ORIGINAL PAPER

A NEW VERSION OF ENERGY FOR SLANT HELIX WITH BENDING ENERGY IN THE LIE GROUPS

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Abstract. *In this work, we research geometrical interpretation involved with the energy on Lie groups. We explore the geometric properties of spherical graphics by way of energy. We apply totally diverse discussion and approach to illustrate bending energy functional for slant helix. Moreover, we have an original and satisfactorily association among energy of the curve in Lie groups.*

Keywords: *Energy, Slant Helix, Lie Groups, Vector Fields, Bending Energy.*

1. INTRODUCTION

Development of Lie group provides to mathematics is that it has three diverse constructions of mathematical type that permit us established a association among these types of diverse varieties. Mainly, this has framework of group. Additionally, components relating to this group form a topological space so that this may be described while becoming a particular case of a topological group. Finally, the components likewise type an analytic manifold.

Lie groups perform a important position not only in physical devices also during mathematical research just like loop groups, gauge groups, and Fourier integral's groups operators that occur as symmetry groups and phase spaces. Lie groups are likewise beneficial for mechanics. Since incomprehensible inviscid fluid motion and rigid body motion correspond to geodesic flow in left (or right) invariant metric described on a Lie group $[1-8]$.

Related principles designed for energy in curvature centered energy is regarded to be at its first phases in development. A few of the legendary fields and landmark research for this theory may be discovered in mathematical physics, membrane chemistry, computer aided geometric design and geometric modeling, shell engineering, biology and thin plate $[9-13]$. One of the well-known functional and related work is bending energy functional, which appeared firstly Bernoulli-Euler elastica formulation for energy [14].

2. MATERIALS AND METHODS

Definition 2.1 *For a parametrized curve* $\gamma : I \subset \mathbb{R} \to R$ *and Frenet apparatus* $(\mathbf{T}, \mathbf{N}, \mathbf{B}, \kappa, \tau)$ *we have*

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$$\boldsymbol{\tau}_{\mathbf{R}} = \frac{1}{2}\langle [\mathbf{T},\mathbf{N}],\mathbf{B}\rangle$$

or equivalently

$$\boldsymbol{\tau}_{\mathbf{R}} = \frac{1}{2\kappa^2\tau}\langle \mathbf{T}'',[\mathbf{T},\mathbf{T}']\rangle + \frac{1}{4\kappa^2\tau}\left\|[\mathbf{T},\mathbf{T}']\right\|^2.$$

For a parametrized arc-lenghted curve $\boldsymbol{\gamma}: I \subset \mathbb{R} \to \mathbf{R}$ in 3-dimensional Lie group. If $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$, $\gamma''''(s)$ are linearly dependent for all $s \in I$ then it is said that $\boldsymbol{\gamma}$ is Frenet curve of osculating order three. It can be constructed to an orthonormal Frenet frame for each Frenet curve of order three in the following way:

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}\\
&\nabla_{\mathbf{T}}\mathbf{N} = (\tau - \tau_{\mathbf{R}})\mathbf{B} - \kappa\mathbf{T}\\
&\nabla_{\mathbf{T}}\mathbf{B} = (\tau_{\mathbf{R}} - \tau)\mathbf{N},
\end{aligned} \tag{2.1}$$

where Lie group $\mathbf{R}$ has the Levi-Civita connection ∇, [20,21].

Proposition 2.2 *For a 3-dimensional Lie group* $\mathbf{R}$ *induced with a bi-invariant metric we have following statements that can be obtained for different Lie groups:*

$(\mathbf{i})$ $\tau_{\mathbf{R}} = 0$ *if* $\mathbf{R}$ *is Abelian group.*
$(\mathbf{ii})$ $\tau_{\mathbf{R}} = 1$ *if* $\mathbf{R}$ *is* SU^2.
$(\mathbf{iii})$ $\tau_{\mathbf{R}} = \frac{1}{2}$ if $\mathbf{R}$ *is* SO^3 [20,24].

3. ENERGY AND SLANT HELIX

Definition 3.1. *Let* $\alpha : I \subset R \to \mathbf{R}$ *be an arc length parametrized curve. Then* α *is called a slant helix if its principal normal vector makes a constant angle with a left-invariant vector field* X *which is unit length [28].*

Definition 3.2. *Let* $\alpha : I \subset R \to \mathbf{R}$ *be an arc length parametrized curve with the Frenet apparatus* $(\mathbf{T},\mathbf{N},\mathbf{B},\kappa,\tau)$. *Then the harmonic curvature function of the curve* α *is defined by [28]*

$$H = \frac{\tau - \tau_{\mathrm{G}}}{\kappa},$$

where $\tau_{\mathrm{G}} = \frac{1}{2} g([\mathbf{T},\mathbf{N}],\mathbf{B})$**.**

Proposition 3.3. *If the curve a is a slant helix in* $\mathbf{R}$*, then the axis of* α *is*

$$\mathbf{X} = [\frac{H\kappa(1+H^2)}{H'}\mathbf{T} + \mathbf{N} + \frac{\kappa(1+H^2)}{H'}\mathbf{B}]\cos\varpi,$$

where $\varpi \neq \frac{\pi}{2}$ *is a constant angle [28].*

Definition 3.4. *For two Riemannian manifolds* (M,ρ) *and* $(N,\tilde{h})$ *the energy of a differentiable map* $f:(M,\rho)\rightarrow(N,\tilde{h})$ *can be defined as*

$$\varepsilon nergy(f)=\frac{1}{2}\int_M\sum_{a=1}^{n}\tilde{h}(df(e_a),df(e_a))v, \tag{3.1}$$

where $\{e_a\}$ *is a local basis of the tangent space and* v *is the canonical volume form in* M [6,7].

4. RESULTS AND DISCUSSION

In the theory of relativity, all the energy moving through an object contributes to the body's total mass that measures how much it can resist to acceleration. Each kinetic and potential energy makes a highly proportional contribution to the mass [23,25,26,27]. In this study not only we compute the energy of surface curves but we also investigate its close correlation with bending energy of elastica which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744. Euler elastica bending energy formula for a space curve in the 3-dimensional Frenet curvature along the curve is known as

$$H_B=\frac{1}{2}\int\kappa^2 ds.$$

Case I: *Energy for tangent spherical image.*

Tangent indicatrix of α *is the parametrized curve* $\beta:\ I\subset R\rightarrow \mathbb{S}^2$ *defined by* $\beta(s^*)=\mathbf{T}(s)$.

Theorem 4.1 *Frenet vector's energy by using Sasaki metric is given by*

$$\varepsilon nergy(\mathbf{T}_\beta)=\frac{1}{2}\int_0^{s^*}(1+(\tau-\tau_{\mathbf{R}})^2+\kappa^2)ds,$$

$$\varepsilon nergy(\mathbf{N}_\beta)=\frac{1}{2}\int_0^{s^*}(1+((\frac{1}{\sqrt{1+H^2}})^{'})^2+(\frac{(\tau_{\mathbf{R}}-\tau)H}{\sqrt{1+H^2}}$$
$$-\frac{\kappa}{\sqrt{1+H^2}})^2+((\frac{H}{\sqrt{1+H^2}})^{'})^2)ds.$$

$$\varepsilon nergy(\mathbf{B}_\beta)=\frac{1}{2}\int_0^{s^*}(1+((\frac{H}{\sqrt{1+H^2}})^{'})^2+((\frac{H\kappa}{\sqrt{1+H^2}})$$
$$+(\frac{1}{\sqrt{1+H^2}})(\tau_{\mathbf{R}}-\tau))^2+((\frac{1}{\sqrt{1+H^2}})^{'})^2)ds.$$

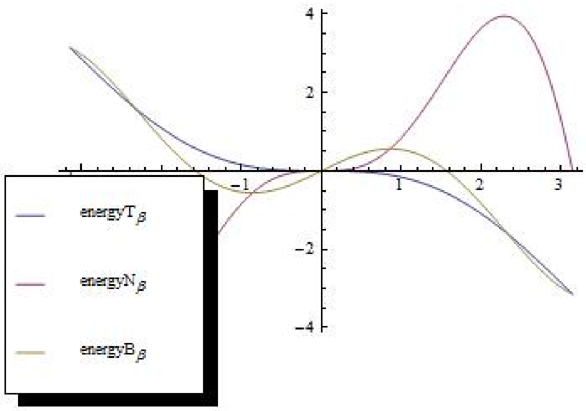
Proof. From definition tangent indicatrix, we have $\mathbf{T}_\beta(s^*)=\mathbf{N}(s)$.

Lemma 4.2.

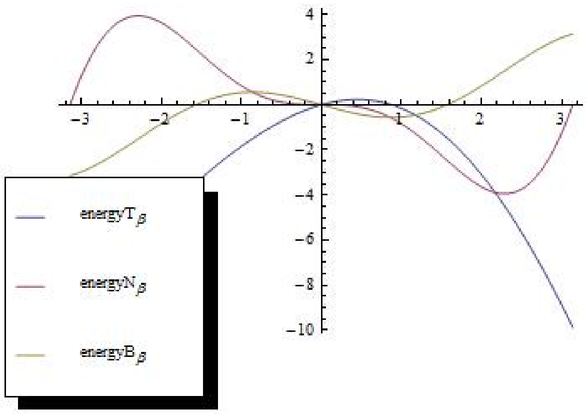
$$\varepsilon nergy(\mathbf{T}_\beta) - H_B = \frac{1}{2}(s^* + \int_0^{s^*} (\tau - \tau_{\mathbf{R}})^2 ds).$$

Now, we consider following results for tangent indicatrix in Lie group.

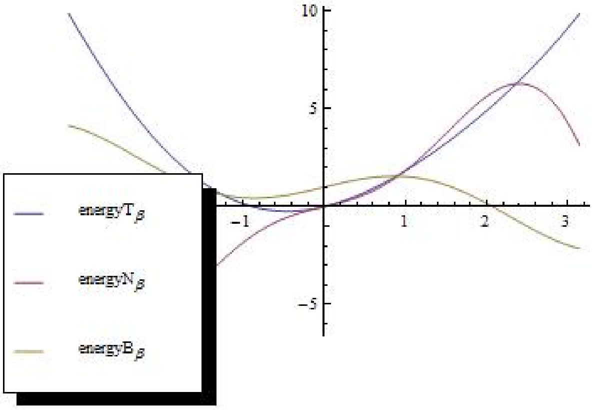
i) *Let* $\mathbf{R}$ *be an Abelian group. Thus we have* $\tau_{\mathbf{R}} = 0$ *and we get following graph respectively for the energy of* $\mathbf{T}_\beta, \mathbf{N}_\beta$ *and* $\mathbf{B}_\beta$.



ii) *Let* $\mathbf{R}$ *be* SU^2. *Thus we have* $\tau_{\mathbf{R}} = 1$ and we get following graph respectively for the energy of $\mathbf{T}_\beta, \mathbf{N}_\beta$ *and* $\mathbf{B}_\beta$.



iii) Let $\mathbf{R}$ be SO^3. *Thus we have* $\tau_{\mathbf{R}} = \frac{1}{2}$ *and we get following graph respectively for the energy of* $\mathbf{T}_\beta, \mathbf{N}_\beta$ *and* $\mathbf{B}_\beta$.



Case II: *Energy for normal spherical image of slant helix.*

Normal indicatrix of α is the parametrized curve $\gamma:\ I\subset R\to \mathbb{S}^2$ defined by $\gamma(s^*)=\mathbf{N}(s)$.

Theorem 4.3 *Frenet vector's energy by using Sasaki metric is given by*

$$\varepsilon nergy\left(\mathbf{T}_{\gamma}\right)=\frac{1}{2}\int_0^{s^*}(1+((\frac{1}{\sqrt{1+H^2}})^{'})^2+(\frac{\kappa}{\sqrt{1+H^2}}+\frac{(\tau_{\mathbf{R}}-\tau)H}{\sqrt{1+H^2}})^2$$

$$+((\frac{H}{\sqrt{1+H^2}})^{'})^2\mathbf{)}ds,$$

$$\varepsilon nergy\left(\mathbf{N}_{\gamma}\right)=\frac{1}{2}\int_0^{s^*}((\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^2+\mathbf{(}\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2})^2+(\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^2$$

$$+((\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^{'}+\kappa\mathbf{(}\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2}))^2+\mathbf{(-(}\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2})^{'}$$

$$+(\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})\left(\tau_{\mathbf{R}}-\tau\right)+\kappa(\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}}))^2+((\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^{'}$$

$$-\mathbf{(}\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2})\left(\tau-\tau_{\mathbf{R}}\right))^2)ds,$$

$$\varepsilon nergy\left(\mathbf{B}_{\beta}\right)=\frac{1}{2}\int_0^{s^*}((\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^2+\mathbf{(}\frac{H^2\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)}-\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)})^2+(\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^2$$

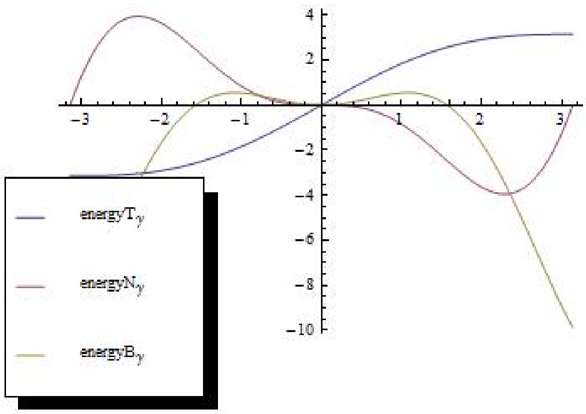
$$+((\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^{'}-\kappa\mathbf{(}\frac{H^2\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)}-\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)}))^2+\mathbf{((}\frac{H^2\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)}$$

$$-\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)})^{'}+\kappa(\frac{H\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})\left(\tau_{\mathbf{R}}-\tau\right)(\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}}))^2+\mathbf{((}\frac{H^2\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)}$$

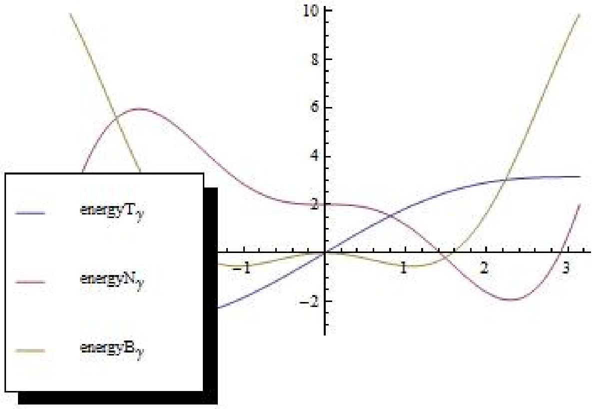
$$-\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2(1+H^2)})\left(\tau-\tau_{\mathbf{R}}\right)-(\frac{\sqrt{1+\sigma_N^2}}{\sigma_N^2\sqrt{1+H^2}})^{'})^2)ds.$$

Now, we consider following results for normal indicatrix in Lie group.

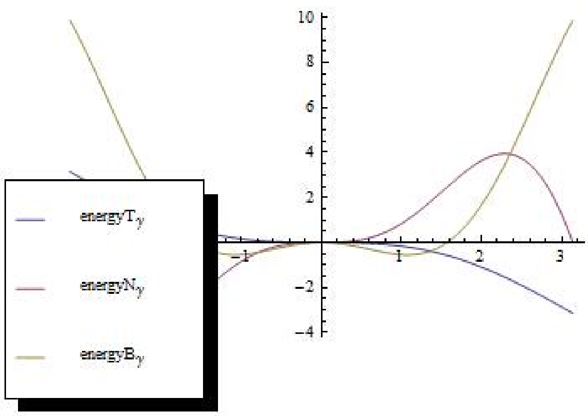
i) *Let* $\mathbf{R}$ *be an Abelian group. Thus we have* $\tau_{\mathbf{R}} = 0$ *and we get following graph respectively for the energy of* $\mathbf{T}_\gamma, \mathbf{N}_\gamma$ *and* $\mathbf{B}_\gamma$**.**



ii) *Let* $\mathbf{R}$ *be* SU^2. *Thus we have* $\tau_{\mathbf{R}} = 1$ and we get following graph respectively for the energy of $\mathbf{T}_\gamma, \mathbf{N}_\gamma$ *and* $\mathbf{B}_\gamma$**.**



iii) Let $\mathbf{R}$ be SO^3. *Thus we have* $\tau_{\mathbf{R}} = \frac{1}{2}$ *and we get following graph respectively for the energy of* $\mathbf{T}_\gamma, \mathbf{N}_\gamma$ *and* $\mathbf{B}_\gamma$**.**



Case III: *Energy for binormal spherical image of slant helix.*

Binormal indicatrix of α is the parametrized curve $\delta: \ I \subset R \to \mathbb{S}^2$ defined by $\delta(s^*) = \mathbf{B}(s)$.

Theorem 4.4 *Frenet vector's energy by using Sasaki metric is given by*

$$\varepsilon nergy(\mathbf{T}_\delta) = \frac{1}{2}\int_0^{s^*} (1 + ((\tau - \tau_{\mathbf{R}}))^2 + (\kappa)^2)ds,$$

$$\varepsilon nergy(\mathbf{N}_\delta) = \frac{1}{2}\int_0^{s^*} (1 + ((\frac{\varepsilon}{\sqrt{1+H^2}})')^2 + ((\frac{\varepsilon}{\sqrt{1+H^2}})\kappa$$
$$+ (\frac{\varepsilon H}{\sqrt{1+H^2}})(\tau_{\mathbf{R}} - \tau))^2 + ((\frac{\varepsilon H}{\sqrt{1+H^2}})')^2)ds,$$

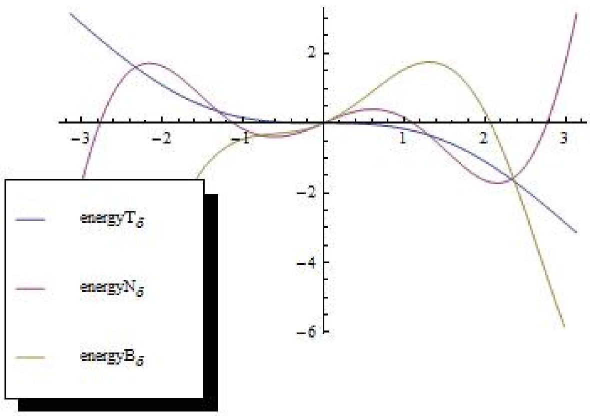
$$\varepsilon nergy(\mathbf{B}_\delta) = \frac{1}{2}\int_0^{s^*} (1 + ((\frac{H}{\sqrt{1+H^2}})')^2 + ((\tau_{\mathbf{R}} - \tau)(\frac{1}{\sqrt{1+H^2}})$$
$$- \kappa(\frac{H}{\sqrt{1+H^2}}))^2 + ((\frac{1}{\sqrt{1+H^2}})')^2)ds.$$

Lemma 4.5.

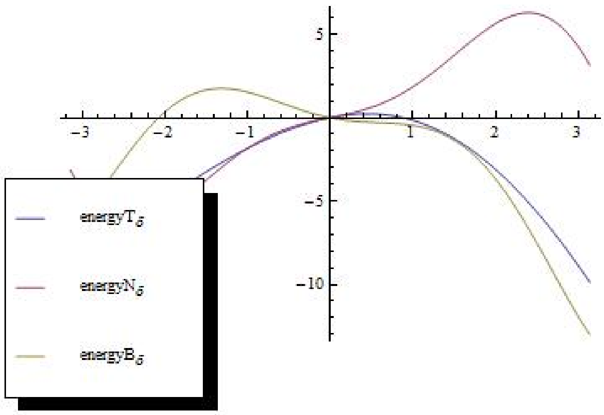
$$\varepsilon nergy(\mathbf{T}_\delta) - H_B = \frac{1}{2}(s^* + \int_0^{s^*} (\tau - \tau_{\mathbf{R}})^2 ds).$$

Now, we consider following results for binormal indicatrix in Lie group.

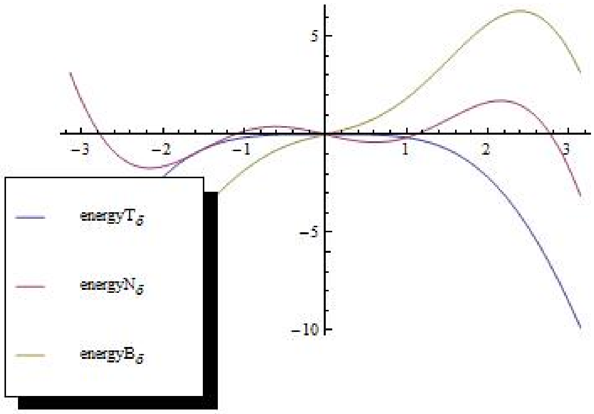
i) *Let* $\mathbf{R}$ *be an Abelian group. Thus we have* $\tau_{\mathbf{R}} = 0$ *and we get following graph respectively for the energy of* $\mathbf{T}_\delta, \mathbf{N}_\delta$ *and* $\mathbf{B}_\delta$.



ii) *Let* $\mathbf{R}$ *be* SU^2. *Thus we have* $\tau_{\mathbf{R}} = 1$ and we get following graph respectively for the energy of $\mathbf{T}_\delta, \mathbf{N}_\delta$ *and* $\mathbf{B}_\delta$.



iii) Let $\mathbf{R}$ be SO^3. *Thus we have* $\tau_{\mathbf{R}} = \frac{1}{2}$ *and we get following graph respectively for the energy of* $\mathbf{T}_\delta, \mathbf{N}_\delta$ *and* $\mathbf{B}_\delta$.



Corollary 4.6. *If the curve a is a slant helix in* $\mathbf{R}$, *then the axis of* α *is*

$$\varepsilon nergy(\mathbf{X}) = \frac{1}{2}\int_0^{s^*}(1+(\cos\varpi(\frac{H\kappa(1+H^2)}{H'})' - \kappa\cos\varpi)^2 + (\cos\varpi(\frac{H\kappa(1+H^2)}{H'})\kappa$$

$$+\cos\varpi\frac{\kappa(1+H^2)}{H'}(\tau_{\mathbf{R}}-\tau))^2 + (\cos\varpi(\frac{\kappa(1+H^2)}{H'})' + \cos\varpi(\tau-\tau_{\mathbf{R}}))^2)ds,$$

where $\varpi \neq \frac{\pi}{2}$ *is a constant angle.*

Lemma 4.7 $\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta$ *have not constant energy in the Lie group* $\mathbf{R}$ *with a bi-invariant metric.*

Proof. From Theorem 4.1 we obtain

$$\tau^2 + \kappa^2 = -1,$$

$$((\frac{1}{\sqrt{1+H^2}})')^2 + (-\frac{\tau H}{\sqrt{1+H^2}} - \frac{\kappa}{\sqrt{1+H^2}})^2 + ((\frac{H}{\sqrt{1+H^2}})')^2 = -1,$$

$$((\frac{H}{\sqrt{1+H^2}})')^2+((\frac{H\kappa}{\sqrt{1+H^2}})+(\frac{1}{\sqrt{1+H^2}})\tau)^2+((\frac{1}{\sqrt{1+H^2}})')^2=-1.$$

This is a contradiction. Thus we have lemma.

Lemma 4.8 $\mathbf{T}_\gamma,\mathbf{N}_\gamma,\mathbf{B}_\gamma,\mathbf{T}_\delta,\mathbf{N}_\delta,\mathbf{B}_\delta$ *have not constant energy in the Lie group* $\mathbf{R}$ *with a bi-invariant metric.*

Now, we consider energy of our spherical images.

Theorem (Main) 4.9

$$\begin{aligned}
\varepsilon nergy(\beta) &= \frac{1}{2}\int_0^{s^*}(1+\kappa^2)ds,\\
\varepsilon nergy(\gamma) &= \frac{1}{2}\int_0^{s^*}(1+\kappa^2+(\tau-\tau_{\mathbf{R}})^2)ds,\\
\varepsilon nergy(\delta) &= \frac{1}{2}\int_0^{s^*}(1+(\tau-\tau_{\mathbf{R}})^2)ds.
\end{aligned}$$

Corollary 4.10.

$$\begin{aligned}
\varepsilon nergy(\beta)-H_B &= \frac{1}{2}s^*,\\
\varepsilon nergy(\gamma)-H_B &= \frac{1}{2}(s^*+\int_0^{s^*}(\tau-\tau_{\mathbf{R}})^2ds).
\end{aligned}$$

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