - \eta(Y)\phi X - \eta(X)\phi Y. \tag{6}$$

An almost contact manifold $\bar{M}$ satisfying (6) is called non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection.

For a non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection, we have

$$\bar{\nabla}_X \boldsymbol{\xi} = X - \phi X - \eta(X)\xi - \phi\left(\left(\bar{\nabla}_\xi \phi\right)X\right). \tag{7}$$

A hypersurface of an almost contact metric manifold $\bar{M}$ $(\emptyset, \xi, \eta, g)$ is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of (1,1) tensor field ϕ defining the contact structure is never tangent to the hypersurface. Let X be a tangent vector on a non-invariant hypersurface of an almost contact metric manifold $\bar{M}$, then ϕX is never tangent to the hypersurface.

Let M be a non-invariant hypersurface of an almost contact metric manifold. Now if we define the following:

$$\phi X = fX + u(X)\hat{N}, \tag{8}$$

$$\phi\hat{N} = -U, \tag{9}$$

$$\xi = V + \lambda\hat{N}, \ \ \lambda = \eta(\hat{N}), \tag{10}$$

$$\eta(X) = v(X), \tag{11}$$

where f is a (1,1) tensor field, u and v are 1-forms, $\hat{N}$ is a unit normal to the hypersurface, $X \in TM$ and u(X) $\neq 0$; then we get an induced a (f, g, u, v, λ)-structure [3] on M satisfying the conditions:

$$f^2 = -I + u \otimes U + v \otimes V, \tag{12}$$

$$fU = -\lambda V, \ \ fV = \lambda U, \tag{13}$$

$$u \circ f = \lambda v, \ \ v \circ f = -\lambda u, \tag{14}$$

$$v(V) = 1 - \lambda^2, v(U) = u(V) = 0, u(U) = 1 - \lambda^2, \tag{15}$$

$$g(fX, fY) = g(X,Y) - u(X)u(Y) - v(X)v(Y), \tag{16}$$

$$g(X, fY) = -g(fX, Y), \quad g(X,U) = u(X), \quad g(X,V) = v(X), \tag{17}$$

for all $X, Y \in TM$, where $\lambda = \eta(\hat{N})$.
The Gauss and Weingarten formulae for a non-invariant hypersurfaces of a nearly Sasakian manifold with semi-symmetric non-metric connection is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)\hat{N}, \tag{18}$$

$$\bar{\nabla}_X \hat{N} = -A_{\hat{N}} X + \lambda X \tag{19}$$

for all $X, Y \in TM$, where $\bar{\nabla}$ and ∇ are the Riemannian and induced Riemannian connections on $\bar{M}$ and M respectively and $\hat{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$. In this formula σ is the second fundamental form on M related to $A_{\hat{N}}$ by

$$\sigma(X,Y) = g(A_{\hat{N}} X, Y) \tag{20}$$

for all $X, Y \in TM$.

3. NON-INVARIANT HYPERSURFACES

Lemma 3.1 If M be a non-invariant hypersurface with (f, g, u, v, λ)- structure of a nearly Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \big((\nabla_X u)Y + (\nabla_Y u)X + \sigma(X, fY) + \sigma(Y, fX)\big)\hat{N} + (\nabla_X f)Y + (\nabla_Y f)X + 2\sigma(X,Y)U - u(X)A_{\hat{N}}Y - u(Y)A_{\hat{N}}X + u(X)\lambda Y + u(Y)\lambda X, \quad (21)$$

$$(\bar{\nabla}_X\eta)Y + (\bar{\nabla}_Y\eta)X = (\nabla_X u)Y + (\nabla_Y u)X - 2\lambda\sigma(X,Y), \quad (22)$$

$$\bar{\nabla}_X\xi = \nabla_X V - \lambda A_{\hat{N}}X + \lambda^2 X + (\sigma(X,V) + X\lambda)\hat{N} \quad (23)$$

for all $X, Y \in TM$.

Proof. By covariant differentiation, we know that

$$(\bar{\nabla}_X\emptyset)Y = \bar{\nabla}_X\emptyset Y - \emptyset(\bar{\nabla}_X Y)$$
$$(\bar{\nabla}_X\emptyset)Y = \bar{\nabla}_X\big(fY + u(Y)\hat{N}\big) - \emptyset(\nabla_X Y + \sigma(X,Y)\hat{N})$$
$$(\bar{\nabla}_X\phi)Y = (\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X,Y)U + u(Y)\lambda X + ((\nabla_X u)Y + \sigma(X, fY))\hat{N}. \quad (24)$$

Similarly,

$$(\bar{\nabla}_Y\phi)X = (\nabla_Y f)X - u(X)A_{\hat{N}}Y + \sigma(X,Y)U + u(X)\lambda Y + ((\nabla_Y u)X + \sigma(Y, fX))\hat{N}. \quad (25)$$

From (24) and (25),we have

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = (\nabla_X f)Y + (\nabla_Y f)X - u(Y)A_{\hat{N}}X - u(X)A_{\hat{N}}Y + 2\sigma(X,Y) + u(Y)\lambda X + u(X)\lambda Y + \big((\nabla_X u)Y + (\nabla_Y u)X + \sigma(Y, fX) + \sigma(X, fY)\big)\hat{N}.$$

Also,

$$(\bar{\nabla}_X\eta)Y = \bar{\nabla}_X\eta(Y) - \eta(\bar{\nabla}_X Y).$$

Using Gauss formula, we get

$$(\bar{\nabla}_X\eta)Y = (\nabla_X v)Y - \lambda\sigma(X,Y). \quad (26)$$

Similarly,

$$(\bar{\nabla}_Y\eta)X = (\nabla_Y v)X - \lambda\sigma(X,Y). \quad (27)$$

Adding (26) and (27), we get

$$(\bar{\nabla}_X\eta)Y + (\bar{\nabla}_Y\eta)X = (\nabla_X v)Y + (\nabla_Y v)X - 2\lambda\sigma(X,Y).$$

Further consider,

$$\bar{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi)\hat{N}, \quad \text{then}$$
$$\bar{\nabla}_X\xi = \nabla_X V + \lambda\nabla_X\hat{N} + (\nabla_X\lambda)\hat{N} + \sigma(X,V)\hat{N}$$
$$\bar{\nabla}_X\xi = (\nabla_X V - \lambda A_{\hat{N}}X + \lambda^2 X) + (\sigma(X,V) + X\lambda)\hat{N}.$$

Theorem 3.2. If M be a non-invariant hypersurface with (f, g, u, v, λ)- structure of a nearly Sasakian manifold $\bar{M}$, with semi-symmetric non-metric connection, then

$$\sigma(X,\xi)U = -fX + f^2X - u(X)U + f^2((\bar{\nabla}_\xi\emptyset)X) - u((\bar{\nabla}_\xi\emptyset)X)U + f(\nabla_X\xi), \quad (28)$$

$$u(\nabla_X\xi) = u(X) - u(fX) - u(f((\bar{\nabla}_\xi\emptyset)X)) \quad (29)$$

for all $X, Y \in TM$.

Proof. Let us consider

$$(\bar{\nabla}_X\emptyset)\xi = \bar{\nabla}_X\emptyset\xi - \emptyset(\bar{\nabla}_X\xi)$$
$$(\bar{\nabla}_X\emptyset)\xi = -\emptyset(X - fX - u(X)\hat{N} - \eta(X)\xi - f((\bar{\nabla}_\xi\emptyset)X) - u((\bar{\nabla}_\xi\emptyset)X)\hat{N})$$

$$(\overline{\nabla}_X \emptyset)\xi = -\big(fX + u(X)\widehat{N}\big) + f^2X + u(fX)\widehat{N} - u(X)U$$
$$+f^2((\overline{\nabla}_\xi \emptyset)X) + u((\overline{\nabla}_\xi \emptyset)X)U - u(f((\overline{\nabla}_\xi \emptyset)X))\widehat{N}. \quad (30)$$

Since, we know the relation

$$(\overline{\nabla}_X \emptyset)\xi = -\phi(\nabla_X \xi) + \sigma(X,\xi)U. \quad (31)$$

Comparing (30) and (31) and equating tangential and normal part, we get the desired results. Hence theorem is proved.

Theorem 3.3. If M be a non-invariant hypersurface with (f, g, u, v, λ)- structure of a nearly Sasakian manifold $\overline{M}$, with semi-symmetric non-metric connection, then

$$(\nabla_X f)Y + (\nabla_Y f)X = 2g(X,Y)V - v(X)Y - v(Y)X - v(X)\emptyset Y$$
$$-v(Y)\emptyset X - 2\sigma(X,Y)U + u(Y)A_{\widehat{N}}X + u(X)A_{\widehat{N}}Y - u(Y)\lambda X - u(X)\lambda Y, \quad (32)$$
$$(\nabla_X u)Y + (\nabla_Y u)X = 2\lambda g(X,Y) - \sigma(X,fY) - \sigma(fX,Y) \quad (33)$$

for all $X, Y \in TM$.

Proof. In view of (21) and (6),we have

$$\Big((\nabla_X u)Y + (\nabla_Y u)X + \sigma(X,fY)\widehat{N} + \sigma(Y,fX)\Big)\widehat{N} + (\nabla_X f)Y + (\nabla_Y f)X +$$
$$2\sigma(X,Y)U - u(X)A_{\widehat{N}}Y - u(Y)A_{\widehat{N}}X + u(X)\lambda Y + u(Y)\lambda X$$
$$= 2g(X,Y)V + +2\lambda g(X,Y)\widehat{N} - v(Y)X - v(X)Y - v(Y)\phi X - v(X)\phi Y.$$

Equating tangential and normal components of above equations, we can obtain (32) and (33) respectively.
Hence theorem is proved.

Theorem 3.4. If M be a non-invariant hypersurface with (f, g, u, v, λ)- structure of a nearly Sasakian manifold $\overline{M}$, with semi-symmetric non-metric connection, then

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X = 2\lambda g(X,Y)\widehat{N} + 2g(X,Y)V - v(X)Y - v(Y)X$$
$$-v(X)\emptyset Y - v(Y)\emptyset X \quad (34)$$

for all $X, Y \in TM$.

Proof. Consider,

$$(\overline{\nabla}_X \emptyset)Y = \overline{\nabla}_X \emptyset Y - \emptyset(\overline{\nabla}_X Y)$$
$$(\overline{\nabla}_X \emptyset)Y = \nabla_X fY + \sigma(X,fY)\widehat{N} + \overline{\nabla}_X u(Y)\widehat{N} - f(\nabla_X Y) - u(\nabla_X Y)\widehat{N} - \sigma(X,Y)\emptyset\widehat{N}$$
$$(\overline{\nabla}_X \emptyset)Y = (\nabla_X f)Y + \big((\nabla_X u)Y + \sigma(X,fY)\big)\widehat{N} - u(Y)A_{\widehat{N}}X$$
$$+u(Y)\lambda X + \sigma(X,Y)U. \quad (35)$$

Similarly,

$$(\overline{\nabla}_Y \emptyset)X = (\nabla_Y f)X + \big((\nabla_Y u)X + \sigma(Y,fX)\big)\widehat{N} - u(X)A_{\widehat{N}}Y$$
$$+u(X)\lambda Y + \sigma(X,Y)U. \quad (36)$$

Adding (35) and (36), we have

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X = \big((\nabla_X u)Y + (\nabla_Y u)X + \sigma(Y,fX) + \sigma(X,fY)\big)\widehat{N} + (\nabla_X f)Y$$
$$+(\nabla_Y f)X - u(Y)A_{\widehat{N}}X - u(X)A_{\widehat{N}}Y + u(X)\lambda Y + u(Y)\lambda X + 2\sigma(X,Y)U. \quad (37)$$

Putting (32) and (33) in (37), we get

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X = 2\lambda g(X,Y)\widehat{N} + 2g(X,Y)V - v(X)Y - v(Y)X - v(X)\emptyset Y - v(Y)\emptyset X$$

Hence theorem is proved.

Theorem 3.5. If M be a totally umbilical non-invariant hypersurface with (f, g, u, v, λ)-structure of a nearly Sasakian manifold $\overline{M}$, with semi-symmetric non-metric connection. Then, it is totally geodesic if and only if

$$u((\overline{\nabla}_\xi \emptyset)X) + \lambda v(X) + u(X) + \lambda X = 0. \quad (38)$$

In particular, if nearly Sasakian manifold with semi-symmetric non-metric connection admits a contact structure then (38) can be expressed as

$$u(X) + \lambda v(X) + \lambda X = 0 \tag{39}$$

for all $X, Y \in TM$.

Proof. From (10), we have

$$\overline{\nabla}_X \xi = \overline{\nabla}_X (V + \lambda \widehat{N})$$
$$= \overline{\nabla}_X V + (\overline{\nabla}_X \lambda) N + \lambda(\overline{\nabla}_X N).$$

Using (18) & (19), we get

$$\overline{\nabla}_X \xi = (\nabla_X V + \lambda^2 X - A_{\widehat{N}} X) + (X\lambda + \sigma(X, V))\mathrm{N}. \tag{40}$$

From (7) and (40)

$$\nabla_X V - \lambda A_{\widehat{N}} X + \lambda^2 X + (\sigma(X,V) + \lambda X)\widehat{N}$$
$$= X - fX - u(X)\widehat{N} - v(X)(V + \lambda \widehat{N}) - f((\overline{\nabla}_\xi \emptyset)X) - u((\overline{\nabla}_\xi \emptyset)X)\widehat{N}.$$

Equating normal part, we have

$$\sigma(X,V) = -u((\overline{\nabla}_\xi \emptyset)X) - \lambda v(X) - u(X) - \lambda X. \tag{41}$$

Now, if M is totally umbilical, then $A_{\widehat{N}} = \zeta I$, where ζ is Kahlerian metric and (20) reduces to

$$\sigma(X,Y) = g(A_{\widehat{N}} X, Y)$$
$$= g(\zeta X, Y).$$

Therefore, $\sigma(X,Y) = \zeta g(X, Y)$

$$\sigma(X,\xi) = \zeta g(X, \xi)$$
$$= \zeta \eta(X)$$
$$\sigma(X,\xi) = \zeta v(X). \tag{42}$$

So, (41) reduces as

$$\zeta v(X) = u((\overline{\nabla}_\xi \emptyset)X) - \lambda v(X) - u(X) - \lambda X. \tag{43}$$

If M is totally umbilical, that is $\zeta = 0$, then above becomes

$$u((\overline{\nabla}_\xi \emptyset)X) - \lambda v(X) - u(X) - \lambda X = 0. \tag{44}$$

Now, if nearly Sasakian manifold with semi-symmetric non-metric connection is equipped with contact structure then above can be written as

$$\lambda v(X) + u(X) + \lambda X = 0.$$

Hence theorem is proved.

Theorem 3.6. If M be a non-invariant hypersurface with (f, g, u, v, λ)- structure of a nearly Sasakian manifold $\overline{M}$, with semi-symmetric non-metric connection. If f is parallel, then we have

$$\sigma(X,Y) = \frac{\mu - 3\lambda(1-\lambda^2)}{(1-\lambda^2)^2} u(X)u(Y) - \frac{2}{1-\lambda^2}(v(X)u(Y) + v(Y)u(X)), \tag{45}$$

where $\mu = \sigma(U,U) = g(A_{\widehat{N}} U, U)$.

Also, M is totally geodesic if and only if

$$u((\overline{\nabla}_\xi \emptyset)X) + \lambda v(X) - u(X) + \lambda X = 0. \tag{46}$$

Proof. Since f is parallel then equation (32) reduces to

$$2\sigma(X,Y)U = 2g(X,Y)V + u(X)A_{\widehat{N}} Y + u(Y)A_{\widehat{N}} X - v(X)Y - v(Y)X$$
$$-v(X)\emptyset Y - v(Y)\emptyset X - u(X)\lambda Y - u(Y)\lambda X.$$

Applying u both sides, we get

$$2\sigma(X,Y)u(U) = 2g(X,Y)u(V) + u(X)u(A_{\widehat{N}} Y) + u(Y)u(A_{\widehat{N}} X)$$
$$-v(X)u(Y) - v(Y)u(X) - v(X)u(\emptyset Y)$$
$$-v(Y)u(\emptyset X) - u(X)u(\lambda Y) - u(Y)u(\lambda X)$$
$$2(1-\lambda^2)\sigma(X,Y) = u(X)u(A_{\widehat{N}} Y) + u(Y)u(A_{\widehat{N}} X) - v(X)u(Y)$$
$$-v(Y)u(X) - 2\lambda u(Y)u(X). \tag{47}$$

In view of (47), we have

$$2(1-\lambda^2)\sigma(X,U)=u(X)u(A_{\tilde{N}}U)+u(U)u(A_{\tilde{N}}X)-v(X)u(U)$$
$$-v(U)u(X)-2\lambda u(U)u(X).$$

As,
$$h(X,Y)=g(A_{\tilde{N}}X,Y)$$
$$h(X,U)=g(A_{\tilde{N}}X,U)=u(A_{\tilde{N}}X).$$

So, above equation becomes

$$u(A_{\tilde{N}}X)=\left(\frac{\mu}{1-\lambda^2}-2\lambda\right)u(X)-v(X), \quad (48)$$

where $\mu=\sigma(U,U)$.

Following in similar way, we get

$$u(A_{\tilde{N}}Y)=\left(\frac{\mu}{1-\lambda^2}-2\lambda\right)u(Y)-v(Y). \quad (49)$$

In view of equations (47), (48) and (49) ,we get

$$\sigma(X,Y)=\frac{\mu-3\lambda(1-\lambda^2)}{(1-\lambda^2)^2}u(X)u(Y)-\frac{2}{1-\lambda^2}\big(v(X)u(Y)+v(Y)u(X)\big).$$

Next, from (41) and (45), we have

$$u((\overline{\nabla}_\xi\emptyset)X)+\lambda v(X)-u(X)+\lambda X=0.$$

Further, if nearly Sasakian manifold with semi-symmetric non-metric connection posses contact structure then

$$\lambda v(X)-u(X)+\lambda X=0.$$

Hence theorem is proved.

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