*ORIGINAL PAPER*

# ON SPHERICAL ELASTIC CURVES: SPHERICAL INDICATRIX ELASTIC CURVES

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***Abstract.*** *In this paper, we study spherical elastic curves corresponding spherical indicatrix of regular curves with non-vanishing curvature in Euclidean 3-space. From classical variational problem of elastic curves, we derive two Euler-Lagrange equations associated to actions of bending energy functional defined on tangent spherical indicatrix of curves in Euclidean 3-space. We show that the solution of the equation system obtaining with respect to curvature and torsion of the curve corresponds to general helix which is often studied in geometry and we arrange a classification expressing curves whose tangent spherical indicatrix are elastic curve. Finally, we make similar calculations for curves whose principal normal and binormal spherical indicatrix are elastic and we give an example for tangent spherical indicatrix elastic curves.*

***Keywords:*** *Elastic curve, Frenet frame, Spherical indicatrix.*

## 1. INTRODUCTION

Any function defined from a space curve to suitable sphere in Euclidean space is called the spherical indicatrix (image) of the curve. This is a nice way to envision the motion of the curve on a sphere by using components of the Frenet Frame. Furthermore, the movement of a spherical indicatrix describes the changes in the original direction of the curve [5].

The spherical indicatrix of a curve in Euclidean space emerges in three types: tangent indicatrix (tangential indicatrix, tangent spherical indicatrix or tangent spherical image), principle normal indicatrix and binormal indicatrix of the curve [1]. In this paper, we focus on tangent indicatrix of a regular curve for finding some spherical elastic curves by a different method. This method can be applied the other imagine types of curves by using same calculations. But we do not dwell on variational and differential calculations of the problem of finding curves whose principle normal and binormal indicatrix are elastic curve since the same procedure would repeat, however we give necessary results of them in the paper.

The problem of elastic curve is a variational problem associated with the total squared curvature functional

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$$F(\gamma)=\int_{\gamma}(\kappa^2+\lambda)ds \tag{1.1}$$

defined on regular curves of a fixed length satisfying given first boundary data [6]. Here $\gamma$ is a space curve. The Lagrange multiplier $\lambda$ has been included partly. We call $\gamma$ a free elastic curve if $\lambda=0$ [2].

Langer and Singer derived two Euler-Lagrange equations corresponding to critical points of the functional (1.1) given as follows

$$\begin{aligned} 2\kappa''+\kappa^3-2\kappa\tau^2-\lambda\kappa &= 0, \\ \kappa\tau'+2\kappa'\tau &= 0 \end{aligned}$$

and solved this equations [4]. On the other hand, Brunnett and Crouch classify the forms of spherical elastic curve based on the differential equation

$$2\kappa_g''+\kappa_g^3+\left(\frac{2}{r^2}-\rho\right)\kappa_g=0$$

where $\kappa_g$ is the geodesic curvature, $\rho$ is the tension parameter and $r$ is the radius of the sphere [3]. In this paper, we present a different approach to spherical elastic curve from Brunnett and Crouch.

This paper is organized as follows: Section 1 contains some preliminaries about frame fields of spherical indicatrix curves. Section 2 deals with integration of tangent indicatrix elastic curves and we derive motion equations corresponding to curvature energy functional in this section. We also solve these equations by using classical techniques of differential geometry and we give a classification for curves whose tangent indicatrix is an elastic curve. Then we give some solutions of principle normal and binormal indicatrix elastic curves on the sphere. Finally, we examine the results on an instance in Euclidean 3-space.

## 2. GEOMETRICAL SET-UP

Let $\gamma=\gamma(s):\mathbf{I}\to\mathbf{R}^3$ be a unit speed curve in Euclidean 3-space $\mathbf{R}^3$. $T=T(s)=\gamma'$ denotes the unit tangent vector to $\gamma$, $N=\dfrac{\gamma''}{\|\gamma''\|}$ is the unit binormal vector and $B=T\times N$ is the binormal vector. Then we have the Frenet frame $\{T,N,B\}$ along the curve $\gamma$ and Frenet equations are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix}=\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.1}$$

where $\kappa>0$ and $\tau$ are curvature and torsion of $\gamma$, respectively [5].

The spherical indicatrix of a regular curve is an application of the Frenet formulas such that this application allows us to define a new curve by using the unit tangent, principle normal and binormal vectors of the original curve. This is a nice geometric properties illuminate some aspect of behavior of the curve. For example the tangent indicatrix of a unit speed curve $\gamma$ is a curve $\sigma \approx T$ with the same Euclidean coordinates as $T = \gamma'$. Thus, $\sigma$ lies on a unit sphere and the motion of $\sigma$ represents the turning of $\gamma$. Although the original curve $\gamma$ has unit speed, we can not expect that $\sigma$ does also [5].

Now, we define the set of all smooth tangent indicatrix curves in the following

$$\Omega = \{\sigma \,|\, \sigma(0) = a_1,\ \sigma(l) = a_2, \sigma'(0) = a_1', \sigma'(l) = a_2'\},$$

where $a_1$ and $a_2$ are points in $\mathbf{R}^3$. and $a_1'$ and $a_2'$ are non-zero vectors. We defined the subset of the unit speed curves

$$\Omega_u = \{\sigma \in \Omega \,\|\, \|\sigma'\| \equiv 1\}.$$

Then $F^{\Lambda} : \Omega \to \mathbf{R}$ is defined by

$$F^{\Lambda}(\sigma) = \frac{1}{2}\int_{\sigma}\left(\left\|\frac{d^2\sigma}{ds_t^2}\right\| + \Lambda\left(\left\|\frac{d\sigma}{ds}\right\|^2 - 1\right)\right)ds_t.$$

Lagrange multiplier principle says a minimum of $F^{\Lambda}$ on $\Omega_u$ is a stationary point for $F^{\Lambda}$ for some $\Lambda$ which is a Lagrange multiplier standing for the length constraint.

In order to find critical point of the functional $F^{\Lambda}$, we will need to determined following derivatives of $\sigma(s_t) \in \Omega_u$:

$$\frac{d\sigma(s_t)}{ds} = \frac{dT}{ds},$$
$$\frac{d\sigma(s_t)}{ds_t}\frac{ds_t}{ds} = \kappa(s)N(s). \tag{2.2}$$

So the following equality can be easily seen from (2.2)

$$\frac{ds_t}{ds} = \kappa(s), \tag{2.3}$$

and we continue to calculations by using (2.3), we obtain

$$\frac{d^2\sigma}{ds_t^2} = -T(s) + \frac{\tau(s)}{\kappa(s)}B(s), \tag{2.4}$$

$$\frac{d^3\sigma}{ds_t^3} = -\left(1 + \left(\frac{\tau(s)}{\kappa(s)}\right)^2\right)N(s) + \left(\frac{\tau(s)}{\kappa(s)}\right)'\frac{1}{\kappa(s)}B(s), \tag{2.5}$$

where $s_t$ is the arc length parameter of $\sigma$.

## 3. CRITICAL POINTS OF THE CURVATURE ENERGY ACTIONS ON SPHERICAL INDICATRIX CURVES

In this section, we investigate critical points of the functional (1.1) defined on spherical indicatrix curves under some boundary conditions. We firstly examine the tangent indicatrix elastic curves and later we give some results other types spherical indicatrix elastic curves.

Since the problem of elastic curve is a variational problem, we need to determine a variation. Then, we denote by $\sigma$ a variation $\sigma(w,s_t)=\sigma_w(s_t)$ with $\sigma(0,s_t)=\sigma(s_t)$. Associated with such a variation is the variation vector field $W=W(s_t)=\left(\frac{\partial\sigma}{\partial w}\right)(0,s_t)$ along the curve $\sigma(s_t)$. We can write $\sigma(s_t)$, $W(w,s_t)$ with the obvious meaning [7].

We suppose that $\sigma$ is a critical point of the functional $F^{\Lambda}$. If we denote by a variation $\sigma$ with a variational vector field $W$, then we have

$$\partial F^{\Lambda}(W)=\frac{\partial}{\partial s}F^{\Lambda}(\sigma+\delta W)\Big|_{\delta=0}=0, \tag{3.1}$$

(see [4] and [6] ). By using a standard argument involving some integrations by parts, we obtain the first variation formula (3.1) of the functional as follows

$$\int_0^l < E(\sigma),W > ds_t+\left(<\frac{d^2\sigma}{ds_t^2},\frac{dW}{ds_t}>+<\Lambda\frac{d\sigma}{ds_t}-\frac{d^3\sigma}{ds_t^3},W>\right)\Bigg|_0^l=0,$$

where

$$E(\sigma)=\frac{d^4\sigma}{ds_t^4}-\frac{d}{ds_t}(\Lambda\frac{d\sigma}{ds_t}). \tag{3.2}$$

By the first integration of Equation (3.2), we obtain

$$\frac{d^3\sigma}{ds_t^3}-(\Lambda\frac{d\sigma}{ds_t})\equiv A,\quad A=const. \tag{3.3}$$

By using (2.5), (2.2) and the Frenet equations (2.1), Equation (3.3) is calculated as follows

$$E(\sigma)=-\left(1+\Lambda+\left(\frac{\tau}{\kappa}\right)^2\right)N+\left(\frac{\tau}{\kappa}\right)'\frac{1}{\kappa}B=A. \tag{3.4}$$

The first derivative of Equation (3.4) can be found as

$$\left(1+\Lambda+\left(\frac{\tau}{\kappa}\right)^2\right)T-\left(\frac{\Lambda'}{\kappa}+3\frac{\tau}{\kappa^2}\left(\frac{\tau}{\kappa}\right)'\right)N$$

$$+\left(-\frac{\tau}{\kappa}\left(1+\Lambda+\left(\frac{\tau}{\kappa}\right)^2\right)+\frac{1}{\kappa}\left(\left(\frac{\tau}{\kappa}\right)'\frac{1}{\kappa}\right)'\right)B \quad = \quad 0.$$

Since $T$, $N$, and $B$ are lineer independent, we have following Euler-Lagrange equations

$$1+\left(\frac{\tau}{\kappa}\right)^2+\Lambda=0, \tag{3.5}$$

$$-3\frac{\tau}{\kappa^2}\left(\frac{\tau}{\kappa}\right)'-\frac{\Lambda'}{\kappa}=0 \tag{3.6}$$

and

$$-\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)\frac{\tau}{\kappa}+\left(\left(\frac{\tau}{\kappa}\right)'\frac{1}{\kappa}\right)'\frac{1}{\kappa}-\Lambda\frac{\tau}{\kappa}=0. \tag{3.7}$$

If we combine (3.5) and (3.7), we obtain

$$\left(\left(\frac{\tau}{\kappa}\right)'\frac{1}{\kappa}\right)'\frac{1}{\kappa}=0. \tag{3.8}$$

From (3.5), we find $\Lambda'=-\left(1+\left(\frac{\tau}{\kappa}\right)^2\right)'$, and so Equation (3.6) can be written as

$$\left(\frac{\tau}{\kappa}\right)'\frac{\tau}{\kappa^2}=0. \tag{3.9}$$

Equations (3.8) and (3.9) give us the characterization of the tangent indicatrix elastic curve. We solve these Euler-Lagrange equations as follows

$$\frac{\tau}{\kappa}=const.$$

This shows that the tangential indicatrix elastic curve is determined to the curvature and the torsion of the original curve. Then we can give the following theorem.

**Theorem 1** *If tangent indicatrix of a regular curve in Euclidean 3-space is an elastic curve, the equation* $\frac{\tau}{\kappa}=const.$ *must be satisfied, that is, the curve must be general helix.*

Now, we give result of Theorem 1 in the following.

**Corollary 1** *The tangent indicatrix of a planar curve* $(\kappa=const>0, \tau=0)$ *and a circular helix* $(\kappa=const>0, \tau=const>0)$ *in* $\mathbf{R}^3$*. are elastic curves on the unit-sphere.*

We solve the problem of finding the principle normal and binormal indicatrix elastic curves by using similar method. In the following theorem, we exhibit some results.

**Theorem 2** *Any regular curve whose the principal normal indicatrix is elastic can be determined by Euler-Lagrange equation:*

$$
\begin{aligned}
&\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(-\left(\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}+\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)' \\
&-\left(\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\kappa}{(\kappa^2+\tau^2)\tau}-\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\kappa^2+\tau^2}\right)\left(\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\kappa}{\kappa^2+\tau^2}+\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\tau}{\kappa^2+\tau^2}\right)' \\
&-\frac{1}{\kappa^2+\tau^2}\frac{\kappa}{\tau}\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-\left(\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)' \\
&+\left(-\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\kappa}{\kappa^2+\tau^2}+\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\kappa^2}{(\kappa^2+\tau^2)\tau}\right)\left(\left(-\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\kappa^2+\tau^2}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}+\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right) \\
&-\left(-\frac{\kappa}{\kappa^2+\tau^2}\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'+\left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{\tau}{\kappa^2+\tau^2}\right)\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}-\left(\left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)'\frac{1}{\sqrt{\kappa^2+\tau^2}}\right)=0.
\end{aligned}
$$

We can find a special solution of this equation as follows.

**Corollary 2** *A spherical principle normal indicatrix of a general helix is an elastic curve.*

**Theorem 3** *If spherical binormal indicatrix of a regular curve in Euclidean 3-space is an elastic curve, the equation* $\frac{\tau}{\kappa}=const.$ *must be satisfied, that is, the curve must be general helix.*

Now, we give an example for tangent indicatrix elastic curves.

**Example 1** Let $\beta(s)=\left(a\cos\frac{s}{\sqrt{a}},a\sin\frac{s}{\sqrt{a}},\frac{bs}{\sqrt{a}}\right)$ be an unit-speed circular helix. Curvature and torsion of $\beta(s)$ are $\kappa(s)=1>0$ and $\tau(s)=\frac{b}{a}=const.>0,$ so tangent indicatrix of the circular helix is

$$\sigma\approx\beta'(s)=\left(-\sqrt{a}\sin\frac{s}{\sqrt{a}},\sqrt{a}\cos\frac{s}{\sqrt{a}},\frac{b}{\sqrt{a}}\right).$$

Tangent indicatrix of the circular helix with $\|\sigma'\|=1$ is a circle cut from unit sphere by the plane $z=\frac{b}{\sqrt{a}}$. Consequently the tangent indicatrix of $\beta$ is an elastic curve.

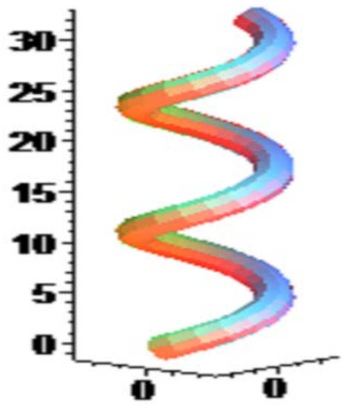


**Figure 1. Circular Helix Correspondig to Example 1.**

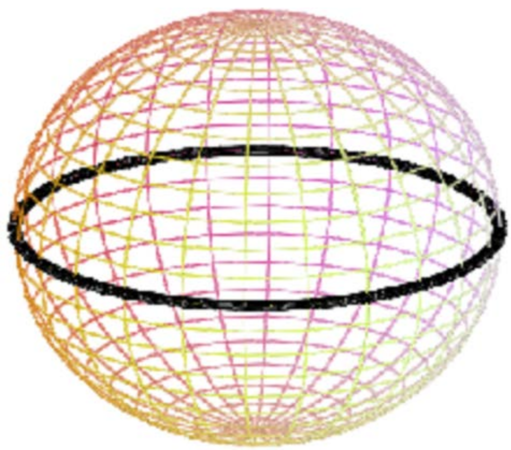

**Figure 2. Tangent Indicatrix Elastic Curve Corresponding to Example 1.**

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