ORIGINAL PAPER

PADOVAN AND PELL-PADOVAN QUATERNIONS

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Abstract. *In this paper, we define Padovan and Pell-Padovan quaternions. We give Binet-like formulas, generating functions and sums formulas. Moreover we give the matrix representation of Padovan and Pell-Padovan quaternions.*

Keywords: *Padovan numbers, Pell-Padovan numbers, Padovan quaternions, Pell-Padovan quaternions.*
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1. INTRODUCTION

A quaternion is defined by

$$q = a + ib + jc + kd,$$

where a,b,c and d are real numbers and i, j, k are the standart orthonormal basis in R^3. Let $q_1 = a_1 + ib_1 + jc_1 + kd_1$ and $q_2 = a_2 + ib_2 + jc_2 + kd_2$ be any two quaternions. Then addition, equality and multiplication by scalar of two quaternions are defined by

$$q_1 + q_2 = (a_1 + a_2) + i(b_1 + b_2) + j(c_1 + c) + k(d_1 + d_2),$$

$$q_1 = q_2 \ \ only\ if\ \ a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$$

and for α ϵ R

$$\alpha q_1 = \alpha a_1 + i\alpha b_1 + j\alpha c_1 + k\alpha d_1$$

We note that the quaternion multiplication is defined using the rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

The conjugate and norm of a quaternion are defined by

$$q^* = a - ib - jc - kd$$

and

$$N(q) = a^2 + b^2 + c^2 + d^2.$$

The Padovan sequence is the sequence of integers P_n defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

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$$P_n = P_{n-2} + P_{n-3}$$

for all $n \geq 3$. The first few values of P_n are 1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,…

Pell-Padovan sequence is defined by the initial values $R_0 = R_1 = R_2 = 1$ and the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3} \; for \; all \; n \geq 3.$$

The first few values of Pell-Padovan numbers are

1,1,1,3,3,7,9,17,25,43,67,111,177,289,…

In 1963[4], Horadam defined n th Fibonacci and Lucas quaternions. Moreover some properties of Fibonacci and Lucas quaternions can be found [2, 3, 5]. Nurkan and Gven in [6] introduced dual Fibonacci quaternions and dual Lucas quaternions. Further interesting results of Pell quaternions, Pell-Lucas and Jacobsthal Quaternions can be found [1, 7, 8]. Tasci D and Yalcin N.F. studied [9] Fibonacci p-quaternions.

In this paper we define and study the Padovan quaternions and Pell-Padovan quaternions. Moreover we give their properties also using matrix representation.

2. PADOVAN QUATERNIONS

Firstly we give the definition of Padovan quaternions.

Definition 2.1. The Padovan quaternions are defined by

$$QP_n = P_n + i\, P_{n+1} + j\, P_{n+2} + k\, P_{n+3,}$$

where P_n is the n th Padovan number.

Theorem 2.2. For $n \geq 0$, the Binet-like Formula for the Padovan quaternions is

$$QP_n = a\alpha r_1^n + b\beta r_2^n + c\gamma r_3^n,$$

where $r_1, r_2 and\; r_3$ are the root of the equation $x^3 - x - 1 = 0$, and

$$a = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, b = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, c = \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)},$$
$$\alpha = 1 + ir_1 + jr_1^2 + kr_1^3, \beta = 1 + ir_2 + jr_2^2 + kr_2^3, \gamma = 1 + ir_3 + jr_3^2 + kr_3^3.$$

Proof. Consider the Binet like Formula of Padovan sequence is

$$P_n = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)} r_1^n + \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)} r_2^n + \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)} r_3^n$$

and

$$QP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$$

Then the prof is easily seen.

The following theorem is related with the generating function of the Padovan quaternions.

Theorem 2.3. The generating function of the Padovan quaternions is

$$g(x) = \frac{(1+i+j+2k)+(1+i+2j+2k)x+(i+j+k)x^2}{1-x^2-x^3}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} QP_n\, x^n = QP_0 + QP_1 x + QP_2 x^2 + \cdots + QP_n x^n + \cdots$$

be generating function of the Padovan quaternions. On the other hand, since

$$x^2 g(x) = QP_0 x^2 + QP_1 x^3 + QP_2 x^4 + \cdots + QP_{n-2} x^n + \cdots$$

and

$$x^3 g(x) = QP_0 x^3 + QP_1 x^4 + QP_2 x^5 + \cdots + QP_{n-3} x^n + \cdots$$

we write

$$(1 - x^2 - x^3)g(x) = QP_0 + QP_1 x + (QP_2 - QP_0)x^2 + (QP_3 - QP_1 - QP_0)x^3$$

$$+ \cdots + (QP_n - QP_{n-2} - QP_{n-3})x^n + \cdots$$

Now using $QP_0 = 1 + i + j + 2k, QP_1 = 1 + i + 2j + 2k, QP_2 = 1 + 2i + 2j + 3k$ and $QP_n - QP_{n-2} - QP_{n-3} = 0$, we obtain

$$g(x) = \frac{(1+i+j+2k)+(1+i+2j+2k)x+(i+j+k)x^2}{1-x^2-x^3}.$$

So the proof is complete.

Theorem 2.4.

$$\sum_{m=0}^{n} QP_m = QP_{n+2} + QP_{n+3} - (2 + 3i + 4j + 5k).$$

Proof. (By induction on n) If $n = 0$ and $n = 1$ then the result is obviously true. We assume that it is true for $n \in Z^+$. Then we shall show that

$$\sum_{m=0}^{n+1} QP_m = QP_{n+3} + QP_{n+4} - (2 + 3i + 4j + 5k)$$

Indeed we have

$$\sum_{m=0}^{n+1} QP_m = \sum_{m=0}^{n} QP_m + QP_{n+1}.$$

Using induction's hypothesis we obtain

$$\sum_{m=0}^{n+1} QP_m = QP_{n+2} + QP_{n+3} - (2 + 3i + 4j + 5k) + QP_{n+1}$$

Other hand, by the Definition 2.1, since

$$QP_{n+2} + QP_{n+1} = QP_{n+4}$$

we have

$$\sum_{m=0}^{n+1} QP_m = QP_{n+3} + QP_{n+4} - (2 + 3i + 4j + 5k).$$

Theorem 2.5.

$i)$ $$\sum_{m=0}^{n} QP_{2m-1} = QP_{2n+2} - (1 + i + j + 2k)$$

$ii)$ $$\sum_{m=0}^{n} QP_{2m} = QP_{2n+3} - (1 + i + j + 2k)$$

$iii)$ $$\sum_{m=0}^{n} QP_{2m+1} = QP_{2n+4} - (1 + i + j + 2k).$$

Proof. The theorem is proved by induction n.

Now we give the matrix representation of Padovan quaternions.

Theorem 2.6 Let for $n \geq 1$ be integer. Then

$$\begin{bmatrix} QP_{n+2} & QP_{n+1} & QP_n \\ QP_{n+1} & QP_n & QP_{n-1} \\ QP_n & QP_{n-1} & QP_{n-2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QP_2 & QP_1 & QP_0 \\ QP_1 & QP_0 & QP_{-1} \\ QP_0 & QP_{-1} & QP_{-2} \end{bmatrix}.$$

Proof. The proof is seen by induction on n.

Theorem 2.7. Let for $n \geq 1$ be integer. Then

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QP_0 \\ QP_1 \\ QP_2 \end{bmatrix} = \begin{bmatrix} QP_n \\ QP_{n+1} \\ QP_{n+2} \end{bmatrix}.$$

3. PELL-PADOVAN QUATERNIONS

Definition 3.1. The n th Pell-Padovan quaternion is defined by

$$QR_n = R_n + i\,R_{n+1} + j\,R_{n+2} + k\,R_{n+3},$$

where R_n is the n th Pell-Padovan number.

Theorem 3.2. The generating function for Pell-Padovan quaternions is

$$f(x) = \frac{(1+i+j+3k)+(1+i+3j+3k)x+(-1+i+j+k)x^2}{1-2x^2-x^3}.$$

Proof. Let

$$f(x) = \sum_{n=0}^{\infty} QR_n\, x^n = QR_0 + QR_1 x + QR_2 x^2 + \cdots + QR_n x^n + \cdots$$

be generating function of the Pell-Padovan quaternions. On the other hand, since

$$2x^2 f(x) = 2QR_0 x^2 + 2QR_1 x^3 + 2QR_2 x^4 + \cdots + 2QR_{n-2} x^n + \cdots$$

and

$$x^3 f(x) = QR_0 x^3 + QR_1 x^4 + QR_2 x^5 + \cdots + QR_{n-3} x^n + \cdots$$

we write

$$\begin{aligned}(1-2x^2-x^3)f(x) &= QR_0 + QR_1 x + (QR_2 - 2QR_0)x^2 + (QR_3 - 2QR_1 - QR_0)x^3 \\ &\quad + \cdots + (QR_n - 2QR_{n-2} - QR_{n-3})x^n + \cdots\end{aligned}$$

Now using $QR_n = 2QR_{n-2} + QR_{n-3}$, $n \geq 3$ we get

$$f(x) = \frac{QR_0 + QR_1 x + (QR_2 - 2QR_0)x^2}{1-2x^2-x^3}.$$

or

$$f(x) = \frac{(1+i+j+3k)+(1+i+3j+3k)x+(-1+i+j+k)x^2}{1-2x^2-x^3}.$$

So the proof is complete.

Theorem 3.3. The Binet- like formula for Pell-Padovan quaternions is

$$\begin{aligned}QR_n &= \frac{2}{\sqrt{5}}[(\alpha-1)(1+i\alpha+j\alpha^2+k\alpha^3)]\alpha^n \\ &\quad - \frac{2}{\sqrt{5}}[(\beta-1)(1+i\beta+j\beta^2+k\beta^3)]\beta^n + (-1+i+j+k)\gamma^n\end{aligned}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1+\sqrt{5}}{2} \; and \; \gamma = -1$$

are the roots of equation $x^3 - 2x - 1 = 0$.

Proof. Using

$$R_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 2\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + \gamma^{n+1}$$

and

$$QR_n = R_n + i\,R_{n+1} + j\,R_{n+2} + k\,R_{n+3},$$

it is easily seen the proof.

Theorem 3.4.

$$\sum_{m=0}^{n} QR_m = \frac{1}{2}[QR_n + QR_{n+1} + QR_{n+2} - (1 + 3i + 5j + 7k)].$$

Proof. (By induction on n)) For $n = 0$ and $n = 1$ are true. Now we assume that it is true for $n \in Z^+$. Then we shall show that it is true for $n + 1$,

$$\sum_{m=0}^{n+1} QR_m = \sum_{m=0}^{n} QR_m + QR_{n+1}$$
$$= \frac{1}{2}[QR_n + QR_{n+1} + QR_{n+2} + 2QR_{n+1} - (1 + 3i + 5j + 7k)],$$

On the other hand since

$$QR_{n+3} = 2QR_{n+1} + QR_n$$

we have

$$\sum_{m=0}^{n+1} QR_m = \frac{1}{2}[QR_{n+1} + QR_{n+2} + QR_{n+3} - (1 + 3i + 5j + 7k)].$$

So the theorem is proved.

Lemma 3.5.

$i)\ \sum_{m=0}^{n} R_{2m} = R_{2n+1} - n$

$ii)\ \sum_{m=0}^{n} R_{2m-1} = R_{2n} + (n - 1)$

$iii)\ \sum_{m=0}^{n} R_{2m+1} = R_{2n+1} + R_{2n} + (n - 1)$

$iv)\ \sum_{m=0}^{n} R_{2m+2} = 2R_{2n+1} + R_{2n} + (n - 2)$

$v)\ \sum_{m=0}^{n} R_{2m+3} = 3R_{2n+1} + 2R_{2n} + (n - 2)$

$vi)\ \sum_{m=0}^{n} R_{2m+4} = 5R_{2n+1} + 3R_{2n} + (n - 5).$

Proof. The proof of i) ,ii), iii), iv), v) and vi) are seen by induction on n.
Now using above the lemma 3.5, we give the following theorems.

Theorem 3.6.

$$\sum_{m=0}^{n} QR_{2m} = (R_{2n+1} - n) + i\big(R_{2n+1} + R_{2n} + (n-1)\big)$$

$$+j\left(2R_{2n+1} + R_{2n} + (n-2)\right) + k\big(3R_{2n+1} + 2R_{2n} + (n-2)\big)$$

Theorem 3.7.

$$\sum_{m=0}^{n} QR_{2m+1} = \big(R_{2n+1} + R_{2n} + (n-1)\big) + i\left(2R_{2n+1} + R_{2n} + (n-2)\right)$$

$$+j\big(3R_{2n+1} + 2R_{2n} + (n-2)\big) + k\big(5R_{2n+1} + 3R_{2n} + (n-5)\big).$$

Now we give the matrix representation of Pell-Padovan quaternions.

Theorem 3.8 Let for $n \geq 1$ be integer. Then

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QR_2 & QR_1 & QR_0 \\ QR_1 & QR_0 & QR_{-1} \\ QR_0 & QR_{-1} & QR_{-2} \end{bmatrix} = \begin{bmatrix} QR_{n+2} & QR_{n+1} & QR_n \\ QR_{n+1} & QR_n & QR_{n-1} \\ QR_n & QR_{n-1} & QR_{n-2} \end{bmatrix}$$

Proof. The proof is seen by induction on n.

Theorem 3.9. Let $n \geq 1$ be integer. In this case

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QR_2 \\ QR_1 \\ QR_0 \end{bmatrix} = \begin{bmatrix} QR_{n+2} \\ QR_{n+1} \\ QR_n \end{bmatrix}.$$

Proof. The proof of theorem is seen by induction on n.

Conflict of Interests: *The authors declare that there is no conflict of interests regarding the publication of this paper.*

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