ORIGINAL PAPER

KIRCHHOFF ELASTIC RODS IN MINKOWSKI 3−SPACE

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Abstract. *We study Kirchhoff elastic rod whose centerlines are non-null curves in the Minkowski 3−space. In particular, we obtain the differential equation describing non-null Kirchhoff elastic rod centerlines. Afterwards, we solve this differential equation in terms of Jacobi elliptic functions for three different cases.*

Keywords: *Kirchhoff elastic rods, Jacobi elliptic functions, Minkowski 3-space, variational calculus.*

1. INTRODUCTION

The classical mathematical models of equilibrium configurations of thin elastic rod are the elastic curves and the Kirchhoff elastic rods. Since the elastic curve is a critical curve of the energy of bending only, it is the simplest model. But the Kirchhoff elastic rod is a more complicated model and is a critical framed curve of the energy with the effects of both bending and twisting. The curve obtained by eliminating the frame of a Kirchhoff elastic rod is called a Kirchhoff elastic rod centerline. So, a Kirchhoff elastic rod centerline is a generalization of an elastic curve [10].

As mentioned above, the elastic curve is a minimizing the integral of the squared curvature among with specified boundary conditions. Many authors have studied the elastic curves in Euclidean, non-Euclidean spaces and a Riemannian manifold to date from the time of James Bernoulli in 1690s [3-6, 13, 17, 19, 22].

Kirchhoff rods in $\mathbb{R}^3$ have been studied by some authors [7-10, 14, 18, 19] after the study of Kirchhoff [11]. Kirchhoff rod centerlines in $\mathbb{R}^3$ by taking cylindrical coordinates are explicitly expressed in terms of Jacobi elliptic function and elliptic integrals in Euclidean $3-$space $\mathbb{R}^3$ by Langer and Singer [14], Shi and Hearst [18]. By using these explicit expressions, Ivey and Singer [6] completely classified the closed elastic rod centerlines in $\mathbb{R}^3$ and determined their knot types. Kirchhoff elastic rods are also studied by Kawakubo [7-10]. He generalized the Kirchhoff elastic rods in the $3-$dimensional Euclidean space $\mathbb{R}^3$ to a Riemannian manifold [7]. In 2004, he studied Kirchhoff elastic rods in the three-sphere S^3 [8]. Later, in 2008, he investigated the Kirchhoff elastic rods in the three-dimensional space forms [9]. Lastly, he examined Kirchhoff rod centerlines in five-dimensional space forms which are fully immersed and not helices [10].

It is important to work non-null curves because timelike curves correspond to the path of an observer moving at less than the speed of light and spacelike curves correspond to the

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geometric equivalent of moving faster than light. Besides, the motion of a particle exposed only to gravity is modelled on a timelike geodesic in space-time [2].

In this paper, our purpose indicate the differential equation of Kirchhoff elastic rod whose centerlines are non-null curves in Minkowski $3-$space $\mathbb{R}_1^3$ and solve this differential equation in terms of Jacobi elliptic functions (see, for Jacobi elliptic functions [1]).

2. PRELIMINARIES

Let $\mathbb{R}_1^3$ denote a Minkowski $3-$space with the Lorentzian metric given by

$$<.,.>=-dx_1^2+dx_2^2+dx_3^2,$$

where (x_1,x_2,x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. A vector x of $\mathbb{R}_1^3$ is said to be spacelike if $<x,x>>0$ or $x=0$, timelike if $<x,x><0$ and lightlike (or null) if $<x,x>=0$ and $x\neq 0$.

For any $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{R}_1^3$, the Lorentzian vector product of x and y is defined by

$$x\times y=\left(x_3y_2-x_2y_3,\,x_3y_1-x_1y_3,\,x_1y_2-x_2y_1\right).$$

Let

$$\begin{aligned}\gamma:\ I\subset\mathbb{R}&\rightarrow\mathbb{R}_1^3\\ s&\rightarrow\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s))\end{aligned}$$

be a smooth unit-speed curve in Minkowski $3-$space $\mathbb{R}_1^3$, where I is an open interval. A curve γ is said spacelike (resp., timelike or lightlike) at $s\in I$ if $\gamma'(s)$ is a spacelike (resp., timelike or lightlike) for all $s\in I$.

Now, we consider a curve γ in Minkowski $3-$space $\mathbb{R}_1^3$, parametrized by arc length s, $0\leq s\leq l$. At a point $\gamma(s)$ of γ, let $T=\gamma'(s)$ denote the unit tangent vector to γ, let $N(s)$ denote the unit principal normal and $\varepsilon_2 B(s)=T(s)\times N(s)$ is the unit binormal vector. Then $\{T,N,B\}$ is an orthonormal basis for all vectors at $\gamma(s)$ on γ which is called the Frenet frame along γ. The derivative equations of Frenet frame $\{T,N,B\}$ are

$$\begin{pmatrix}T'\\N'\\B'\end{pmatrix}=\begin{pmatrix}0&\varepsilon_1\kappa&0\\-\varepsilon_0\kappa&0&\varepsilon_2\tau\\0&-\varepsilon_1\tau&0\end{pmatrix}\begin{pmatrix}T\\N\\B\end{pmatrix}, \tag{1}$$

where $\varepsilon_0=<T,T>=\pm 1$, $\varepsilon_1=<N,N>=\pm 1$ and $\varepsilon_2=<B,B>=\pm 1$. Also κ and τ are the curvature and the torsion of γ, respectively (see [15, 16]).

3. VARIATION FORMULAS

Let $\gamma(t): I \to \mathbb{R}_1^3$ be a non-null curve in Minkowski $3-$ space $\mathbb{R}_1^3$. $V = V(t)$ will denote the tangent vector to γ, T the unit tangent vector and v the speed $v(t) = \|V(t)\| = |<V(t), V(t)>|^{\frac{1}{2}}$.

A variation of the curve γ is defined by

$$\begin{array}{llll} \gamma: & (-\delta,\delta)\times I & \to & \mathbb{R}_1^3 \\ & (w,t) & \to & \gamma(w,t) = \gamma_w(t) \end{array}$$

with $\gamma(0,t) = \gamma(t)$. Associated with such a variation is the variation vector field $W = \left.\frac{\partial \gamma_w}{\partial w}\right|_{w=0}$ along the curve $\gamma(t)$. More generally, we use $V = V(w,t)$, $W = W(w,t)$, $T = T(w,t)$, $v = v(w,t)$ etc., with the usual meaning. V velocity vector is $V(0,t) = \frac{\partial \gamma}{\partial t} = v(0,t)T(0,t)$.

Now, we consider the functional is given by

$$\mathcal{F}(\gamma) = \int_0^l \left(\lambda_1 + \lambda_2\tau + \frac{\lambda_3}{2}\kappa^2\right) ds = \int_0^1 \left(\lambda_1 + \lambda_2\tau + \frac{\lambda_3}{2}\kappa^2\right) v dt.$$

We will calculate the first variation of $\mathcal{F}$ in the direction of the variation vector field W (see, for calculus of variations [20]). For this, we will need derivatives of v, κ and τ in the direction of W.

We have the structural equations

$$X(Y) - Y(X) = [X,Y], \tag{2}$$

$$X(Y(Z)) - Y(X(Z)) - [X,Y](Z) = 0 \tag{3}$$

for the vector fields X, Y and Z in $\mathbb{R}_1^3$ (see, [4, 13, 16, 21]). If we use equations (1), (2) and (3), we have following lemma.

Lemma 1. ([4, 5, 13, 21]) Using the above notation, the following allegations are true:

- $[W,V] = 0$,
- $W(v) = \varepsilon_0 v < W_s, T >$,
- $[W,T] = -\varepsilon_0 < W_s, T > T$,
- $W(\kappa) = < W_{ss}, N > -2\varepsilon_0\kappa < W_s, T >$,
- $W(\tau) = \varepsilon_1(< W_{ss}, \frac{B}{\kappa} >)_s - \varepsilon_0 < W_s, \tau T - \kappa B >$.

4. DIFFERENTIAL EQUATION OF NON-NULL KIRCHHOFF ELASTIC ROD CENTERLINES

In this section, we will obtain differential equation determining the non-null Kirchhoff elastic rod centerlines in Minkowski $3-$space $\mathbb{R}_1^3$. Now, we consider a non-null Kirchhoff elastic rod centerline as a critical point of the functional

$$\mathcal{F}(\gamma)=\lambda_1\int_0^l ds+\lambda_2\int_0^l \tau ds+\frac{\lambda_3}{2}\int_0^l \kappa^2 ds$$

in Minkowski $3-$space $\mathbb{R}_1^3$ of curves

$$\gamma:[0,\ell]\to\mathbb{R}_1^3,\qquad \|\gamma'(s)\|=1$$

$$\gamma(0)=P_0,\qquad \gamma(l)=P_\ell,\qquad \gamma'(0)=V_0,\qquad \gamma'(l)=V_\ell.$$

Using Lemma 1 one obtains the first variation of $\mathcal{F}$ in the direction of W

$$\delta\mathcal{F}(\gamma)[W]=\int_0^l\left\{(\lambda_3\kappa W(\kappa)+\lambda_2 W(\tau))v+\left[\lambda_1+\lambda_2\tau+\frac{\lambda_3}{2}\kappa^2\right]W(v)\right\}dt$$
$$=\int_0^l\{\lambda_3\kappa<W_{ss},N>-\frac{3}{2}\varepsilon_0\lambda_3\kappa^2<W_s,T>+\varepsilon_1\lambda_2(<W_{ss},\frac{B}{\kappa}>)_s$$
$$+\varepsilon_0\lambda_2\kappa<W_s,B>+\varepsilon_0\lambda_1<W_s,T>\}ds. \qquad (4)$$

Later, if partial integration is used equation (4), we have

$$\delta\mathcal{F}(\gamma)[W]=\int_0^l\left(\lambda_3\kappa_{ss}+\frac{1}{2}\varepsilon_0\varepsilon_1\kappa(\lambda_3\kappa^2-2\lambda_1)-\varepsilon_1\kappa\tau\left(-\varepsilon_0\lambda_2+\varepsilon_2\lambda_3\tau\right)\right)<W,N>ds$$
$$+\int_0^l(\kappa_s(2\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)+\varepsilon_2\lambda_3\kappa\tau_s)<W,B>ds \qquad (5)$$
$$+(\frac{\varepsilon_1\lambda_2}{\kappa}<W_{ss},B>+\lambda_3\kappa<W_s,N>)|_0^l$$
$$-\left(<W,\frac{-2\varepsilon_0\lambda_1+\varepsilon_0\lambda_3\kappa^2}{2}T+\lambda_3\kappa_s N+\kappa(\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)B>\right)|_0^l.$$

The formula (5) can be written as

$$\delta\mathcal{F}(\gamma)[W]=\int_0^l<W,\xi[\gamma]>ds+(\Psi[\gamma,W])|_0^l,$$

in terms of Euler and boundary operators $\xi[\gamma]$ and $\Psi[\gamma,W]$, where l is the length of γ.

The term $\xi[\gamma]$ is

$$\xi[\gamma]=\left(\lambda_3\kappa_{ss}+\frac{1}{2}\varepsilon_0\varepsilon_1\kappa(\lambda_3\kappa^2-2\lambda_1)-\varepsilon_1\kappa\tau\left(-\varepsilon_0\lambda_2+\varepsilon_2\lambda_3\tau\right)\right)N$$

$$+(\kappa_s(2\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)+\varepsilon_2\lambda_3\kappa\tau_s)B$$

and $\Psi[\gamma,W]$ is

$$\Psi[\gamma,W]=(\frac{\varepsilon_1\lambda_2}{\kappa}<W_{ss},B>+\lambda_3\kappa<W_s,N>)|_0^l$$
$$-\left(<W,\frac{-2\varepsilon_0\lambda_1+\varepsilon_0\lambda_3\kappa^2}{2}T+\lambda_3\kappa_sN+\kappa(\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)B>\right)|_0^l .$$

Thus, under suitable boundary conditions, γ is a critical point of $\mathsf{F}(\gamma)$ if and only if the following Euler-Lagrange equation $\xi[\gamma]=0$ is satisfied:

$$\lambda_3\kappa_{ss}+\frac{1}{2}\varepsilon_0\varepsilon_1\kappa(\lambda_3\kappa^2-2\lambda_1)-\varepsilon_1\kappa\tau(-\varepsilon_0\lambda_2+\varepsilon_2\lambda_3\tau)=0,$$

$$\kappa_s(2\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)+\varepsilon_2\lambda_3\kappa\tau_s=0.$$

In this case, the first variation formula reduces to

$$\delta\mathsf{F}(\gamma)[W]=(\Psi[\gamma,W])|_0^l=(\frac{\varepsilon_1\lambda_2}{\kappa}<W_{ss},B>+\lambda_3\kappa<W_s,N>-<J,W>)|_0^l . \tag{6}$$

Here, we have set

$$J=\frac{-2\varepsilon_0\lambda_1+\varepsilon_0\lambda_3\kappa^2}{2}T+\lambda_3\kappa_sN+\kappa(\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)B. \tag{7}$$

Formula (6) puts us in a position to apply the Noether to constants of motion along γ. First we consider translational their infinitesimal counterparts, the constant vector fields clearly zero, then we have

$$0=\delta\mathsf{F}(\gamma)[W]=(<J,W>)|_0^l=<J(l),W(l)>-<J(0),W(0)>.$$

The variation formulas continue to hold when l is replaced with any intermediate l', $0<l'<l$. It follows that $<J,W>$ is constant on $[0,l]$. Yet, W is an arbitrary constant field, so we have the following theorem:

Theorem 2. $J=\frac{-2\varepsilon_0\lambda_1+\varepsilon_0\lambda_3\kappa^2}{2}T+\lambda_3\kappa_sN+\kappa(\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)B$ is a constant vector field along non-null Kirchhoff elastic rod in equilibrium.

Take into consideration J corresponds to $-F$, where F is the (constant) force along the rod (see, [12]). The converse to this result is an immediate consequence of the observation that $J_s=\xi[\gamma]$ for any non-null curve γ.

Proposition 3. If $J=\frac{-2\varepsilon_0\lambda_1+\varepsilon_0\lambda_3\kappa^2}{2}T+\lambda_3\kappa_sN+\kappa(\varepsilon_2\lambda_3\tau-\varepsilon_0\lambda_2)B$ is a constant vector field along a non-null curve γ, then γ satisfies the Euler equations $\xi[\gamma]$.

Corollary 4. The curvature and torsion of a non-null Kirchhoff elastic rod centerline provide the following pair of equations:

$$\text{constant} = \mu^2 = \varepsilon_0 \frac{(2\lambda_1 - \lambda_3\kappa^2)^2}{4} + \varepsilon_1\lambda_3^2\kappa_s^2 + \varepsilon_2\kappa^2(\varepsilon_2\lambda_3\tau - \varepsilon_0\lambda_2)^2$$

and

$$0 = [\kappa^2(2\varepsilon_2\lambda_3\tau - \varepsilon_0\lambda_2)]_s. \tag{8}$$

In the above corollary, if the equation (8) is integrated, we have

$$\kappa^2(2\varepsilon_2\lambda_3\tau - \varepsilon_0\lambda_2) = c = constant. \tag{9}$$

Using equations (7) and (9) we obtain the differential equation describing Kirchhoff elastic rod whose centerline is non-null curve in Minkowski $3-$space as follow:

$$\mu^2 = < J, J > = \varepsilon_0 \frac{(2\lambda_1 - \lambda_3\kappa^2)^2}{4} + \varepsilon_1\lambda_3^2\kappa_s^2 + \varepsilon_2 \frac{(c - \varepsilon_0\lambda_2\kappa^2)^2}{4\kappa^2}. \tag{10}$$

5. SOLUTION OF DIFFERENTIAL EQUATION

Solving the differential equation (10) are given in terms of three parameters p, w, κ_0 and Jacobi elliptic function $sn(x, p)$. Making the change of variable $u = \kappa^2$, we arrive

$$P(u) = u_s^2 = -\varepsilon_0\varepsilon_1 u^3 - \frac{\varepsilon_1(\varepsilon_2\lambda_2^2 - 4\varepsilon_0\lambda_1\lambda_3)}{\lambda_3^2}u^2 - \frac{\varepsilon_1(4\varepsilon_0\lambda_1^2 - 2\varepsilon_0\varepsilon_2 c\lambda_2 - 4\mu^2)}{\lambda_3^2}u - \frac{\varepsilon_1\varepsilon_2 c^2}{\lambda_3^2}. \tag{11}$$

We then consider the three possible cases for non-null Kirchhoff elastic rod centerlines.

Case 1: (*Timelike Kirchhoff elastic rod centerlines*) Let us consider the case $\varepsilon_0 = -1$, $\varepsilon_1 = \varepsilon_2 = 1$. Then, from (11) we have

$$P(u) = u_s^2 = u^3 - \frac{(\lambda_2^2 + 4\lambda_1\lambda_3)}{\lambda_3^2}u^2 - \frac{(-4\lambda_1^2 + 2c\lambda_2 - 4\mu^2)}{\lambda_3^2}u - \frac{c^2}{\lambda_3^2}. \tag{12}$$

This $(u_s)^2 = P(u) = 0$ equation is solved by using Jacobi elliptic functions. The cubic polynomial $P(u)$ satisfies $P(0) = -\frac{c^2}{\lambda_3^2} \leq 0$. Then, the curvature is non-zero. Moreover, if $u = \kappa^2$ is a non-constant solution to (12), it must clearly take on values at which $P(u) > 0$. Now, we may assume $P(u)$ has three real roots α_1, α_2, α_3 satisfying $0 < \alpha_3 < \alpha_2 < \alpha_1$. Then, we can write equation (12) in the form

$$(u_s)^2-(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)=0$$

and the solution can be written in terms of the Jacobi elliptic function as

$$u=u(s)=\alpha_1+q^2sn^2(rs,p),$$

where

$$p^2=\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1},\ q^2=\alpha_2-\alpha_1,\ r=\sqrt{\frac{q^2}{4p^2}}=\frac{1}{2}\sqrt{\alpha_3-\alpha_1}.$$

Of course, α_1, α_2, α_3 are related to the coefficients of $P(u)$ by

$$\frac{4\lambda_1\lambda_3+\lambda_2^2}{\lambda_3^2}=\alpha_1+\alpha_2+\alpha_3,$$

$$\frac{c^2}{\lambda_3^2}=\alpha_1\alpha_2\alpha_3,$$

$$\frac{4\mu^2-2c\lambda_2+4\lambda_1^2}{\lambda_3^2}=\alpha_1\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3.$$

The parameter κ_0 is maximum curvature, p and w with $0\le p\le w\le 1$ control the shape. The square of maximum curvature for γ timelike Kirchhoff elastic rod centerlines are $u(0)=\alpha_1=\kappa_0^2$. Then, the formula for the curvature is

$$\kappa^2=\kappa_0^2+\frac{p^2}{w^2}sn^2(t,p)\quad \text{with}\quad t=\frac{s}{2w}.$$

The parameters p, w and κ_0 are determined by the coefficients by the following relations:

$$\frac{4\lambda_1\lambda_3+\lambda_2^2}{\lambda_3^2}=3\kappa_0^2+\frac{1}{w^2}\left(p^2+1\right),$$

$$\frac{c^2}{\lambda_3^2}=\kappa_0^2\left(\frac{p^2}{w^2}+\kappa_0^2\right)\left(\frac{1}{w^2}+\kappa_0^2\right),$$

$$\frac{4\mu^2-2c\lambda_2+4\lambda_1^2}{\lambda_3^2}=\kappa_0^2\left(\frac{1}{w^2}(p^2+1)+2\kappa_0^2\right)+\left(\frac{p^2}{w^2}+\kappa_0^2\right)\left(\frac{1}{w^2}+\kappa_0^2\right).$$

Case 2: (*Spacelike Kirchhoff elastic rod centerlines with timelike principal normal*) In this case $\varepsilon_1=-1$, $\varepsilon_0=\varepsilon_2=1$. Then, using (11) we obtain

$$P(u)=u_s^2=u^3+\frac{\left(\lambda_2^2-4\lambda_1\lambda_3\right)}{\lambda_3^2}u^2+\frac{\left(4\lambda_1^2-2c\lambda_2-4\mu^2\right)}{\lambda_3^2}u+\frac{c^2}{\lambda_3^2}. \tag{13}$$

The cubic polynomial $P(u)$ satisfies $P(0)=\frac{c^2}{\lambda_3^2}>0$. Then the curvature may be zero. We can assume $P(u)>0$ has three real roots α_1, α_2, α_3 satisfying $\alpha_3<0<\alpha_2<\alpha_1$. Then, we can write equation (13) in the form

$$(u_s)^2-(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)=0$$

and the solution can be written in terms of the Jacobi elliptic function as

$$u=u(s)=\alpha_1+q^2 sn^2(rs,p),$$

where

$$p^2=\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1},\, q^2=\alpha_2-\alpha_1,\, r=\sqrt{\frac{q^2}{4p^2}}=\frac{1}{2}\sqrt{\alpha_3-\alpha_1}.$$

The real roots α_1, α_2, α_3 are also related to the coefficients of $P(u)$ by the equations

$$\frac{4\lambda_1\lambda_3-\lambda_2^2}{\lambda_3^2}=\alpha_1+\alpha_2+\alpha_3,$$

$$-\frac{c^2}{\lambda_3^2}=\alpha_1\alpha_2\alpha_3,$$

$$-\frac{4\mu^2+2c\lambda_2-4\lambda_1^2}{\lambda_3^2}=\alpha_1\alpha_3+\alpha_1\alpha_2+\alpha_2\alpha_3.$$

Here, the formula for the curvature is

$$\kappa^2=\kappa_0^2+\frac{p^2}{w^2}sn^2(t,p) \quad \text{with} \quad t=\frac{s}{2w}$$

as Case 1. The parameters p, w and κ_0 are determined by the coefficients by the following relations:

$$\frac{4\lambda_1\lambda_3-\lambda_2^2}{\lambda_3^2}=3\kappa_0^2+\frac{1}{w^2}\left(p^2+1\right),$$

$$\frac{c^2}{\lambda_3^2}=-\kappa_0^2\left(\frac{p^2}{w^2}+\kappa_0^2\right)\left(\frac{1}{w^2}+\kappa_0^2\right),$$

$$\frac{-4\mu^2-2c\lambda_2+4\lambda_1^2}{\lambda_3^2}=\kappa_0^2\left(\frac{1}{w^2}(p^2+1)+2\kappa_0^2\right)+\left(\frac{p^2}{w^2}+\kappa_0^2\right)\left(\frac{1}{w^2}+\kappa_0^2\right).$$

Case 3: (*Spacelike Kirchhoff elastic rod centerlines with spacelike principal normal*) Another case is $\varepsilon_2=-1$, $\varepsilon_0=\varepsilon_1=1$. Then, using (11) we obtain

$$P(u) = u_s^2 = -u^3 + \frac{\left(\lambda_2^2 + 4\lambda_1\lambda_3\right)}{\lambda_3^2}u^2 - \frac{\left(4\lambda_1^2 + 2c\lambda_2 - 4\mu^2\right)}{\lambda_3^2}u + \frac{c^2}{\lambda_3^2}. \quad (14)$$

The cubic polynomial $P(u)$ satisfies $P(0) = \frac{c^2}{\lambda_3^2} > 0$. Then the curvature may be zero. We may assume $P(u) > 0$ has three real roots α_1, α_2, α_3 satisfying $\alpha_1 < 0 < \alpha_2 < \alpha_3$. Then, we can write equation (14) in the form

$$(u_s)^2 + (u + \alpha_1)(u - \alpha_2)(u - \alpha_3) = 0$$

and the solution can be written in terms of the Jacobi elliptic function as

$$u = u(s) = \alpha_3(1 - q^2 sn^2(rs, p)),$$

where

$$p^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1},\ q^2 = \frac{\alpha_3 - \alpha_2}{\alpha_3},\ r = \sqrt{\frac{\alpha_3 q^2}{4p^2}} = \frac{1}{2}\sqrt{\alpha_3 + \alpha_1}.$$

Also, α_1, α_2, α_3 are related to the coefficients of $P(u)$ by

$$-\frac{4\lambda_1\lambda_3 + \lambda_2^2}{\lambda_3^2} = \alpha_1 - \alpha_2 - \alpha_3,$$

$$-\frac{c^2}{\lambda_3^2} = \alpha_1\alpha_2\alpha_3,$$

$$\frac{4\mu^2 - 2c\lambda_2 - 4\lambda_1^2}{\lambda_3^2} = \alpha_1\alpha_3 + \alpha_1\alpha_2 - \alpha_2\alpha_3.$$

The parameter κ_0 is maximum curvature, p and w with $0 \le p \le w \le 1$ control the shape. The square of maximum curvature for γ spacelike Kirchhoff elastic rod is $u(0) = \alpha_3 = \kappa_0^2$. Then, the formula for the curvature is

$$\kappa^2 = \kappa_0^2(1 - \frac{p^2}{w^2} sn^2(z, p)) \quad \text{with} \quad z = \frac{\kappa_0 s}{2w}.$$

The parameters p, w and κ_0 are determined by the coefficients by the following relations:

$$\frac{4\lambda_1\lambda_3 + \lambda_2^2}{\lambda_3^2} = \frac{\kappa_0^2}{w^2}(3w^2 - p^2 - 1),$$

$$\frac{c^2}{\lambda_3^2} = -\frac{\kappa_0^6}{w^4}(1 - w^2)(w^2 - p^2),$$

$$\frac{4\mu^2 - 2c\lambda_2 - 4\lambda_1^2}{\lambda_3^2} = \frac{\kappa_0^4}{w^4}(2w^2 - 3w^4 + 2w^2p^2 - p^2).$$

The shape of the non-null Kirchhoff elastic rod centerlines depends on λ_1, λ_2, λ_3 and the constant of integration c, with $\mu > 0$ determined by (10).

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REFERENCES

[1] Abramowitz, M., Stegun, I. A., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. U. S. Government Printing Office, Washington, D. C., 1964.

[2] Beem, J. K., Ehrlich, P. E., Easley, K. L., *Global Lorentzian Geometry,* Marcel Dekker Inc. New York, 1996.

[3] Brunnett, G., Crouch, P. E., *Elastic Curves on the Sphere*, Naval Postgraduate School, Monterey, California, 1993.

[4] Huang, R., Shang, D., *Applied Mathematics and Mechanics*, **30**(9), 1193, 2009.

[5] Huang, R., Junyan, Y., *International Journal of Geometric Methods in Modern Physics,* **13**(4), 2016.

[6] Ivey, T. A., Singer, D. A., *Proceedings of the London Mathematical Society*, **79**(2), 429, 1999.

[7] Kawakubo, S., *Tohoku Mathematical Journal*, **54**(2), 179, 2002.

[8] Kawakubo, S., *Tohoku Mathematical Journal*, **56**(2), 205, 2004.

[9] Kawakubo, S., *Journal of the Mathematical Society of Japan*, **60**(2), 551, 2008.

[10] Kawakubo, S., *Journal of Geometry and Physics*, **76**, 158, 2014.

[11] Kirchhoff, G., *J. Reine Angew. Math.*, **56,** 285., 1859.

[12] Landau, L., Lifshitz, E., *Theory of Elasticity*, Pergamon Press, Elmsford, NY, Oxford, 1959.

[13] Langer, J., Singer, D. A., *Journal of Differential Geometry* , **20**(1), 1, 1984.

[14] Langer, J., Singer, D. A., *SIAM Review*, **38**(4), 605, 1996.

[15] Lopez, R., *International Electronic Journal of Geometry,* **7**(1), 44, 2014.

[16] O'Neill, B., *Semi-Riemann Geometry with Applications to Relativity*, Academic Pres. New York, 1993.

[17] Sager, I., Abazari, N., Ekmekci, N., Yaylı, Y., *Int. J. Contemp. Math. Sciences,* **6**(7), 309, 2011.

[18] Shi, Y., Hearst J., *Journal of Chemical Physics*, **101**(6), 5186, 1994.

[19] Singer, D., *Lectures on elastic curves and rods*, AIP Conf. Proc. 1002, Amer. Inst. Phys., Melville. New York, 2008.

[20] Weinstock, R., *Calculus of variations with applications to physics and engineering,* Dover Pub. Inc. New York, 1974.

[21] Yay, Y., *Kirchhoff Elastic Rods in Minkowski* $3-$ *Space*, Msc Thesis, Süleyman Demirel University, Isparta, 2010.

[22] Yücesan, A., Oral, M., *International Journal of Geometric Methods in Modern Physics,* **8**(1), 1, 2011.