ORIGINAL PAPER

THE LAGUERRE TYPE *d*- ORTHOGONAL POLYNOMIALS

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Abstract. *This paper is mainly connected with the theory of generating functions for some Laguerre type d-orthogonal polynomials. In this study, we present some theorems giving multilateral and multilinear generating functions for the Laguerre type d-orthogonal polynomials. We also give their sepecial cases.*

Keywords: *Laguerre type d-orthogonal polynomials, generating function, multilinear and multilateral generating functions.*

1. INTRODUCTION

A natural generalization of an arbitrary number of p numerator and q denominator parameters $(p,q \in \mathrm{N}_0 = \{0\} \cup \mathrm{N})$ is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$ {}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p; \\ \beta_1,...,\beta_q;\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} $$
$$ = {}_pF_q\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta_q;z\right). $$

Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$ (\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in C \setminus Z_0^-) $$
$$ = \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in C \setminus \{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathrm{N};\ \lambda \in C \end{cases} $$

and Z_0^- denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function. Thus, if a numerator paramater is a negative integer or 0, the ${}_pF_q$ series terminates led to a generalized hypergeometric polynomial of type

$$ {}_{p+1}F_q(-n,(a_p);(\alpha_q+1);x) = \sum_{m=0}^{n} \frac{(-n)_m (a_1)_m ... (a_p)_m}{(\alpha_1+1)_m ... (\alpha_q+1)_m} \frac{x^m}{m!}, \tag{1.1} $$

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where $\alpha_j \neq -1,-2,...; \quad j=1,...,q.$

Taking $p=0$ and $q=d$ in (1.1), Ben Cheikh *at al.* obtained the Laguerre type *d*-orthogonal polynomials as follows [4]

$$l_n^{(\alpha_d)}(x) = {}_1F_d(-n;(\alpha_d+1);x),$$

where $\boldsymbol{\alpha}_d = (\alpha_1,\alpha_2,...,\alpha_d)$ and their generating function relation is generated by

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} l_n^{(\alpha_d)}(x) t^n = (1-t)^{-\lambda} {}_1F_d\left(\lambda;(\alpha_d+1);\frac{-xt}{1-t}\right)$$
$$\left(\lambda \in \mathrm{C} \quad and \quad |t|<1\right). \tag{1.2}$$

In (1.2) taking $d=1$, the generating function (1.2) reduces classical generating function for Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right).$$

In [1], another generating function for Laguerre type d-orthogonal polynomials given by

$$\sum_{n=0}^{\infty} l_n^{(\alpha_d)}(x) \frac{t^n}{n!} = e^t {}_0F_d(-;(\alpha_d+1);-xt). \tag{1.3}$$

They have

$$l_n^{\vec{\alpha}_d(\alpha)}(x) = \frac{n!}{(\alpha+1)_{dn}} Z_n^{\alpha}\left(dx^{\frac{1}{d}},d\right),$$

where $\vec{\alpha}_d(\alpha) = \left(\frac{\alpha+1}{d}-1,...,\frac{\alpha+d}{d}-1\right), \ \alpha>-1,$ and

$$Z_n^{\alpha}\left(dx^{\frac{1}{d}},d\right) = \frac{(\alpha+1)_{dn}}{n!} {}_1F_d\left(-n;\frac{\alpha+1}{d},...,\frac{\alpha+d}{d};x\right), \quad n \geq 0. \tag{1.4}$$

If we choose $d=1$ in (1.4), these polynomials reduce Laguerre polynomials as follows:

$$Z_n^{\alpha}(x,1) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n;\alpha+1;x) = L_n^{\alpha}(x).$$

On the other hand, Varma *at al.* [9] investigated *d*-orthogonality for the another kind of Laguerre type *d*-orthogonal polynomials which are generated by

$$\sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d) \frac{t^n}{n!} = (1-t)^{-(1+\alpha)d} \exp\left\{x\left[1-(1-t)^{-d}\right]\right\}, \tag{1.5}$$

where $|t|<1$. In (1.5) taking $d=1$, these polynomials reduce Laguerre polynomials.

This paper mainly concerns is to obtain theorems giving generating function relations for all of Laguerre type *d*-orthogonal polynomials and discuss their special cases.

2. GENERATING FUNCTIONS

Theorem 2.1. *We have the following generating function for the Laguerre type d-orthogonal polynomials defined by (1.4):*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{\left(\alpha+1\right)_{dn}}Z_n^{\alpha}\left(dx^{\frac{1}{d}},d\right)t^n=(1-t)^{-\lambda}\,{}_1F_d\left(\lambda;\frac{\alpha+1}{d},\dots,\frac{\alpha+d}{d};\frac{-xt}{1-t}\right), \tag{2.1}$$

where $\lambda\in\mathrm{C}$ *and* $|t|<1$.

Proof. Let T denote the first member of assertion (2.1). Using (1.4), we have

$$\begin{aligned}
T&=\sum_{n=0}^{\infty}(\lambda)_n\,{}_1F_d\left(-n;\frac{\alpha+1}{d},\dots,\frac{\alpha+d}{d};x\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\lambda)_n(-n)_m}{\left(\frac{\alpha+1}{d}\right)_m\cdots\left(\frac{\alpha+d}{d}\right)_m}\frac{x^m}{m!}\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\lambda)_n(-1)^m}{\left(\frac{\alpha+1}{d}\right)_m\cdots\left(\frac{\alpha+d}{d}\right)_m}\frac{x^m}{(n-m)!}\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\lambda)_{n+m}(-1)^m}{\left(\frac{\alpha+1}{d}\right)_m\cdots\left(\frac{\alpha+d}{d}\right)_m}\frac{\left(xt\right)^m}{m!}\frac{t^n}{n!}\\
&=\sum_{m=0}^{\infty}\frac{(\lambda)_m(-1)^m}{\left(\frac{\alpha+1}{d}\right)_m\cdots\left(\frac{\alpha+d}{d}\right)_m}\frac{\left(xt\right)^m}{m!}\sum_{n=0}^{\infty}\frac{\left(\lambda+m\right)_n}{n!}t^n\\
&=(1-t)^{-\lambda}\,{}_1F_d\left(\lambda;\frac{\alpha+1}{d},\dots,\frac{\alpha+d}{d};\frac{-xt}{1-t}\right)
\end{aligned}$$

which completes the proof.

Theorem 2.2. *Following generating function for Laguerre type d-orthogonal polynomials in (1.4) holds true:*

$$\sum_{n=0}^{\infty}Z_n^{\alpha}\left(dx^{\frac{1}{d}},d\right)\frac{t^n}{\left(\alpha+1\right)_{dn}}=e^t\,{}_0F_1\left(-,\frac{\alpha+1}{d},\dots,\frac{\alpha+d}{d};-xt\right),\quad |t|<1.$$

Proof. Let T denote the first member of assertion (2.2). We have

$$T=\sum_{n=0}^{\infty}{}_1F_d\left(-n;\frac{\alpha+1}{d},\ldots,\frac{\alpha+d}{d};x\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-n)_m}{\left(\frac{\alpha+1}{d}\right)_m\ldots\left(\frac{\alpha+d}{d}\right)_m}\frac{x^m}{m!}\frac{t^n}{n!},$$
$$T=\sum_{m=0}^{\infty}\frac{1}{\left(\frac{\alpha+1}{d}\right)_m\ldots\left(\frac{\alpha+d}{d}\right)_m}\frac{\left(-xt\right)^m}{m!}\sum_{n=0}^{\infty}\frac{t^n}{n!}$$
$$=e^t\,{}_0F_1\left(-,\frac{\alpha+1}{d},\ldots,\frac{\alpha+d}{d};-xt\right),$$

which completes the proof.

Theorem 2.3. *Following generating function for the Laguerre type d-orthogonal polynomials defined by (1.5) holds true:*

$$\sum_{n=0}^{\infty}P_{m+n}^{(\alpha)}(x;d)\frac{t^n}{n!}=\left(1-t\right)^{-m-(\alpha+1)d}\exp\left\{x\left(1-(1-t)^{-d}\right)\right\}P_m^{(\alpha)}\left(\frac{x}{\left(1-t\right)^d};d\right).$$

Proof. Replacing t by $t+v$ in (1.5), we have

$$\sum_{n=0}^{\infty}P_n^{(\alpha)}(x;d)\tfrac{(t+v)^n}{n!}=(1-t-v)^{-(1+\alpha)d}\exp\left\{x\left[1+(1-t-v)^{-d}\right]\right\}$$
$$\sum_{n=0}^{\infty}P_n^{(\alpha)}(x;d)\tfrac{1}{n!}\sum_{m=0}^{n}\binom{m}{n}t^{n-m}v^m$$
$$=(1-t)^{-(1+\alpha)d}\left(1-\tfrac{v}{1-t}\right)^{-(1+\alpha)d}\exp\left\{x\left(1-\tfrac{1}{(1-t)^d}\right)\right\}\exp\left\{\tfrac{x}{(1-t)^d}\left(1-\tfrac{1}{\left(1-\frac{v}{1-t}\right)^d}\right)\right\}$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+n}{n}P_{n+m}^{(\alpha)}(x;d)\tfrac{t^nv^m}{(n+m)!}$$
$$=(1-t)^{-(1+\alpha)d}\exp\left\{x\left(1-\tfrac{1}{(1-t)^d}\right)\right\}\sum_{m=0}^{\infty}\left(1-t\right)^{-m}P_m^{(\alpha)}\left(\tfrac{x}{(1-t)^d};d\right)\tfrac{v^m}{m!}$$
$$\sum_{n=0}^{\infty}P_{m+n}^{(\alpha)}(x;d)\tfrac{t^n}{n!}=\left(1-t\right)^{-m-(\alpha+1)d}\exp\left\{x\left(1-(1-t)^{-d}\right)\right\}P_m^{(\alpha)}\left(\tfrac{x}{(1-t)^d};d\right),$$

which completes the proof.

Lemma 2.4. *Laguerre type d-orthogonal polynomials* $P_n^{(\alpha)}(x;d)$ *have the following addition formula*

$$P_n^{(\alpha+\beta+1)}(x+y;d)=\sum_{k=0}^{n}\binom{n}{k}P_{n-k}^{(\alpha)}(x;d)P_k^{(\beta)}(y;d).$$

Proof. Using (1.5), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} P_n^{(\alpha+\beta+1)}(x+y;d)\frac{t^n}{n!} \\
&= (1-t)^{-(1-(1+\alpha+\beta))d}\exp\left\{(x+y)\left[1-(1-t)^{-d}\right]\right\} \\
&= (1-t)^{-(1+\alpha)d}\exp\left\{x\left(1-(1-t)^{-d}\right)\right\}(1-t)^{-(1+\beta)d}\exp\left\{y\left(1-(1-t)^{-d}\right)\right\} \\
&= \sum_{n=0}^{\infty} P_n^{(\alpha)}(x;d)\frac{t^n}{n!}\sum_{k=0}^{\infty} P_k^{(\beta)}(y;d)\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_n^{(\alpha)}(x;d)P_k^{(\beta)}(y;d)\frac{t^{n+k}}{n!k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_{n-k}^{(\alpha)}(x;d)P_k^{(\beta)}(y;d).\frac{t^n}{(n-k)!k!},
\end{aligned}
$$

and equating the coefficients of t^n which completes the proof.

3. MULTILINEAR AND MULTILATERAL GENERATING FUNCTIONS

In this section, firstly we derive several families of bilinear and bilateral generating functions for the Laguerre type *d*-orthogonal polynomials by using the similar method considered in [2, 5].

Theorem 3.1. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

where $(a_k \neq 0,\ \mu,\psi \in C)$ *and*

$$\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\xi\right) := \sum_{k=0}^{[n/p]} a_k\left(\lambda\right)_{n-pk}\, l_{n-pk}^{(\alpha_d)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\frac{\xi^k}{(n-pk)!}.$$

Then, for $p \in \mathrm{N};$ *we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n = (1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\frac{-xt}{1-t}\right)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.1}$$

provided that each member of (3.1) exists.

Proof. For convenience, let S denote the first member of the assertion (2.1). Then,

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_k\left(\lambda\right)_{n-pk}\,l_{n-pk}^{(\alpha_d)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\,\frac{t^{n-pk}}{(n-pk)!}.$$

Replacing n by $n+pk$, we may write that

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\left(\lambda\right)_n\,l_n^{(\alpha_d)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\,\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{\left(\lambda\right)_n}{n!}l_n^{(\alpha_d)}(x)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k\\&=(1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\frac{-xt}{1-t}\right)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\end{aligned}$$

which completes the proof.

By using a similar idea, we also get the next results immediately.

Theorem 3.2. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

where $(a_k\neq 0,\ \mu,\psi\in C)$ *and*

$$\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\xi\right):=\sum_{k=0}^{[n/p]}a_k l_{n-pk}^{(\alpha_d)}(x)\Omega_{\mu+\psi k}(y_1,...,y_r)\frac{\xi^k}{(n-pk)!}.$$

Then, for $p\in\mathrm{N}$; *we have*

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n=e^t\,{}_0F_d(-;(\alpha_d+1);-xt)\Lambda_{\mu,\psi}(y_1,...,y_r;\eta)\qquad(3.2)$$

provided that each member of (3.2) exists.

Theorem 3.3. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r\in\mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

where $(a_k\neq 0,\ \mu,\psi\in C)$ *and*

$$\Theta_{n,p}^{\mu,\psi}\left(x;y_1,...,y_r;\xi\right):=\sum_{k=0}^{[n/p]}a_k\frac{\left(\lambda\right)_{n-pk}}{\left(\alpha+1\right)_{d(n-pk)}}Z_{n-pk}^{\alpha}\left(dx^{\frac{1}{d}},d\right)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k$$

Then, for $p \in \mathrm{N}$; *we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x; y_1,...,y_r; \frac{\eta}{t^p}\right) t^n = (1-t)^{-\lambda} {}_1F_d\left(\lambda; (\alpha_d+1); \frac{-xt}{1-t}\right) \Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.3}$$

provided that each member of (3.3) exists.

Theorem 3.4. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

where $(a_k \neq 0,\ \mu,\psi \in C)$ *and*

$$\Theta_{n,p}^{\mu,\psi}\left(x; y_1,...,y_r;\xi\right) := \sum_{k=0}^{[n/p]} \frac{a_k}{(\alpha+1)_{d(n-pk)}} Z_{n-pk}^{\alpha}\left(dx^{\frac{1}{d}}, d\right) \Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k.$$

Then, for $p \in \mathrm{N}$; *we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x; y_1,...,y_r; \frac{\eta}{t^p}\right) t^n = e^t {}_0F_1\left(-, \frac{\alpha+1}{d},...,\frac{\alpha+d}{d}; -xt\right) \Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.4}$$

provided that each member of (3.4) exists.

Theorem 3.5. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1,...,y_r)\zeta^k$$

where $(a_k \neq 0,\ \mu,\psi \in C)$ *and*

$$\Theta_{n,p}^{\mu,\psi}\left(x; y_1,...,y_r;\xi\right) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\alpha)}(x;d) \Omega_{\mu+\psi k}(y_1,...,y_r) \frac{\xi^k}{(n-pk)!}.$$

Then, for $p \in \mathrm{N}$; *we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x; y_1,...,y_r; \frac{\eta}{t^p}\right) t^n = (1-t)^{-(1+\alpha)d} \exp\left\{x\left[1-(1-t)^{-d}\right]\right\} \Lambda_{\mu,\psi}(y_1,...,y_r;\eta) \tag{3.5}$$

provided that each member of (3.5) exists.

Theorem 3.6. *Corresponding to an identically non-vanishing function* $\Omega_{\mu}(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{\mu,\psi}^{n,p}(x+y;z_1,...,z_r;u) := \sum_{k=0}^{[n/p]} a_k P_{n-pk}^{(\alpha+\beta+1)}(x+y;d)\Omega_{\mu+\psi k}(z_1,...,z_r)\frac{u^k}{(n-pk)!}, \tag{3.6}$$

where $(a_k \neq 0,\ \mu,\psi \in C,\ n,p \in \mathrm{N})$ *. Then, for* $p \in \mathrm{N};$ *we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l P_{n-k}^{(\alpha)}(x;d)P_{k-pl}^{(\beta)}(y;d)\Omega_{\mu+\psi l}(z_1,...,z_r)u^l = \Lambda_{\mu,\psi}^{n,p}(x+y;z_1,...,z_r;u) \tag{3.7}$$

provided that each member of (3.7) exists.

Proof. Let T denote the left-hand side of equality (3.7). Then we have

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l P_{n-k-pl}^{(\alpha)}(x;d)P_k^{(\beta)}(y;d)\Omega_{\mu+\psi l}(z_1,...,z_r)u^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl} P_{n-k-pl}^{(\alpha)}(x;d)P_k^{(\beta)}(y;d)\right)\Omega_{\mu+\psi l}(z_1,...,z_r)u^l \end{aligned}$$

and using equalities (2.4) and (3.6), we get

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]} a_l P_{n-pl}^{(\alpha+\beta+1)}(x+y;d)\Omega_{\mu+\psi l}(z_1,...,z_r)u^l \\ &= \Lambda_{\mu,\psi}^{n,p}(x+y;z_1,...,z_r;u), \end{aligned}$$

which completes the proof.

Theorem 3.7. *Corresponding to an identically non-vanishing function* $\Omega_{\mu}(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathrm{N})$ *and of complex order* μ *, let*

$$\Lambda_{m,q}(x;y_1,...,y_r;t) := \sum_{n=0}^{\infty} a_n P_{m+nq}^{(\alpha)}(x;d)\Omega_{\mu+pn}(y_1,...,y_r)t^n,$$

where $(a_n \neq 0,\ \mu \in C)$ and

$$N_{n,m,q}^{p,\mu}(y_1,...,y_r;z) := \sum_{k=0}^{[n/q]} a_k \Omega_{\mu+pk}(y_1,...,y_r)\frac{z^k}{(n-qk)!}.$$

Then, for every nonnegative integer m,

$$\sum_{n=0}^{\infty} P_{m+n}^{(\alpha)}(x;d) N_{n,m,q}^{p,\mu}(y_1,...,y_r;z)t^n$$
$$= (1-t)^{-m-(1+\alpha)d} \exp\left\{x\left[1-(1-t)^{-d}\right]\right\} \Lambda_{m,q}\left(\frac{x}{(1-t)^d}; y_1,...,y_r; \frac{zt^q}{(1-t)^q}\right) \tag{3.8}$$

provided that each member of (3.8) exists.
Proof. Let T denote the left-hand side of equality (3.8). Then we have

$$T = \sum_{n=0}^{\infty} P_{m+n}^{(\alpha)}(x;d) \sum_{k=0}^{[n/q]} a_k \Omega_{\mu+pk}(y_1,...,y_r) \frac{z^k}{(n-qk)!} t^n$$
$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} P_{m+n+qk}^{(\alpha)}(x;d) \frac{t^n}{n!} \right) a_k \Omega_{\mu+pk}(y_1,...,y_r) \left(zt^q\right)^k$$
$$= \sum_{k=0}^{\infty} (1-t)^{-m-qk-(\alpha+1)d} \exp\left\{x\left(1-(1-t)^{-d}\right)\right\}$$
$$\times P_{m+qk}^{(\alpha)}\left(\frac{x}{(1-t)^d}; d\right) a_k \Omega_{\mu+pk}(y_1,...,y_r)\left(zt^q\right)^k$$

$$T = (1-t)^{-m-(1+\alpha)d} \exp\left\{x\left[1-(1-t)^{-d}\right]\right\}$$
$$\times \sum_{k=0}^{\infty} a_k P_{m+qk}^{(\alpha)}\left(\frac{x}{(1-t)^d}; d\right) \Omega_{\mu+pk}(y_1,...,y_r) \left(\frac{zt^q}{(1-t)^q}\right)^k$$
$$= (1-t)^{-m-(1+\alpha)d} \exp\left\{x\left[1-(1-t)^{-d}\right]\right\}$$
$$\times \Lambda_{m,q}\left(\frac{x}{(1-t)^d}; y_1,...,y_r; \frac{zt^q}{(1-t)^q}\right),$$

which completes the proof.

4. SPECIAL CASES

As an application of the above theorems, when the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_s)$, $k \in \mathrm{N}_0$, $s \in \mathrm{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$s = r, \quad \Omega_{\mu+\psi k}(y_1,...,y_r) = u_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 3.1, where the Erkus-Srivastava polynomials $u_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ generated by [5]

$$\sum_{n=0}^{\infty} u_n^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)\frac{t^n}{n!} = \prod_{j=1}^{r}\{(1-x_i t^{m_j})^{-\alpha_i}\}. \tag{4.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for Laguerre type d-orthogonal polynomials $l_n^{(\alpha_d)}(x)$ and Erkus-Srivastava polynomials.

Corollary 4.1. *If*

$$\Lambda_{\mu,\psi}(y;\zeta) := \sum_{k=0}^{\infty} a_k u_{\mu+\psi k}^{(\alpha_1,\ldots,\alpha_r)}(y_1,\ldots,y_r)\zeta^k \ , a_k \neq 0, \quad \mu,\psi \in C ,$$

then, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \,(\lambda)_{n-pk}\, l_{n-pk}^{(\alpha_d)}(x) u_{\mu+\psi k}^{(\alpha_1,\ldots,\alpha_r)}(y_1,\ldots,y_r)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!} \\ &= (1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\frac{-xt}{1-t}\right)\Lambda_{\mu,\psi}(y_1,\ldots,y_r;\eta) \end{aligned} \tag{4.2}$$

provided that each member of (4.2) exists.

Remark 4.1. *Using the generating relation (4.1) for the Erkus-Srivastava polynomials and getting* $a_k = \frac{1}{k!}$, $\mu = 0$, $\psi = 1$ *in Corollary 4.1, we find that*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} (\lambda)_{n-pk}\, l_{n-pk}^{(\alpha_d)}(x) u_k^{(\alpha_1,\ldots,\alpha_r)}(y_1,\ldots,y_r)\tfrac{\eta^k}{k!}\frac{t^{n-pk}}{(n-pk)!} \\ &= (1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\tfrac{-xt}{1-t}\right)\prod_{j=1}^{r}\{(1-y_i t^{m_j})^{-\alpha_i}\}. \end{aligned}$$

If we set $r=1$ and $\Omega_{\mu+\psi k}(y) = Z_{\mu+\psi k}^{\alpha}\left(dy^{\frac{1}{d}},d\right)$, $(y>0)$ in Theorem 3.3, we have the bilinear generating function relation for the Laguerre type d-orthogonal polynomials $Z_{\mu+\psi k}^{\alpha}\left(dy^{\frac{1}{d}},d\right)$.

Corollary 4.2. *If*

$$\Lambda_{\mu,\psi}(y;\zeta) := \sum_{k=0}^{\infty} a_k Z_{\mu+\psi k}^{\alpha}\left(dy^{\frac{1}{d}},d\right)\zeta^k \ , a_k \neq 0, \quad \mu,\psi \in C ,$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k \frac{(\lambda)_{n-pk}}{(\alpha+1)_{d(n-pk)}} Z_{n-pk}^{\alpha}\left(dx^{\frac{1}{d}},d\right) Z_{\mu+\psi k}^{\alpha}\left(dy^{\frac{1}{d}},d\right)\frac{\eta^k}{t^{pk}}t^n$$
$$=(1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\frac{-xt}{1-t}\right)\Lambda_{\mu,\psi}(y;\eta) \tag{4.3}$$

provided that each member of (4.3) exists.

Remark 4.2. *Using the generating relation (2.1) for the Laguerre type d-orthogonal polynomials and getting* $a_k=\frac{(\gamma)_k}{(\alpha+1)_{dk}}$, $\mu=0$, $\psi=1$ *in Corollary 4.2,we find that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} \frac{(\lambda)_{n-pk}}{(\alpha+1)_{d(n-pk)}}\frac{(\gamma)_k}{(\alpha+1)_{dk}} Z_{n-pk}^{\alpha}\left(dx^{\frac{1}{d}},d\right) Z_{k}^{\alpha}\left(dy^{\frac{1}{d}},d\right)\frac{\eta^k}{k!}\frac{t^{n-pk}}{(n-pk)!}$$
$$=(1-t)^{-\lambda}\,{}_1F_d\left(\lambda;(\alpha_d+1);\tfrac{-xt}{1-t}\right)(1-t)^{-\gamma}\,{}_1F_d\left(\gamma;(\alpha_d+1);\tfrac{-yt}{1-t}\right).$$

If we set $r=1$ and $\Omega_{\mu+\psi k}(y)=P_{\mu+pk}^{(\alpha)}(y;d)$, $(y>0)$ in Theorem 3.7, we have the bilinear generating function relation for the Laguerre type *d*-orthogonal polynomials $P_{\mu+pk}^{(\alpha)}(y;d)$.

Corollary 4.3. *If*

$$\Lambda_{m,q}(x;y_1,\dots,y_r;t):=\sum_{k=0}^{\infty} a_k P_{m+qk}^{(\alpha)}(x;d)t^k\ P_{\mu+pk}^{(\alpha)}(y_1,\dots,y_r;d), a_k\neq 0,\quad \mu,\psi\in C\,,$$

and

$$N_{n,m,q}^{p,\mu}(y_1,\dots,y_r;z):=\sum_{k=0}^{[n/q]} a_k P_{\mu+pk}^{(\alpha)}(y_1,\dots,y_r;d)\frac{z^k}{(n-qk)!}$$

then, we have

$$\sum_{n=0}^{\infty} P_{m+n}^{(\alpha)}(x;d) N_{n,m,q}^{p,\mu}(y_1,\dots,y_r;z)t^n$$
$$=(1-t)^{-m-(1+\alpha)d}\exp\left\{x\left[1-(1-t)^{-d}\right]\right\}\Lambda_{m,q}\left(\tfrac{x}{(1-t)^d},d;y_1,\dots,y_r;\tfrac{zt^q}{(1-t)^q}\right) \tag{4.4}$$

provided that each member of (4.4) exists.

Furthermore, for every suitable choice of the coefficients a_i $(i\in \mathrm{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1,\dots,y_s)$, $(s\in \mathrm{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, and 3.7 can be applied in order to derive various families of multilinear and multilateral generating functions for the Laguerre type *d*-orthogonal polynomials.

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