ORIGINAL PAPER

BOUNDS FOR THE LAPLACIAN EIGENVALUE OF GRAPHS USING 2-ADJACENCY

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Abstract. *Let G be a simple connected graph. We redefine adjacency for any two vertices as defined if there is connected two edges between any two vertices, i and j. This vertices are called two adjacency and denoted by $i \sim_2 j$. In this study, we defined 2-adjacency Laplacian matrix and denoted by $L^{\sim_2}(G)$ from by above definition and we find a lower bound for 2-spectral radius which is denoted by $\mu^{\sim_2}(G)$ and present an open problem $L^{\sim_2}(G)$.*

Keywords: *2-adjacency, energy, bound.*

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and the edge set $E = E(G)$. We assume that the vertices degree are ordered such that $d_1 \geq d_2 \geq \cdots \geq d_n$ where d_i is the degree of v_i for $i = 1,2, \dots$,n and the average of the degrees of the vertices adjacent to v_i is denoted by m_i. Let N_i be the neighbor set of the vertex $v_i \in V$. Let $D(G)$ be the diagonal matrix of vertex degrees of a graph G. Also, let $A(G)$ be the adjacency matrix of G and denoted by $A(G) = (a_{ij})$ be defined as the $n \times n$ matrix (a_{ij}), where

$$a_{ij} = \begin{cases} 1 & ; \quad v_i v_j \in E \\ 0 & ; \quad otherwise \end{cases}$$

The Laplacian matrix of a graph G is defined as $L(G) = (l_{ij})$, where

$$l_{ij} = \begin{cases} d_i & ; \quad i = j \\ -1 & ; \quad i \sim j \\ 0 & ; \quad otherwise \end{cases}$$

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The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Gersgorin's theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$. We assume that the Laplacian eigenvalues are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Especially, the largest eigenvalue of $L(G)$ is called Laplacian spectral radius of G, denoted by $\mu(G)$.

Since the 1980's many researchers have investigated upper bounds for $\mu(G)$, among the known upper bounds for $\mu(G)$ are the following:

1. Anderson and Morley's bound [1]:

$$\mu(G) \leq \max\{d_i + d_j : v_i v_j \in E\}. \quad (1)$$

2. Li and Zhang's bound which is better than bound [8]:

$$\mu(G) \leq 2 + \sqrt{(r-2)(s-2)} \quad (2)$$

where $r = \max\{d_i + d_j : v_i v_j \in E\}$ and $s = \max\{d_i + d_j : v_i v_j \in E - \{v_k v_h\}\}$ with $v_k v_h \in E(G)$ such that $d_k + d_h = r$.

3. Merris's bound [10]:

$$\mu(G) \leq \max\{d_i + m_i : v_i \in V(G)\} \quad (3)$$

where m_i is the average of the degrees of the vertices of G adjacent to v_i, which is called average 2-degree of vertex v_i.

2. MAIN RESULTS

Let G be a simple connected graph. We redefine adjacency for any two vertices i and j. And then we called 2-adjacency this vertices. If this vertices are 2-adjacency then denoted by $i \sim_2 j$.

In this study we defined 2-adjacency Laplacian matrix and denoted by $L^{\sim 2}(G)$ from by above definition and we find a lower bound for 2-spectral radius which is denoted by $\mu^{\sim 2}(G)$ and present an open problem $L^{\sim 2}(G)$. Then we expand to k-adjacency of two vertices and redefine of k-Laplacian matrix and obtain new bound of k-spectral Laplacian eigenvalue of a graph.

Definition 2.1 If there are two connected edges between vertices of the i and j in graph, then this vertices are called 2-adjacent and denoted by $i \sim_2 j$.

Definition 2.2 The 2-degree of any vertex i in graph is the number of 2-adjacency incident to the vertex i and the 2-degree of a vertex i is denoted by $d_i^{\sim 2}$.

We can define the k-adjacency by generalized 2-adjacency concept further.

Definition 2.3 If there are k connected edges between vertices of the i and j in graph, then this vertices are called k-adjacent and denoted by $i \sim_k j$.

Definition 2.4 The k-degree of any vertex i in graph is the number of k-adjacency incident to the vertex i and the k-degree of a vertex i is denoted by $d_i^{\sim k}$.

Definition 2.5 Given a graph $G = (V, E)$ with $|V| = n$ the 2-adjacency degree matrix $D^{\sim 2}(G)$ for G is a $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{cases} d_i^{\sim 2} & ; \quad i = j \\ 0 & ; \quad otherwise \end{cases}$$

where the 2-degree $d_i^{\sim 2}$ of a vertex i is the number of 2-adjacency incident to the vertex i.

Definition 2.6 Given a simple, connected graph G with n vertices, its 2-adjacency Laplacian matrix $L^{\sim 2}(G) = \left(l_{ij}^{\sim 2}\right)_{n \times n}$ defined as:

$$\left(l_{ij}^{\sim 2}\right) = \begin{cases} d_i^{\sim 2} & ; \quad i = j \\ -1 & ; \quad i \sim_2 j \\ 0 & ; \quad otherwise \end{cases}$$

where $d_i^{\sim 2}$ is 2-degree of the vertex i.

Because the 2-adjacency Laplacian matrix is real, symmetric matrix, its eigenvalues are real and we denoted by

$$0 = \mu_n^{\sim 2} \leq \cdots \leq \mu_2^{\sim 2} \leq \mu_1^{\sim 2}.$$

$\mu_1^{\sim 2}$ denote the largest eigenvalue of $L^{\sim 2}(G)$. Also, every row sum and column sum of $L^{\sim 2}$ is zero. There are equality

$$L^{\sim 2}(G) = D^{\sim 2}(G) - A^{\sim 2}(G)$$

for 2-adjacency Laplacian matrix.

Theorem 2.7 If G is a simple, connected graph and $\mu_1^{\sim 2}(G)$ is the Laplacian 2-spectral radius, then

$$\mu_1^{\sim 2}(G) > \max_{i\sim_2 j} \sqrt{(d_i^{\sim 2} - p_i)(d_j^{\sim 2} - p_j)} \tag{4}$$

where p_i, t_k are $p_i = \frac{\sum_{k\sim_2 i} t_k}{t_i}$, $t_k = \frac{\sum_{k\sim_2 j} d_j}{d_k^{\sim 2}}$ respectively.

Proof: Let $X = (x_1, x_2, .., x_n)^T$ be an eigenvector corresponding to the largest eigenvalue $\mu_1^{\sim 2}$ of $T^{-1}(G)\, L^{\sim 2}(G)T(G)$ where $T(G)$ is the block diagonal matrix $diag(t_1, t_2, .., t_n)$,$t_i = \frac{\sum_{i\sim_2 j} d_j}{d_i^{\sim 2}}$, $i = 1,2, \dots, n$. We can assume that x_i is the largest eigencomponent 1 and the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $x_k \leq 1, \forall k$.

We consider the matrix $T^{-1}(G)\, L^{\sim 2}(G)T(G)$. Now the $(i,j)^{th}$ element of $T^{-1}(G)\, L^{\sim 2}(G)T(G)$ is

$$\begin{cases} d_i^{\sim 2} & ; \quad i = j \\ -1 & ; \quad i \sim_2 j \\ 0 & ; \quad otherwise. \end{cases}$$

We have

$$\{ T^{-1}(G) L^{\sim 2}(G)T(G)\}X = \mu_1^{\sim 2}(G)X \tag{5}$$

From the i th equation of (5), we have

$$\mu_1^{\sim 2} x_i = d_i^{\sim 2} x_i - \sum_{k\sim_2 i} \{x_k\}$$

$$\mu_1^{\sim 2} > d_i^{\sim 2} - p_i \tag{6}$$

Similarly, from the j th equation of (5), we have

$$\mu_1^{\sim 2} x_j = d_j^{\sim 2} x_j - \sum_{k \sim_2 j} \{x_k\}$$

$$\mu_1^{\sim 2} > d_j^{\sim 2} - p_j. \tag{7}$$

From (6) and (7), we get

$$\mu_1^{\sim 2}(G) > \max_{i \sim_2 j} \sqrt{(d_i^{\sim 2} - p_i)(d_j^{\sim 2} - p_j)}.$$

Definition 2.8 Given a simple, connected graph G with n vertices, its 2-adjacency self-Laplacian matrix $L^{\approx}(G) = \left(l_{ij}^{\approx}\right)_{n \times n}$ defined as:

$$\left(l_{ij}^{\approx}\right) = \begin{cases} d_i^{\sim 2} & ; \quad i = j \\ -2 & ; \quad i \sim_2 j \\ 0 & ; \quad otherwise \end{cases}$$

where $d_i^{\sim 2}$ is 2-degree of the vertex i.

Because the 2-adjacency self-Laplacian matrix is real, symmetric matrix, its eigenvalues are real and we denoted by

$$0 = \mu_n^{\approx} \leq \cdots \leq \mu_2^{\approx} \leq \mu_1^{\approx}.$$

$\mu_1^{\approx}$ denote the largest eigenvalue of $L^{\approx}(G)$. Also, every row sum and column sum of $L^{\approx}$ is zero. There are equality

$$L^{\approx}(G) = D^{\sim 2}(G) - A^{\approx}(G)$$

for 2-adjacency self-Laplacian matrix.

Theorem 2.9 If G is a simple, connected graph and $\mu_1^{\approx}(G)$ is the 2-self Laplacian spectral radius, then

$$\mu_1^{\approx}(G) > \max_{i\sim_2 j}\sqrt{(d_i^{\sim 2} - 2p_i)(d_j^{\sim 2} - 2p_j)} \tag{8}$$

where p_i, t_k are $p_i = \frac{\sum_{k\sim_2 i} t_k}{t_i}$, $t_k = \frac{\sum_{k\sim_2 j} d_j}{d_k^{\sim 2}}$ respectively.

Proof: Let $X = (x_1, x_2, .., x_n)^T$ be an eigenvector corresponding to the largest eigenvalue $\mu_1^{\approx}$ of $T^{-1}(G)\, L^{\approx}(G)T(G)$ where $T(G)$ is the block diagonal matrix $diag(t_1, t_2, .., t_n)$,$t_i = \frac{\sum_{i\sim_2 j} d_j}{d_i^{\sim 2}}$, $i = 1,2, \ldots, n$. We can assume that x_i is the largest eigencomponent 1 and the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $x_k \leq 1$, $\forall k$.

We consider the matrix $T^{-1}(G)\, L^{\approx}(G)T(G)$. Now the $(i,j)^{th}$ element of $T^{-1}(G)\, L^{\approx}(G)T(G)$ is

$$\begin{cases} d_i^{\sim 2} & ; \quad i = j \\ -2\dfrac{t_j}{t_i} & ; \quad i \sim_2 j \\ 0 & ; \quad otherwise. \end{cases}$$

We have

$$\{T^{-1}(G)L^{\approx}(G)T(G)\}X = \mu_1^{\approx}(G)X \tag{9}$$

From the i th equation of (9), we have

$$\mu_1^{\approx} x_i = d_i^{\sim 2} x_i - 2\sum_{k\sim_2 i}\frac{t_k}{t_i}$$

$$\mu_1^{\approx} > d_i^{\sim 2} - 2p_i \tag{10}$$

Similarly, from the j th equation of (9), we have

$$\mu_1^{\approx} x_j = d_j^{\sim 2} x_j - 2\sum_{k\sim_2 j}\frac{t_k}{t_j}$$

$$\mu_1^{\approx} > d_j^{\sim 2} - 2p_j. \tag{11}$$

From (10) and (11), we get

$$\mu_1^{\approx}(G) > \max_{i\sim_2 j}\sqrt{(d_i^{\sim 2} - 2p_i)(d_j^{\sim 2} - 2p_j)}$$

Conjecture 2.10 Let G be a simple, connected graph. Then

$$\mu_1^{\sim 2}(G) \leq \mu_1^{\approx}(G). \tag{12}$$

When we generalize for k-adjacency to found bound, we obtain following result.

Conjecture 2.11 If G is a simple, connected graph and $\mu_1^{\sim k}(G)$ is the k-Laplacian spectral radius, then

$$\mu_1^{\sim k}(G) > \max_{i\sim_k j}\sqrt{(d_i^{\sim k} - kp_i)(d_j^{\sim k} - kp_j)} \tag{13}$$

where p_i, t_i are $p_i = \frac{\sum_{i\sim_k j} t_j}{t_i}$, $t_i = \frac{\sum_{i\sim_k j} d_j}{d_i^{\sim k}}$ respectively.

Specially, if we get $k = 1$, Conjecture 2.11 will turn to Merris's bound (4).

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