ORIGINAL PAPER

A NOVEL METHOD FOR RELATIVISTIC ENERGY OF A UNIT VECTOR FIELD IN LIE GROUPS VIA GALILEAN TRANSFORMATION

TALAT KORPINAR[1], RIDVAN CEM DEMIRKOL[1]

Manuscript received: 18.08.2017; Accepted paper: 13.12.2017;
Published online: 30.03.2018.

Abstract. *In this paper, we firstly present a new approach for the computation of the energy of the moving particle in a Lie group* $\mathbf{R}$ *for a given reference system* K. *In this approach, we use geometrical descriptions of the curvature and torsion functions of the curve for the calculation. Then we consider a second reference system* K^*, *which moves relative to* K *for an arbitrary direction with a uniform velocity under the Galilean transformation. Finally, we compute the relativistic energy of the moving particle considering relative reference system* K^*.

Mathematics Subject Classifications: *53C80, 53A04.*
Keywords: *Energy, Relative Frame, Galilean Transformation, Unit Vector Fields.*

1. INTRODUCTION

Relativity principle quest for the ultimate understanding of how does an event perceive in one state or place from the other. If the event taking place in one state or place of motion is observed differently from the other, a question appears on how should laws of physics are implemented to this relative concept? Before the theory of special relativity of Einstein, the relativity principle of the motion of a particle, whose velocity much less than velocity of light, is determined by the Galilean relativity.

In the theory of the classical mechanics, time difference is the same between any two happenings in every coordinate system since time is assumed to be absolute. This is one of the major distinctions between the Galilean relativity and modern relativity theory, which states that velocity of light is the same for all observers in vacuum [1-3].

In [4], we had already investigated energy of the moving particle lying on any inertial frame of reference in a Lie group **R.** Now, we examine relativistic energy of the observed moving particle lying on the relative reference frame, which moves with uniform velocity such that it is not comparable to the velocity of light with respect to the first inertial frame in a Lie group **R.** By doing this, we have a chance to compare to the energy and relativistic energy of the moving particle and observed moving particle.

Some advantages of choosing a Lie group can be listed as follows. Firstly, Lie group is not only a smooth manifold but also an algebraic group for which the operation of a group is given by composition of smooth invertible functions. Secondly, transformations of the reference time, location, state or orientation of uniform translation at uniform velocity are

[1]Mus Alparslan University, Department of Mathematics, 49250 Mus, Turkey.
E-mail: talatkorpinar@gmail.com; rcdemirkol@gmail.com.

known as Galilean transformations, which forms a Lie group except for reflections. Further, we also know that Galilean group $G(3)$ preserves consequences of relative orientation, measuring time and length intervals in different inertial frames of motion associated with each other by Galilean transformations.

Foundations of the Lie group is based on continuous transformation group, which can be demonstrated as the rigid motion of a ball continuously with a painted design on it in three dimensional space. Another field of study that has the mechanics of continuum is elasticity theory in which solids are thought as nearly rigid materials and continuous such that they can be deformable slightly with an external force. Using this close relation, we give a correlation between the relativistic energy of the observed moving particle and bending energy of elastica, which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744 [5-7].

Some studies that motivate us working on this topic can be listed as the following. Unit vector field's energy was studied by Wood [8]. Gil-Medrano [9], worked on the relation between energy and volume of vector fields. Other studies [10, 11] investigated energy distributions and corrected energy distributions on Riemannian manifolds. Altin [12] computed energy of Frenet vector fields for given non-lightlike curves in semi-Euclidean space. Körpinar [13] discussed timelike biharmonic particle's energy in Heisenberg spacetime. Kumar and Srivastava [14] presented changes in curvature and torsion under the relative motion.

2. PRELIMINARIES

2.1. FRENET FRAME IN LIE GROUPS

Let $\mathbf{R}$ be a Lie group with a bi-invariant metric $\langle , \rangle$ and ∇ be a Levi-Civita connection of $\mathbf{R}$. If h shows Lie algebra of $\mathbf{R}$, then we get isomorphism between h and $T_j\mathbf{R}$, where j is natural element of $\mathbf{R}$. For a bi-invariant metric $\langle , \rangle$ on $\mathbf{R}$ it is obtained that

$$\langle \mathbf{K}, [\mathbf{L}, \mathbf{Y}] \rangle = \langle [\mathbf{K}, \mathbf{L}], \mathbf{Y} \rangle,$$

and

$$D_{\mathbf{K}}\mathbf{L} = \frac{1}{2}[\mathbf{K}, \mathbf{L}]$$

for all $\mathbf{K}, \mathbf{L}, \mathbf{Y} \in h.$

Let $\boldsymbol{\gamma} : I \subset \mathsf{R} \to \mathbf{R}$ be an arc-lengthed curve and $\{\mathbf{K}_1, \mathbf{K}_2, ..., \mathbf{K}_n\}$ be a orthonormal basis of $h.$ Then for any two vector fields $\mathbf{P}$ and $\mathbf{Q}$ it is written that $\mathbf{P} = \sum_{a=1}^{n} p_a \mathbf{K}_a$ and $\mathbf{Q} = \sum_{a=1}^{n} q_a \mathbf{K}_a$ along the curve $\boldsymbol{\gamma}$, where $p_a : I \to \mathsf{R}$ and $q_a = I \to \mathsf{R}$ are smooth functions. If $\mathbf{P}$ and $\mathbf{Q}$ are any two vector fields, then Lie bracket is written in the following form.

$$[\mathbf{P}, \mathbf{Q}] = \sum_{a=1}^{n} p_a q_a [\mathbf{K}_a, \mathbf{K}_b].$$

Covariant derivative of $\mathbf{P}$ along the curve $\boldsymbol{\gamma}$ is stated by

$$D_{\gamma'}\mathbf{P} = \mathbf{P}' + \frac{1}{2}\left[\mathbf{e}^{\mu}_{(0)}, \mathbf{P}\right],$$

where $\mathbf{P}' = \sum_{a=1}^{n} \frac{d\mathbf{p}}{dt}\mathbf{K}_a$ and $\mathbf{e}^{\mu}_{(0)} = \boldsymbol{\gamma}'$ [15 – 18]. Frenet apparatus of the curve $\boldsymbol{\gamma}$ is represented by elements $\left(\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu}_{(1)}, \mathbf{e}^{\mu}_{(2)}, \kappa, \tau\right)$ in a 3-dimensional Lie group **R.**

Characterization of the intrinsic geometric properties of a parameterized curve $\boldsymbol{\gamma}$ can most subtly be determined by using Serret-Frenet equations. Frenet tetrad frame is consisted of three orthonormal vectors $\mathbf{e}^{\mu}_{(\theta)}$, assuming the curve $\boldsymbol{\gamma}$ is sufficiently smooth at each point. The index within the parenthesis is the tetrad index that describes particular member of the tetrad. In particular, $\mathbf{e}^{\mu}_{(0)}$ is the unit tangent vector, $\mathbf{e}^{\mu}_{(1)}$ is the unit normal, and $\mathbf{e}^{\mu}_{(2)}$ is the unit binormal vector of the curve Γ, respectively. Orthonormality conditions are summarized by $\mathbf{e}^{\mu}_{(\theta)}\mathbf{e}^{\mu}_{(\beta)} = \eta_{\theta\beta}$, where $\eta_{\theta\beta}$ is Euclidean metric such that: $\mathrm{diag}(1,1,1)$. For non-negative coefficients κ, τ, and vectors $\mathbf{e}^{\mu}_{(i)}(i = 0,1,2)$ following equations and properties are satisfied [16,17].

$$\begin{aligned}
\frac{\nabla \mathbf{e}^{\mu}_{(0)}}{ds} &= \kappa \mathbf{e}^{\mu}_{(1)}, \\
\frac{\nabla \mathbf{e}^{\mu}_{(1)}}{ds} &= -\kappa \mathbf{e}^{\mu}_{(0)} + (\tau - \tau_{\mathbf{R}})\mathbf{e}^{\mu}_{(2)}, 2.1 \\
\frac{\nabla \mathbf{e}^{\mu}_{(2)}}{ds} &= (\tau_{\mathbf{R}} - \tau)\mathbf{e}^{\mu}_{(1)},
\end{aligned} \tag{1}$$

where Lie group **R** has the Levi-Civita connection ∇ and

$$\boldsymbol{\tau}_{\mathbf{R}} = \frac{1}{2}\left\langle \left[\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu}_{(1)}\right], \mathbf{e}^{\mu}_{(2)} \right\rangle$$

or equivalently

$$\boldsymbol{\tau}_{\mathbf{R}} = \frac{1}{2\kappa^2\tau}\left\langle \mathbf{e}^{\mu''}_{(0)}, \left[\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu'}_{(0)}\right] \right\rangle + \frac{1}{4\kappa^2\tau}\left\| \left[\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu'}_{(0)}\right] \right\|^2.$$

Proposition 2.1 *Let* **R** *be a 3-dimensional Lie group induced with a bi-invariant metric, then we have following statements;*

(i) $\tau_{\mathbf{R}} = 0$ *if* **R** *is Abelian group.*

(ii) $\tau_{\mathbf{R}} = 1$ *if* **R** *is* SU^2.

(iii) $\tau_{\mathbf{R}} = \frac{1}{2}$ if **R** *is* SO^3 [16, 19].

2.2. ENERGY ON THE UNIT VECTOR FIELD IN $\mathbb{R}^3$

We firstly give the fundamental definitions and propositions, which are used to compute the energy of the unit vector field.

Definition 2.2 *Let* (M,ρ) *and* $(N,\tilde{h})$ *be two Riemannian manifolds then energy of a differentiable map* $f:(M,\rho)\to(N,\tilde{h})$ *can be defined as*

$$\varepsilon nergy(f)=\frac{1}{2}\int_M\sum_{a=1}^{n}\tilde{h}(df(e_a),df(e_a))v, \tag{2.2}$$

where $\{e_a\}$ *is a local basis of the tangent space and* v *is the canonical volume form in* M [8].

Proposition 2.3 *Let* $Q:T(T^1M)\to T^1M$ *be the connection map. Then following two conditions hold.*

i) $\omega\circ Q=\omega\circ d\omega$ *and* $\omega\circ Q=\omega\circ\tilde{\omega}$, *where* $\tilde{\omega}:T(T^1M)\to T^1M$ *is the tangent bundle projection;*

ii) *for* $\rho\in T_xM$ *and a section* $\xi:M\to T^1M$; *we have*

$$Q(d\xi(\rho))=\nabla_\rho\xi, \tag{2.3}$$

where ∇ *is the Levi-Civita covariant derivative* [8, 12].

Definition 2.4 *Let* $\varsigma_1,\varsigma_2\in T_\xi(T^1M)$, *then we define*

$$\rho_S(\varsigma_1,\varsigma_2)=\rho(d\omega(\varsigma_1),d\omega(\varsigma_2))+\rho(Q(\varsigma_1),Q(\varsigma_2)). \tag{2.4}$$

This yields a Riemannian metric on TM . *As known* ρ_S *is called the Sasaki metric that also makes the projection* $\omega:T^1M\to M$ *a Riemannian submersion.*

Theorem 2.5 *Let* $\gamma(s)$ *be a unit speed curve defined on 3-dimensional Lie group then we can derive following relations on the energy of tangent, normal, and binormal vectors respectively.*

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)}=\frac{1}{2}(s+\int_0^s\kappa^2ds),$$

$$\varepsilon nergy\mathbf{e}^{\mu}_{(1)}=\frac{1}{2}(s+\int_0^s\left(\kappa^2+(\tau-\tau_{\mathbf{R}})^2\right)ds),$$

$$\varepsilon nergy\mathbf{e}^{\mu}_{(2)}=\frac{1}{2}(s+\int_0^s(\tau-\tau_{\mathbf{R}})^2ds),$$

where κ,τ are curvature and torsion of the curve $\gamma(s)$, [12].

Proof. From (2.2) and (2.3) we know that

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)} = \frac{1}{2}\int_0^s \rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right)ds.$$

Using the Eq. (2.4) we also have

$$\rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right) = \rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})), d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}))) + \rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})), Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we get

$$d(\omega)\circ d(\mathbf{e}^{\mu}_{(0)}) = d(\omega\circ\mathbf{e}^{\mu}_{(0)}) = d(id_C) = id_{TC}.$$

We also induce following equality by the previous statement.

$$Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})) = \nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)} = \kappa\mathbf{e}^{\mu}_{(1)}.$$

Thus, we find from (2.1)

$$\rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right) = \rho\left(\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu}_{(0)}\right) + \rho\left(D_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}, D_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)$$
$$= 1+\kappa^2.$$

So, we can easily obtain

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)} = \frac{1}{2}(s + \int_0^s \kappa^2 ds).$$

This completes the proof.

3. RELATIVISTIC ENERGY ON THE MOVING INERTIAL FRAME

Let K be the first system that contains a moving particle in a 3-dimensional Lie gorup $\mathbf{R}$ such that it has three position vectors with a time parameter. Then we can define another moving relative system K^* corresponding to K such that it moves with uniform velocity v relatively to the K on the direction of $\vec{\mathbf{v}} = (a\mathbf{e}^{\mu}_{(0)} + b\mathbf{e}^{\mu}_{(1)} + c\mathbf{e}^{\mu}_{(2)})v,$ where $a,b,c \in \mathrm{R}$ and $\left\|\vec{\mathbf{v}}\right\| = v.$

Let assume that motion of the moving particle in the 3-dimensional Lie gorup $\mathbf{R}$ corresponds to a space curve Γ, whose Frenet frame characterization is defined in (2.1).

Under the Galilean transformation we can transfer this construction to the moving relative inertial system K^* as the following.

$$\Gamma^*\left(s^*\right) = \Gamma\left(s\right) - \overrightarrow{\mathbf{v}}\, s,\ s = s^*, \tag{3.1}$$

where s, s^* are time parameters for the moving particle in the inertial sytem of K and relative inertial system K^*, respectively. We will use s in the moving relative inertial system K^* instead of s^* due to the equality given in (3.1) . Thus, we obtain the observed curve $\Gamma^*(s)$ for the relative system as

$$\Gamma^*(s) = \Gamma(s) - v(a\mathbf{e}^{\mu}_{(0)} + b\mathbf{e}^{\mu}_{(1)} + c\mathbf{e}^{\mu}_{(2)})s. \tag{3.2}$$

Theorem 3.1 *The Frenet frame elements for the observed curve* $\boldsymbol{\gamma}^*(s)$ *are stated in terms of the element of the first system as the following.*

$$\begin{aligned}
&\mathbf{e}^{\mu^*}_{(0)} = (1 + v(b\kappa s - a))\mathbf{e}^{\mu}_{(0)} + (v(-a\kappa s + c(\tau - \tau_{\mathbf{R}})s - b))\mathbf{e}^{\mu}_{(1)} \\
&+ (v(-b(\tau - \tau_{\mathbf{R}})s - c))\mathbf{e}^{\mu}_{(2)}, \\
&\mathbf{e}^{\mu^*}_{(1)} = \frac{1}{\kappa^*} v(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau - \tau_{\mathbf{R}})s)\mathbf{e}^{\mu}_{(0)} \\
&+ \frac{1}{\kappa^*}(\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau - \tau_{\mathbf{R}})' s + 2c(\tau - \tau_{\mathbf{R}}) \\
&+ b(\tau - \tau_{\mathbf{R}})^2 s))\mathbf{e}^{\mu}_{(1)} + \frac{1}{\kappa^*} v(c(\tau - \tau_{\mathbf{R}})^2 s - a\kappa(\tau - \tau_{\mathbf{R}})s \\
&- 2b(\tau - \tau_{\mathbf{R}}) - b(\tau - \tau_{\mathbf{R}})' s)\mathbf{e}^{\mu}_{(2)}, \\
&\mathbf{e}^{\mu^*}_{(2)} = \frac{1}{\kappa^*}((v^2(-a\kappa s + c(\tau - \tau_{\mathbf{R}})s - b)(c(\tau - \tau_{\mathbf{R}})^2 s - a\kappa(\tau - \tau_{\mathbf{R}})s \\
&- 2b(\tau - \tau_{\mathbf{R}}) - b(\tau - \tau_{\mathbf{R}})' s) + v(b(\tau - \tau_{\mathbf{R}})s + c)(\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s \\
&+ c(\tau - \tau_{\mathbf{R}})' s + 2c(\tau - \tau_{\mathbf{R}}) + b(\tau - \tau_{\mathbf{R}})^2 s)))\mathbf{e}^{\mu}_{(0)} \\
&- \frac{1}{\kappa^*}(v(1 + vb\kappa s - va)(c(\tau - \tau_{\mathbf{R}})^2 s - a\kappa(\tau - \tau_{\mathbf{R}})s - 2b(\tau - \tau_{\mathbf{R}}) \\
&- b(\tau - \tau_{\mathbf{R}})' s) + v^2(b(\tau - \tau_{\mathbf{R}})s + c)(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau - \tau_{\mathbf{R}})s))\mathbf{e}^{\mu}_{(1)} \\
&+ \frac{1}{\kappa^*}(((1 + v(b\kappa s - a))(\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau - \tau_{\mathbf{R}})' s \\
&+ 2c(\tau - \tau_{\mathbf{R}}) + b(\tau - \tau_{\mathbf{R}})^2 s)) \\
&- v^2(-a\kappa s + c(\tau - \tau_{\mathbf{R}})s - b)(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau - \tau_{\mathbf{R}})s))\mathbf{e}^{\mu}_{(2)},
\end{aligned} \tag{3.3}$$

where κ, τ are curvature and torsion of the $\boldsymbol{\gamma}(s)$ and κ^* is the curvature of the observed curve $\boldsymbol{\gamma}^*(s)$. We can also express the curvature and torsion of the observed curve $\boldsymbol{\gamma}^*(s)$ by using the first system as follows.

$$\begin{aligned}
&\kappa^* = ((bv(\kappa' s + 2\kappa) + vs(a\kappa^2 - \kappa c(\tau - \tau_{\mathbf{R}})))^2 + (av(-2\kappa - \kappa' s) + cv((\tau - \tau_{\mathbf{R}})' s \\
&+ 2(\tau - \tau_{\mathbf{R}})) + bvs(\kappa^2 + (\tau - \tau_{\mathbf{R}})^2) + \kappa)^2 + (bv(-2(\tau - \tau_{\mathbf{R}})
\end{aligned}$$

$$-\left(\tau-\tau_{\mathbf{R}}\right)' s)+\left(\tau-\tau_{\mathbf{R}}\right)vs(c\left(\tau-\tau_{\mathbf{R}}\right)-a\kappa))^2)^{\frac{1}{2}},$$

$$\begin{aligned}
\tau^* = \frac{1}{u}&((v^2\left(-a\kappa s+c\left(\tau-\tau_{\mathbf{R}}\right)s-b\right)(c\left(\tau-\tau_{\mathbf{R}}\right)^2 s-a\kappa\left(\tau-\tau_{\mathbf{R}}\right)s \\
&-2b\left(\tau-\tau_{\mathbf{R}}\right)-b\left(\tau-\tau_{\mathbf{R}}\right)' s)+v\left(b\left(\tau-\tau_{\mathbf{R}}\right)s+c\right)(\kappa+v(b\kappa^2 s-2a\kappa \\
&-a\kappa' s+c\left(\tau-\tau_{\mathbf{R}}\right)' s+2c\left(\tau-\tau_{\mathbf{R}}\right)+b\left(\tau-\tau_{\mathbf{R}}\right)^2 s))(bv\kappa'' s+3bv\kappa' \\
&+3av\kappa^2+3av\kappa\kappa' s-2cv\kappa\left(\tau-\tau_{\mathbf{R}}\right)' s-cv\kappa'\left(\tau-\tau_{\mathbf{R}}\right)s-3cv\kappa\left(\tau-\tau_{\mathbf{R}}\right) \\
&-\kappa^2-b\kappa^3 vs-bv\kappa\left(\tau-\tau_{\mathbf{R}}\right)^2 s)-((v\left(1+vb\kappa s-va\right)(c\left(\tau-\tau_{\mathbf{R}}\right)^2 s \\
&-a\kappa\left(\tau-\tau_{\mathbf{R}}\right)s-2b\left(\tau-\tau_{\mathbf{R}}\right)-b\left(\tau-\tau_{\mathbf{R}}\right)' s) \\
&+v^2\left(b\left(\tau-\tau_{\mathbf{R}}\right)s+c\right)\left(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa\left(\tau-\tau_{\mathbf{R}}\right)s\right))(\kappa'+3bv\kappa\kappa' s \\
&+3bv\kappa^2-3av\kappa'-av\kappa'' s+cv\left(\tau-\tau_{\mathbf{R}}\right)'' s+3cv\left(\tau-\tau_{\mathbf{R}}\right)' \\
&+3bv\left(\tau-\tau_{\mathbf{R}}\right)\left(\tau-\tau_{\mathbf{R}}\right)' s+3bv\left(\tau-\tau_{\mathbf{R}}\right)^2+v(a\kappa^3-c\left(\tau-\tau_{\mathbf{R}}\right)^3)s \\
&+v\kappa\left(\tau-\tau_{\mathbf{R}}\right)\left(a\left(\tau-\tau_{\mathbf{R}}\right)-c\kappa\right)s))+(((1+v(b\kappa s-a))(\kappa+v(b\kappa^2 s \\
&-2a\kappa-a\kappa' s+c\left(\tau-\tau_{\mathbf{R}}\right)' s+2c\left(\tau-\tau_{\mathbf{R}}\right)+b\left(\tau-\tau_{\mathbf{R}}\right)^2 s)) \\
&-v^2(-a\kappa s+c\left(\tau-\tau_{\mathbf{R}}\right)s-b)(b\kappa' s+2b\kappa+a\kappa^2 s \\
&-c\kappa\tau s))(-2av\kappa'\left(\tau-\tau_{\mathbf{R}}\right)s-av\kappa\left(\tau-\tau_{\mathbf{R}}\right)' s-3av\kappa\tau+3cv\left(\tau-\tau_{\mathbf{R}}\right)^2 \\
&+3cv\left(\tau-\tau_{\mathbf{R}}\right)\left(\tau-\tau_{\mathbf{R}}\right)' s-3bv\left(\tau-\tau_{\mathbf{R}}\right)'-bv\left(\tau-\tau_{\mathbf{R}}\right)'' s+\left(\tau-\tau_{\mathbf{R}}\right)\kappa \\
&+bv\kappa^2\left(\tau-\tau_{\mathbf{R}}\right)s+bv\left(\tau-\tau_{\mathbf{R}}\right)^3 s))),
\end{aligned}$$

where

$$\begin{aligned}
u = &(v^2\left(-a\kappa s+c\left(\tau-\tau_{\mathbf{R}}\right)s-b\right)(c\left(\tau-\tau_{\mathbf{R}}\right)^2 s-a\kappa\left(\tau-\tau_{\mathbf{R}}\right)s \\
&-2b\left(\tau-\tau_{\mathbf{R}}\right)-b\left(\tau-\tau_{\mathbf{R}}\right)' s)+v\left(b\left(\tau-\tau_{\mathbf{R}}\right)s+c\right)(\kappa \\
&+v(b\kappa^2 s-2a\kappa-a\kappa' s+c\left(\tau-\tau_{\mathbf{R}}\right)' s+2c\left(\tau-\tau_{\mathbf{R}}\right)+b\left(\tau-\tau_{\mathbf{R}}\right)^2 s)))^2 \\
&-(v\left(1+vb\kappa s-va\right)(c\left(\tau-\tau_{\mathbf{R}}\right)^2 s-a\kappa\left(\tau-\tau_{\mathbf{R}}\right)s-2b\left(\tau-\tau_{\mathbf{R}}\right) \\
&-b\left(\tau-\tau_{\mathbf{R}}\right)' s)+v^2\left(b\left(\tau-\tau_{\mathbf{R}}\right)s+c\right)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa(\tau-\tau_{\mathbf{R}})s))^2 \\
&+(((1+v(b\kappa s-a))(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s+c\left(\tau-\tau_{\mathbf{R}}\right)' s \\
&+2c\left(\tau-\tau_{\mathbf{R}}\right)+b\left(\tau-\tau_{\mathbf{R}}\right)^2 s))-v^2(-a\kappa s+c\left(\tau-\tau_{\mathbf{R}}\right)s-b)(b\kappa' s \\
&+2b\kappa+a\kappa^2 s-c\kappa\left(\tau-\tau_{\mathbf{R}}\right)s))^2.
\end{aligned}$$

Proof. From (3.1) and (3.2) we know that

$$\boldsymbol{\gamma}^* = \boldsymbol{\gamma}-v(a\mathbf{e}^{\mu}_{(0)}+b\mathbf{e}^{\mu}_{(1)}+c\mathbf{e}^{\mu}_{(2)})s.$$

If we take derivative of $\boldsymbol{\gamma}^*$ with respect to s^* parameter bear in the mind the fact that $s = s^*$ and use the Frenet frame given (2.1), then we get

$$\begin{aligned}
\mathbf{e}^{\mu^*}_{(0)} = \left(1+v\left(b\kappa s-a\right)\right)\mathbf{e}^{\mu}_{(0)}&+\left(v\left(-a\kappa s+c\left(\tau-\tau_{\mathbf{R}}\right)s-b\right)\right)\mathbf{e}^{\mu}_{(1)} \\
&+\left(v\left(-b\left(\tau-\tau_{\mathbf{R}}\right)s-c\right)\right)\mathbf{e}^{\mu}_{(2)}.
\end{aligned}$$

We can find the desired result by using standard method and usual definitions of Frenet frame as the following.

$$\mathbf{e}_{(1)}^{\mu^*} = \frac{1}{k^*}\mathbf{e}_{(0)}^{\mu^{*'}} \text{and} \mathbf{e}_{(2)}^{\mu^*} = \mathbf{e}_{(0)}^{\mu^*} \times \mathbf{e}_{(1)}^{\mu^*},$$
$$k^* = \left\|\mathbf{e}_{(0)}^{\mu^{*'}}\right\| \text{and} \tau^* = \frac{\left\lfloor \mathbf{e}_{(0)}^{\mu^*}, \mathbf{e}_{(0)}^{\mu^{*'}} \right\rfloor \cdot \mathbf{e}_{(0)}^{\mu^{*''}}}{\left\|\mathbf{e}_{(0)}^{\mu^*} \times \mathbf{e}_{(0)}^{\mu^{*'}}\right\|^2}.$$

After this introductory theorem, which informs us about the Frenet frame construction on the relative moving inertial system, we are ready to present the core idea of this study. In the theory of relativity, all the energy moving through an object contributes to the body's total mass that measures how much it can resist to acceleration. Each kinetic and potential energy makes a highly proportional contribution to the mass $[13-15]$. In this study not only we compute the relativistic energy of the moving particle in the moving inertial system but we also investigate its close correlation with bending energy of elastica, which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744. Elastica of bending energy formula for a space curve in the 3-dimensional Frenet curvature along the curve is given by

$$H_B = \frac{1}{2}\int \kappa^2 ds,$$

where s is an arclength and $\kappa^2 = \left\|\nabla_{\mathbf{e}_{(0)}^{\mu}} \mathbf{e}_{(0)}^{\mu}\right\|^2$, $[23]$.

Theorem 3.2 *Relativistic energy of the tangent vector of the observed curve* $\boldsymbol{\gamma}^*$ *in the moving relative system* K^* *is stated by*

$$\begin{aligned}\mathcal{E}nergy\mathbf{e}_{(0)}^{\mu^*} &= \frac{1}{2}\int_0^s ((1+v(b\kappa s-a))^2 + (v(-a\kappa + c(\tau-\tau_{\mathbf{R}}))s - vb)^2 \\ &+ (-bv(\tau-\tau_{\mathbf{R}})s - cv)^2 + (v(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau-\tau_{\mathbf{R}})s))^2 \\ &+ (\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau-\tau_{\mathbf{R}})' s + 2c(\tau-\tau_{\mathbf{R}}) \\ &+ b(\tau-\tau_{\mathbf{R}})^2 s))^2 + (v(c(\tau-\tau_{\mathbf{R}})^2 s - a\kappa(\tau-\tau_{\mathbf{R}})s \\ &- 2b(\tau-\tau_{\mathbf{R}}) - b(\tau-\tau_{\mathbf{R}})' s))^2)ds.\end{aligned}$$

Proof. From (2.2) and (2.3) we know that

$$\mathcal{E}nergy\mathbf{e}_{(0)}^{\mu^*} = \frac{1}{2}\int_0^s \rho_S\left(d\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*}), d\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*})\right)ds.$$

Using the Eq. (2.3) we have

$$\rho_S\left(d\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*}), d\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*})\right) = \rho(d\omega(\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*})), d\omega(\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*}))) + \rho(Q(\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*})), Q(\mathbf{e}_{(0)}^{\mu^*}(\mathbf{e}_{(0)}^{\mu^*}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we also get

$$d(\omega)\circ d(\mathbf{e}^{\mu^*}_{(0)}) = d(\omega\circ\mathbf{e}^{\mu^*}_{(0)}) = d(id_C) = id_{TC}.$$

We also know by the previous definition

$$Q(\mathbf{e}^{\mu^*}_{(0)}(\mathbf{e}^{\mu^*}_{(0)})) = \nabla_{\mathbf{e}^{\mu^*}_{(0)}}\mathbf{e}^{\mu^*}_{(0)} = \kappa^*\mathbf{e}^{\mu^*}_{(1)}.$$

Thus, we find from (2.1)

$$\begin{aligned}\rho_S\left(d\mathbf{e}^{\mu^*}_{(0)}(\mathbf{e}^{\mu^*}_{(0)}), d\mathbf{e}^{\mu^*}_{(0)}(\mathbf{e}^{\mu^*}_{(0)})\right) &= \rho\left(\mathbf{e}^{\mu^*}_{(0)},\mathbf{e}^{\mu^*}_{(0)}\right) + \rho\left(\nabla_{\mathbf{e}^{\mu^*}_{(0)}}\mathbf{e}^{\mu^*}_{(0)}, \nabla_{\mathbf{e}^{\mu^*}_{(0)}}\mathbf{e}^{\mu^*}_{(0)}\right)\\
&= (1+v(b\kappa s-a))^2 + (v(-a\kappa + c(\tau-\tau_{\mathbf{R}}))s - vb)^2\\
&\quad + (-bv(\tau-\tau_{\mathbf{R}})s - cv)^2 + (v(b\kappa' s + 2b\kappa + a\kappa^2 s\\
&\quad - c\kappa(\tau-\tau_{\mathbf{R}})s))^2 + (\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s\\
&\quad + c(\tau-\tau_{\mathbf{R}})' s + 2c(\tau-\tau_{\mathbf{R}}) + b(\tau-\tau_{\mathbf{R}})^2 s))^2\\
&\quad + (v(c(\tau-\tau_{\mathbf{R}})^2 s - a\kappa(\tau-\tau_{\mathbf{R}})s\\
&\quad - 2b(\tau-\tau_{\mathbf{R}}) - b(\tau-\tau_{\mathbf{R}})' s))^2.\end{aligned}$$

So we can easily obtain the desired result by using the above statement.

We have following graph, which illustrates a relation between on the variation of the energy of tangent vector field of the curve Γ in the rest system K and the energy of tangent vector field of the observed curve Γ^* in the moving relative system K^* with respect to time.

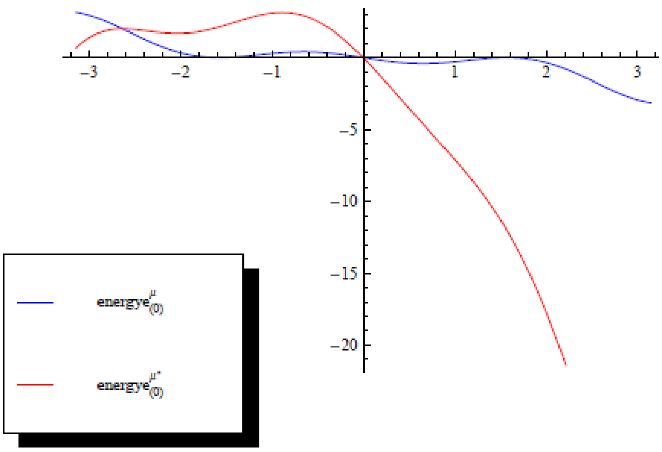

Figure 1. Energy variation graph for tangent vector field of a given curve.

Now we will examine some special motion of the moving inertial system.

Corollary 3.3 *If the relatively moving inertial system* K^* *moves parallel to tangent of the unit speed curve* $\gamma(s)$ *with uniform speed* v*, then we have*

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} = (1-av)^2 \varepsilon nergy\mathbf{e}_{(0)}^{\mu} + \frac{1}{2}a^2v^2\int_0^s (\kappa^2(\kappa^2 + (\tau-\tau_{\mathbf{R}})^2)s^2 + 4\kappa^2 + 4\kappa\kappa' s + \kappa'^2 s^2)ds - av\int_0^s (\kappa^2 - \kappa' s\kappa)ds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} - H_B((1-av)^2 + 4a^2v^2 - 2av) = \frac{1}{2}(1-av)^2 s + \frac{1}{2}a^2v^2\int_0^s (\kappa^2(\kappa^2 + (\tau-\tau_{\mathbf{R}})^2)s^2 + 4\kappa\kappa' s + \kappa'^2 s^2)ds - av\int_0^s \kappa' s\kappa ds.$$

Proof. By the assumption, we know that relative system moves on the direction of $\mathbf{v} = (a\mathbf{e}_{(0)}^{\mu})v,$ where v is the velocity. Then energy of the tangent vector of the observed curve can be computed by using the similar argument as in Theorem 3.2. Second part of the proof is obvious if we consider the formula of the bending energy of elastica.

Corollary 3.4 *If the relatively moving inertial system* K^* *moves parallel to normal of the unit speed curve* $\boldsymbol{\gamma}(s)$ *with uniform speed* $v,$ *then we have*

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} = \varepsilon nergy\mathbf{e}_{(0)}^{\mu} + 4b^2v^2\varepsilon nergy\mathbf{e}_{(1)}^{\mu} - \frac{3}{2}b^2v^2 s + \frac{1}{2}b^2v^2\int_0^s (s^2(\kappa^2 + (\tau-\tau_{\mathbf{R}})^2 + \kappa'^2 + (\tau-\tau_{\mathbf{R}})'^2 + 2\kappa^2(\tau-\tau_{\mathbf{R}})^2 + \kappa^4 + (\tau-\tau_{\mathbf{R}})^4) + s(4\kappa\kappa' + 4(\tau-\tau_{\mathbf{R}})(\tau-\tau_{\mathbf{R}})'))ds + bv\int_0^s \kappa(1+\kappa^2 + (\tau-\tau_{\mathbf{R}})^2)sds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} - H_B(1+4b^2v^2) = \frac{1}{2}s + \frac{1}{2}b^2v^2\int_0^s \left(1 + 4(\tau-\tau_{\mathbf{R}})^2\right)ds + \frac{1}{2}b^2v^2\int_0^s (s^2(\kappa^2 + (\tau-\tau_{\mathbf{R}})^2 + \kappa'^2 + (\tau-\tau_{\mathbf{R}})'^2 + 2\kappa^2(\tau-\tau_{\mathbf{R}})^2 + \kappa^4 + (\tau-\tau_{\mathbf{R}})^4) + s(4\kappa\kappa' + 4(\tau-\tau_{\mathbf{R}})(\tau-\tau_{\mathbf{R}})'))ds + bv\int_0^s \kappa(1+\kappa^2 + (\tau-\tau_{\mathbf{R}})^2)sds.$$

Corollary 3.5 *If the relatively moving inertial system* K^* *moves parallel to binormal of the unit speed curve* $\boldsymbol{\gamma}(s)$ *with uniform speed* $v,$ *then we have*

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} = \varepsilon nergy\mathbf{e}_{(0)}^{\mu} + 4c^2v^2\varepsilon nergy\mathbf{e}_{(2)}^{\mu} - \frac{3}{2}c^2v^2 s + \frac{1}{2}c^2v^2\int_0^s (s^2((\tau-\tau_{\mathbf{R}})^2 + (\tau-\tau_{\mathbf{R}})'^2 + \kappa^2(\tau-\tau_{\mathbf{R}})^2 + (\tau-\tau_{\mathbf{R}})^4) + 4s(\tau-\tau_{\mathbf{R}})(\tau-\tau_{\mathbf{R}})')ds + cv\int_0^s \left(\kappa(\tau-\tau_{\mathbf{R}})' s + 2\kappa(\tau-\tau_{\mathbf{R}})\right)ds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(0)}^{\mu^*} - H_B = \frac{1}{2}(1+c^2v^2)s + \frac{1}{2}c^2v^2\int_0^s (s^2((\tau-\tau_{\mathbf{R}})^2 + (\tau-\tau_{\mathbf{R}})'^2$$
$$+\kappa^2(\tau-\tau_{\mathbf{R}})^2 + (\tau-\tau_{\mathbf{R}})^4) + 4s(\tau-\tau_{\mathbf{R}})(\tau-\tau_{\mathbf{R}})' + 4(\tau-\tau_{\mathbf{R}})^2)ds$$
$$+cv\int_0^s \left(\kappa(\tau-\tau_{\mathbf{R}})' s + 2\kappa(\tau-\tau_{\mathbf{R}})\right)ds.$$

We have following diagram as a demonstration to observe the variation of the energy in the tangent vector field of the observed curve with respect to the motion of relative system. Here $\varepsilon nergy\mathbf{e}_{(0)_T}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(0)_N}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(0)_B}^{\mu^*}$ represent energy of the tangent vector of the observed curve Γ^* in the relative system, which acting parallel to tangent, normal, and binormal of the unit speed curve $\mathbf{\Gamma}$, respectively.

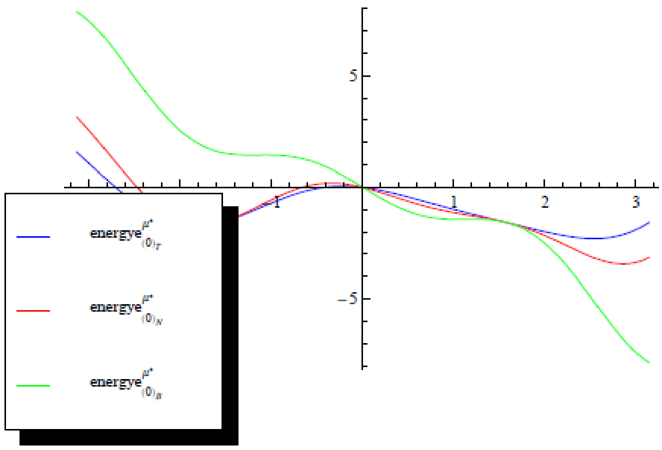

Figure 2. Energy variation graph for tangent vector field of an observed curve.

Theorem 3.6 *Relativistic energy of the normal vector of the observed curve* $\boldsymbol{\gamma}^*$ *in the moving relative system* K^* *is stated by*

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*} = \frac{1}{2}\int_0^s ((1+v(b\kappa s-a))^2 + (v(-a\kappa + c(\tau-\tau_{\mathbf{R}}))s - vb)^2$$
$$+(-bv(\tau-\tau_{\mathbf{R}})s - cv)^2)ds + \frac{1}{2}\int_0^s (\kappa^{*2}[(1+v(b\kappa s-a))^2$$
$$+(v(-a\kappa s + c(\tau-\tau_{\mathbf{R}})s - b))^2 + (v(-b(\tau-\tau_{\mathbf{R}})s - c))^2]$$
$$+(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2(-a\kappa s + c(\tau-\tau_{\mathbf{R}})s - b)(c(\tau-\tau_{\mathbf{R}})s - a\kappa(\tau-\tau_{\mathbf{R}})s$$
$$-2b(\tau-\tau_{\mathbf{R}}) - b(\tau-\tau_{\mathbf{R}})' s) + v(b(\tau-\tau_{\mathbf{R}})s + c)(\kappa$$
$$+v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau-\tau_{\mathbf{R}})' s + 2c(\tau-\tau_{\mathbf{R}})$$
$$+b(\tau-\tau_{\mathbf{R}})^2 s))))^2 + (-\frac{1}{\kappa^*}(v(1+vb\kappa s - va)(c(\tau-\tau_{\mathbf{R}})^2 s$$
$$-a\kappa(\tau-\tau_{\mathbf{R}})s - 2b(\tau-\tau_{\mathbf{R}}) - b(\tau-\tau_{\mathbf{R}})' s)$$
$$+v^2(b(\tau-\tau_{\mathbf{R}})s + c)(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau-\tau_{\mathbf{R}})s)))^2$$

$$+(\frac{1}{\kappa^*}((1+v(b\kappa s-a))(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s)$$
$$+c(\tau-\tau_{\mathbf{R}})' s+2c(\tau-\tau_{\mathbf{R}})+b(\tau-\tau_{\mathbf{R}})^2 s))-v^2(-a\kappa s)$$
$$+c(\tau-\tau_{\mathbf{R}})s-b)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa(\tau-\tau_{\mathbf{R}})s))^2])ds,$$

where κ^*,τ^* are defined as in Theorem 3.1.

Proof. It is obvious if we consider the following equality.

$$\mathbf{e}_{(1)}^{\mu^*\,'}=-\kappa^*\mathbf{e}_{(0)}^{\mu^*}+(\tau^*-\tau_{\mathbf{R}}^*)\mathbf{e}_{(2)}^{\mu^*}.$$

The rest of the proof can be done similarly as in the proof of Theorem 3.2.

We have following graph, which illustrates a relation between on the variation of the energy of normal vector field of the curve Γ in the rest system K and the energy of normal vector field of the observed curve Γ^* in the moving relative system K^* with respect to time.

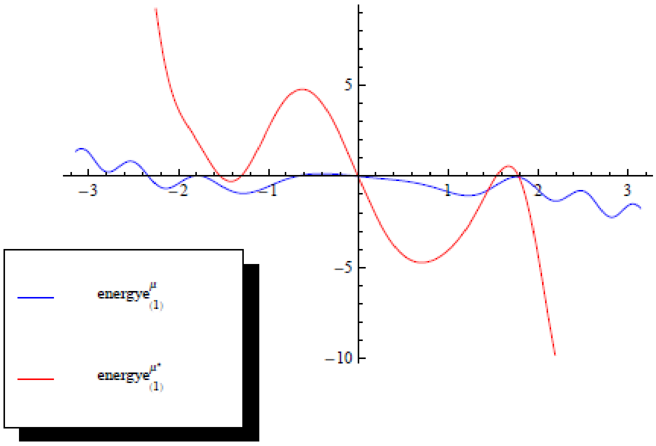

Figure 3. Energy variation graph for normal vector field of a given curve.

Corollary 3.7 *If the relatively moving inertial system K^* moves parallel to tangent of the unit speed curve $\gamma(s)$ with uniform speed v, then we have*

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*}=\frac{1}{2}\int_0^s((1-va)^2+(va\kappa s)^2)ds$$
$$+\frac{1}{2}\int_0^s(\kappa^{*2}[(1+v(-a))^2+(v(-a\kappa s))^2]$$
$$+(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2(-a\kappa s)(-a\kappa(\tau-\tau_{\mathbf{R}})s))^2$$
$$+(-\frac{1}{\kappa^*}(v(1-va)(-a\kappa(\tau-\tau_{\mathbf{R}})s))^2$$
$$+(\frac{1}{\kappa^*}((1+v(-a))(\kappa+v(-2a\kappa-a\kappa' s))-v^2(-a\kappa s)(a\kappa^2 s))^2])ds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*}-H_B^*(1-va)^2=\frac{1}{2}(1-va)^2s+\frac{1}{2}a^2v^2\int_0^s\kappa^2s^2(1+\kappa^{*^2})ds$$
$$+\frac{1}{2}a^4v^4\int_0^s\frac{(\tau^*-\tau_{\mathbf{R}}^*)^2}{\kappa^{*^2}}\kappa^4s^4(\tau-\tau_{\mathbf{R}})^2ds+\frac{1}{2}a^2v^2(1-va)^2\int_0^s\frac{(\tau^*-\tau_{\mathbf{R}}^*)^2}{\kappa^{*^2}}(\kappa^2(\tau-\tau_{\mathbf{R}})^2s^2)ds$$
$$+\frac{1}{2}\int_0^s\frac{(\tau^*-\tau_{\mathbf{R}}^*)^2}{\kappa^{*^2}}((1-va)(\kappa+av(2\kappa-\kappa's))+v^2a^2\kappa^3s^2)^2])ds.$$

Corollary 3.8 *If relatively moving inertial system K^* moves parallel to normal of the unit speed curve $\gamma(s)$ with uniform speed v, then we have*

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*}=\frac{1}{2}\int_0^s((1+vb\kappa s)^2+(vb)^2+(bv(\tau-\tau_{\mathbf{R}})s)^2)ds$$
$$+\frac{1}{2}\int_0^s(\kappa^{*^2}[(1+vb\kappa s)^2+(v(-b))^2+(v(-b(\tau-\tau_{\mathbf{R}})s))^2]$$
$$+(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2(-b)(-2b(\tau-\tau_{\mathbf{R}})-b(\tau-\tau_{\mathbf{R}})'s)$$
$$+v(b(\tau-\tau_{\mathbf{R}})s)(\kappa+v(b\kappa^2s+b(\tau-\tau_{\mathbf{R}})s))))^2$$
$$+(-\frac{1}{\kappa^*}(v(1+vb\kappa s)(-2b(\tau-\tau_{\mathbf{R}})-b(\tau-\tau_{\mathbf{R}})'s)$$
$$+v^2(b(\tau-\tau_{\mathbf{R}})s)(b\kappa's+2b\kappa)))^2+(\frac{1}{\kappa^*}((1+v(b\kappa s))(\kappa$$
$$+v(b\kappa^2s+b(\tau-\tau_{\mathbf{R}})^2s))-v^2(-b)(b\kappa's+2b\kappa))^2])ds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*}-H_B^*(1+b^2v^2)=\frac{1}{2}(1+v^2b^2)s+\frac{1}{2}b^2v^2\int_0^s(\kappa^{*^2}+1)(\kappa^2+(\tau-\tau_{\mathbf{R}})^2)s^2ds$$
$$+bv\int_0^s(\kappa^{*^2}+1)\kappa s ds+\frac{1}{2}\int_0^s(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2(-b)(-2b(\tau-\tau_{\mathbf{R}})-b(\tau-\tau_{\mathbf{R}})'s)$$
$$+v(b(\tau-\tau_{\mathbf{R}})s)(\kappa+v(b\kappa^2s+b(\tau-\tau_{\mathbf{R}})^2s))))^2+(-\frac{1}{\kappa^*}(v(1+vb\kappa s)(-2b(\tau-\tau_{\mathbf{R}})$$
$$-b(\tau-\tau_{\mathbf{R}})'s)+v^2(b(\tau-\tau_{\mathbf{R}})s)(b\kappa's+2b\kappa)))^2$$
$$+(\frac{1}{\kappa^*}((1+v(b\kappa s))(\kappa+v(b\kappa^2s+b(\tau-\tau_{\mathbf{R}})^2s))-v^2(-b)(b\kappa's+2b\kappa))^2])ds.$$

Corollary 3.9 *If relatively moving inertial system K^* moves parallel to binormal of the unit speed curve $\gamma(s)$ with uniform speed v, then we have*

$$\varepsilon nergy(\mathbf{e}_{(1)}^{\mu^*})=\frac{1}{2}\int_0^s(1+(vc(\tau-\tau_{\mathbf{R}})s)^2+(cv)^2)ds$$
$$+\frac{1}{2}\int_0^s(\kappa^{*^2}[1+(vc(\tau-\tau_{\mathbf{R}})s)^2+(vc)^2]$$
$$+(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2c^2(\tau-\tau_{\mathbf{R}})^3s^2+vc(\kappa+v(c(\tau-\tau_{\mathbf{R}})'s$$

$$+2c(\tau-\tau_{\mathbf{R}}))))^2+(-\frac{1}{\kappa^*}(vc(\tau-\tau_{\mathbf{R}})^2 s$$
$$-v^2c^2\kappa(\tau-\tau_{\mathbf{R}})s))^2+(\frac{1}{\kappa^*}(\kappa+v(c(\tau-\tau_{\mathbf{R}})' s$$
$$+2c(\tau-\tau_{\mathbf{R}})))+v^2(c^2(\tau-\tau_{\mathbf{R}})^2 s^2\kappa))^2])ds,$$

or equivalently

$$\varepsilon nergy\mathbf{e}_{(1)}^{\mu^*}-H_B^*(1+c^2v^2)=\frac{1}{2}s(1+c^2v^2)+\frac{1}{2}c^2v^2\int_0^s(\tau-\tau_{\mathbf{R}})^2 s^2(1+\kappa^{*^2})ds$$
$$+\frac{1}{2}\int_0^s(\tau^*-\tau_{\mathbf{R}}^*)^2[(\frac{1}{\kappa^*}(v^2c^2(\tau-\tau_{\mathbf{R}})^3 s^2+vc(\kappa+v(c(\tau-\tau_{\mathbf{R}})' s+2c(\tau-\tau_{\mathbf{R}})))))^2$$
$$+(-\frac{1}{\kappa^*}(vc(\tau-\tau_{\mathbf{R}})^2 s-v^2c^2\kappa(\tau-\tau_{\mathbf{R}})s))^2$$
$$+(\frac{1}{\kappa^*}(\kappa+v(c(\tau-\tau_{\mathbf{R}})' s+2c(\tau-\tau_{\mathbf{R}})))+v^2(c^2(\tau-\tau_{\mathbf{R}})^2 s^2\kappa))^2])ds.$$

We have following diagram as a demonstration to observe changes on the energy of normal vector field of the observed curve with respect to the motion of relative system and time. Here $\varepsilon nergy\mathbf{e}_{(1)_T}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(1)_N}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(1)_B}^{\mu^*}$ represent energy of the normal vector field of the observed curve Γ^* in the relative system, which acting parallel to tangent, normal, and binormal of the unit speed curve Γ, respectively.

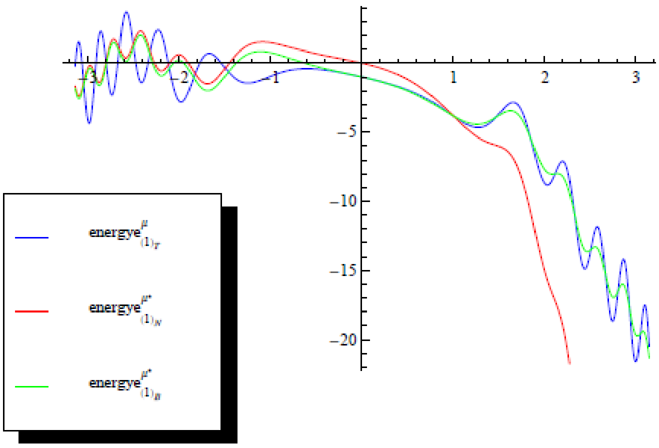

Figure 4. Energy variation graph for normal vector field of an observed curve.

Theorem 3.10 *Relativistic energy of binormal vector of the observed curve* γ^* *in the moving relative system* K^* *is stated by*

$$\varepsilon nergy\mathbf{e}_{(2)}^{\mu^*}=\frac{1}{2}\int_0^s((1+v(b\kappa s-a))^2+(v(-a\kappa+c(\tau-\tau_{\mathbf{R}}))s-vb)^2$$
$$+(-bv(\tau-\tau_{\mathbf{R}})s-cv)^2)ds+\frac{1}{2}\int_0^s\frac{(\tau^*-\tau_{\mathbf{R}}^*)^2}{\kappa^{*^2}}[(v(b\kappa' s+2b\kappa$$
$$+a\kappa^2 s-c\kappa(\tau-\tau_{\mathbf{R}})s))^2+(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s$$

$$+c(\tau-\tau_{\mathbf{R}})'s+2c(\tau-\tau_{\mathbf{R}})+b(\tau-\tau_{\mathbf{R}})^2 s))^2$$
$$+(v(c(\tau-\tau_{\mathbf{R}})^2 s-a\kappa\tau s-2b(\tau-\tau_{\mathbf{R}})-b(\tau-\tau_{\mathbf{R}})' s))^2]ds.$$

where κ^*,τ^* are defined as in Theorem 3.1.

Proof. By the Frenet Equation defined in (2.1), we have

$$\mathbf{e}^{\mu^*}_{(2)}=-(\tau^*-\tau^*_{\mathbf{R}})\mathbf{e}^{\mu^*}_{(1)}.$$

The rest of the proof can be done similarly as proof of Theorem 3.2.

We have following graph, which illustrates a relation between on the variation of the energy of binormal vector field of the curve Γ in the rest system K and the energy of binormal vector field of the observed curve Γ^* in the moving relative system K^* with respect to time.

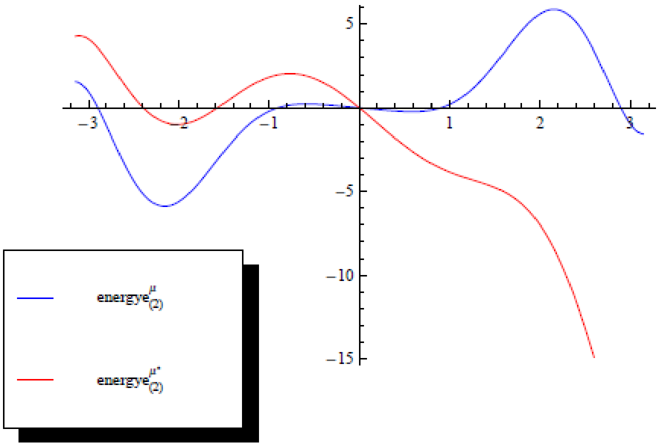

Figure 5. Energy variation graph for binormal vector field of a given curve.

Corollary 3.11 *If relatively moving inertial system* K^* *moves parallel to tangent of the unit speed curve* $\gamma(s)$ *with uniform speed* v, *then we have*

$$\varepsilon nergy\mathbf{e}^{\mu^*}_{(2)}=\frac{1}{2}(1-av)^2 s+\frac{1}{2}a^2v^2\int_0^s\kappa^2 s^2 ds+\frac{1}{2}\int_0^s\frac{(\tau^*-\tau^*_{\mathbf{R}})^2}{\kappa^{*2}}[(va\kappa^2 s)^2$$
$$+(\kappa+v(-2a\kappa-a\kappa' s))^2+(va\kappa(\tau-\tau_{\mathbf{R}})s)^2]ds.$$

Corollary 3.12 *If relatively moving inertial system* K^* *moves parallel to normal of the unit speed curve* $\gamma(s)$ *with uniform speed* v, *then we have*

$$\varepsilon nergy\mathbf{e}^{\mu^*}_{(2)}=\frac{1}{2}(1+v^2b^2)s+\frac{1}{2}b^2v^2\int_0^s(\kappa^2+(\tau-\tau_{\mathbf{R}})^2)s^2 ds$$
$$+bv\int_0^s\kappa s ds+\frac{1}{2}\int_0^s\frac{(\tau^*-\tau^*_{\mathbf{R}})^2}{\kappa^{*2}}[(v(b\kappa' s+2b\kappa))^2$$
$$+(\kappa+v(b\kappa^2 s+b(\tau-\tau_{\mathbf{R}})^2 s))^2$$

$$+\left(v\left(-2b\left(\tau-\tau_{\mathbf{R}}\right)-b\left(\tau-\tau_{\mathbf{R}}\right)' s\right)\right)^2]ds.$$

Corollary 3.13 *If relatively moving inertial system K^* moves parallel to binormal of the unit speed curve $\gamma(s)$ with uniform speed v, then we have*

$$\varepsilon nergy\mathbf{e}_{(2)}^{\mu^*} = \frac{1}{2}\left(1+c^2v^2\right)s + \frac{1}{2}c^2v^2\int_0^s\left(\tau-\tau_{\mathbf{R}}\right)^2 s^2 ds$$
$$+\frac{1}{2}\int_0^s \frac{\left(\tau^*-\tau_{\mathbf{R}}^*\right)^2}{\kappa^{*2}}[(v(-c\kappa(\tau-\tau_{\mathbf{R}})s))^2+(\kappa+v(c(\tau-\tau_{\mathbf{R}})' s$$
$$+2c(\tau-\tau_{\mathbf{R}})))^2+(v(c(\tau-\tau_{\mathbf{R}})^2 s))^2]ds.$$

We have following diagram as a demonstration to observe changes on the energy of binormal vector field of the observed curve with respect to motion of relative system and time. Here $\varepsilon nergy\mathbf{e}_{(2)_T}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(2)_N}^{\mu^*}, \varepsilon nergy\mathbf{e}_{(2)_B}^{\mu^*}$ represent energy of the normal vector of the observed curve Γ^* in the relative system, which acting parallel to tangent, normal, and binormal of the unit speed curve Γ, respectively.

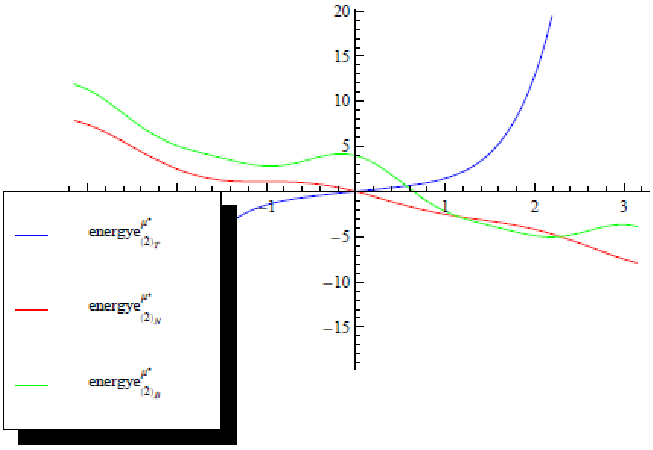

Figure 6. Energy variation graph for binormal vector field of an observed curve.

4. GEOMETRIC AND PHYSICAL CONSEQUENCES

As is widely known, one of the most significant results of Einstein's understandings of gravity on his general relativity theory is that the possibility of time and space may cause edges or holes. This is known as spacetime singularities. Thus, it is still a discussion at the geometric and physical research to investigate the implications and effects of spacetime singularities for our universe.

As can be seen in Theorem 2.4 energy of the moving particle does not have any singularity.

Also, it can be emphasized that the relativistic energy of tangent vector of the observed curve γ^* has no singularity at anywhere.

The energy of moving particle γ in the first inertial frame has no singularity at any point but this is not valid for the observed moving particle γ^* in the relative reference frame. Since relativistic energy of normal and binormal vector of the observed curve has a singularity when $\kappa^* = 0$ or $u = 0.$ We have

$$bv(\kappa' s + 2\kappa) + vs(a\kappa^2 + \kappa c(\tau - \tau_{\mathbf{R}})) = 0,$$
$$av(-2\kappa - \kappa' s) + cv((\tau - \tau_{\mathbf{R}})' s + 2(\tau - \tau_{\mathbf{R}}) + bvs(\kappa^2 + (\tau - \tau_{\mathbf{R}})^2) + \kappa = 0,$$
$$bv(-2(\tau - \tau_{\mathbf{R}}) - (\tau - \tau_{\mathbf{R}})' s) + (\tau - \tau_{\mathbf{R}})vs(c(\tau - \tau_{\mathbf{R}}) - a\kappa) = 0,$$

or equivalently

$$\begin{aligned}
0 &= v^2(-a\kappa s + c(\tau - \tau_{\mathbf{R}})s - b)(c(\tau - \tau_{\mathbf{R}})^2 s - a\kappa(\tau - \tau_{\mathbf{R}})s \\
&- 2b(\tau - \tau_{\mathbf{R}}) - b(\tau - \tau_{\mathbf{R}})' s) + v(b(\tau - \tau_{\mathbf{R}})s + c)(\kappa \\
&+ v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau - \tau_{\mathbf{R}})' s + 2c(\tau - \tau_{\mathbf{R}}) + b(\tau - \tau_{\mathbf{R}})^2 s)) \\
0 &= -v(1 + vb\kappa s - va)(c(\tau - \tau_{\mathbf{R}})^2 s - a\kappa(\tau - \tau_{\mathbf{R}})s - 2b(\tau - \tau_{\mathbf{R}}) \\
&- b(\tau - \tau_{\mathbf{R}})' s) + v^2(b(\tau - \tau_{\mathbf{R}})s + c)(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa(\tau - \tau_{\mathbf{R}})s) \\
0 &= ((1 + v(b\kappa s - a))(\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s + c(\tau - \tau_{\mathbf{R}})' s \\
&+ 2c(\tau - \tau_{\mathbf{R}}) + b(\tau - \tau_{\mathbf{R}})^2 s)) - v^2(-a\kappa s + c(\tau - \tau_{\mathbf{R}})s - b)(b\kappa' s \\
&+ 2b\kappa + a\kappa^2 s - c\kappa(\tau - \tau_{\mathbf{R}})s)
\end{aligned}$$

We also see that as the velocity of the relative system K^* converges to zero relativistic energy of the moving observed particle in the relative system converges to the energy of the particle in the inertial system.

For the future studies, it can also be investigated whether the choice of increasing velocity of the relative system K^* increase the relativistic energy of the moving observed particle or not.

REFERENCES

[1] Synge, J.L., *Relativity: The Special Theory*. Interscience Publishers, New York, 1965.
[2] Einstein, A., *Annalen der Physik*, **17**, 891, 1905.
[3] Einstein, A., *Relativity.The Special and General Theory*. New York. Henry Holt, 1920.
[4] Körpinar, T., Demirkol, R.C., *J.Adv.Phys.*, **6**, 562, 2017.
[5] Miller, P.F., *Galilean Transformation,*VDM Publishing, 2010.
[6] Kolev, B., *J. Nonlinear Math. Phys*., **11**, 480, 2004.
[7] Neumaier, A., Westra, D., *Classical and quantum mechanics via Lie Algebras*, arXiv: 0810.1019v2, 2011.
[8] Wood, C.M., *Geom. Dedic*., **64**, 319, 1997.
[9] Gil Medrano, O., *Differential Geometry and its Applications*, **15**, 137, 2001.
[10] Chacon, P.M., Naveira, A.M., Weston, J.M., *Monatsh. Math*., **133**, 281, 2001.
[11] Chacon, P.M., Naveira, A.M., *Osaka J. Math.,* **41**, 97, 2004.
[12] Altin, A., *Ukrainian Mathematical J.,* **63**(6), 969, 2011.
[13] Körpinar, T., *Int J Phys*., **53**, 3208, 2014.

[14] Kumar, S., Srivastav, S.K., *American Journal of Mathematics and Mathematical Sciences*, **1**(2), 101, 2012.
[15] Milne, J.S., *Algebraic Groups, Lie Groups, and their Arithmetic Subgroups,* www.jmilne.org/math/.
[16] Ciftci, U., *J. Geom. Phys*., **59**, 1597, 2009.
[17] Crouch, P., Silva, L.F., *J. Dynam. Control Systems*, **1**(2), 177, 1995.
[18] Milnor, J., *Advances in Mathematics*, **21**, 293, 1976.
[19] do Esprito-Santo, N., Fornari, S., Frensel, K., Ripoll, J., *Manuscripta Math*., **111**(4), 459, 2003.
[20] Capovilla, R., Chryssomalakos, C., Guven, J., *J. Phys. A. Math. Gen*., **35**, 6571, 2002.
[21] Carmeli, M., *Phys. Rev. B*., **138**, 1003, 1965.
[22] Weber, J., *Relativity and Gravitation*, Interscience, New York, 1961.
[23] Guven, J, Valencia, D.M., Vazquez-Montejo, J., *Phys. A: Math Theory*, **47**, 355201, 2014.