ORIGINAL PAPER

A NOVEL TECHNIQUE FOR SOLVING NONLINEAR WBK EQUATIONS OF FRACTIONAL ORDER

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Abstract. *In this paper Homotopy Analysis Method (HAM) is implemented to obtain analytical solutions of nonlinear Whitham-Broer-Kaup partial differential equations and Cauchy Reaction-diffusion problems of fractional order. Numerical results coupled with the graphical representations reveal the complete compatibility of proposed algorithm for such problems.*

Keywords: *Whitham-Broer-Kaup equations, Homotopy analysis method, Cauchy Reaction-diffusion problems, fractional calculus.*

1. INTRODUCTION

Differential equations arise in almost all areas of the applied and engineering sciences [1-18]. Several numerical and analytical techniques including Homotopy Analysis (HAM), Perturbation, Modified Adomian's Decomposition (MADM), finite difference, Spline, Variational iteration method (VIM) which was first proposed by He [20-21] have been developed to solve such problems, see [1-18] and the references therein. Recently, there many researchers have started working on a very special type of differential equations which is called fractional differential equations [12-18] and are extremely important in member of physical problems related to applied and engineering sciences.Most scientific problems such as Whitham-Broer-Kaup (WBk) equations are inherently of nonlinear.Except a limited number of such problems, most of them do not have analytical solution.In this article we apply a very efficient and reliable technique which is called Homotopy Analysis Method (HAM) [1-10, 15, 16] to obtain analytical solutions of nonlinear Whitham-Broer-Kaup equations and Cauchy Reaction-diffusion problems of fractional order.

2. PRELIMINARIES AND NOTATION

In this segment, we give some fundamental definitions and properties of the fractional calculus theory which will be used additional in this work. For the finite derivative in $[a,b]$ we define the following fractional integral and derivatives.

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Definition 1. A real function $f(x), x>0,$ is said to be in the space $C\mu, \mu \in R,$ If there exists a real number $(p>\mu)$ such that $f(x)=x^p f_1(x),$ where $f_1(x)=C(0,\infty)$ and it is said to be in the space $C_\mu^m \mu$ if $f^m \in C\mu, m \in N.$

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C\mu, \mu \geq -1,$ is defined as

$$J^\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha>0, x>0, J^0(x)=f(x).$$

Some properties of the operator J^a are discussed in the following
For $f \in C\mu, \mu \geq -1,, \alpha, \beta \geq 0$ and $\gamma \geq -1$

$$J^\alpha J^\beta f(x)=J^{\alpha+B} f(x),$$
$$J^\alpha J^B f(x)=J^B J^\alpha f(x),$$
$$J^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}.$$

The Riemann--Liouville derivative has convinced disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator suggested by M. Caputo in his struggle on the theory of viscoelasticity [2].

Definition 3. For m to be the smallest integer that exceeds, α the Caputo time fractional derivative operator of order $\alpha>0$ and defined as

$$D_t^\alpha f(x)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(m-a)}\int_0^x (x-t)^{m-\alpha-1} f(t)dt, -1<<m, m\in N \\ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \alpha=m\end{cases}$$

3. HOMOTOPY ANALYSIS METHOD (HAM) [1-10, 15, 16]

We consider the following equation

$$\widetilde{N}[u(\tau)]=0, \tag{1}$$

where $\widetilde{N}$ is a nonlinear operator, τ denotes dependent variables and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in the similar way. By means of HAM Liao [6-10] constructed zero-order deformation equation

$$(1-p)\mathcal{L}[\emptyset(\tau;p)-u_0(\tau)]=p\hbar\widetilde{N}[\emptyset(\tau;p)], \tag{2}$$

where $\mathcal{L}$ is a linear operator, $u_0(\tau)$ is an initial guess. $\hbar \neq 0$ is an auxiliary parameter and $p \in [0,1]$ is the embedding parameter. It is obvious that when p=0 and 1, it holds

$$\mathcal{L}[\emptyset(\tau;0) - u_0(\tau)] = 0 \quad \Longrightarrow \quad \emptyset(\tau;0) = u_0(\tau), \tag{3}$$

$$\hbar\widetilde{N}[\emptyset(\tau;1)] \; = 0 \; \Longrightarrow \; \emptyset(\tau;1) = \; u(\tau), \tag{4}$$

respectively. The solution $\emptyset(\tau;p)$ varies from initial guess $u_0(\tau)$ to solution $u(\tau)$. Liao [18] expanded $\emptyset(\tau;p)$ in Taylor series about the embedding parameter

$$\emptyset(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m, \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \emptyset(\tau;p)}{\partial p^m}\Big|_{p=0} \tag{6}$$

The convergence of (5) depends on the auxiliary parameter $\hbar$. If this series is convergent at p=1, one has

$$\emptyset(\tau;1) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

Define vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), u_3(\tau), \ldots\ldots\ldots, u_n(\tau)\},$$

If we differentiate the zeroth-order deformation equation Eq. (2) m-times with respect to p and then divide them $m!$ and finally set $p = 0$, we obtain the following mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - X_m u_{m-1}(\tau)] = \hbar\Re_m(\vec{u}_{m-1}), \tag{8}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\widetilde{N}[\emptyset(\tau;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{9}$$

and

$$X_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \tag{10}$$

If we multiply with $\mathcal{L}^{-1}$ each side of Eq. (8), we will obtain the following mth order deformation

$$u_m(\tau) = X_m u_{m-1}(\tau) + \hbar\,\Re_m\,(\vec{u}_{m-1})$$

4. NUMERICAL APPLICATIONS

In this section, we apply Homotopy Analysis Method (HAM) to solve fractional order nonlinear Whitham-Broer-Kaup (WBk) and Cauchy Reaction-diffusion problems. Numerical results are very encouraging.

Problem: 1

Consider the Whitham-Broer-Kaup (WBk) Eqs.

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} + uu_x + \beta u_{xx} = 0, \\ \dfrac{\partial^\alpha v}{\partial t^\alpha} + (uv)_x + \alpha_1 u_{xxx} - \beta v_{xx} = 0. \end{cases} \qquad 0 < \alpha \le 1,$$

where $\alpha,\ \beta$ are constants, with the initial conditions

$$u(x,0) = \omega - 2Bkcoth(k\xi), \qquad v(x,0) = -2B(B+\beta)k^2 csc\text{?}h^2(k\xi),$$

where $B = \sqrt{\alpha_1 + \beta^2}$ and $\xi = x + x_0$ and $\alpha_1, \beta, x_0, k, \omega$ are arbitrary constants.

Now we apply the HAM to solve fractional order WBK Eqs. The solutions of $u(x,t)$ and $v(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2, \dots \dots \dots . .\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n\,, \qquad v(x,t) = \sum_{n=o}^{+\infty} b_n t^n,$$

where a_n and b_n are coefficients. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = \omega - 2Bkcoth(k\xi)\,, \quad v_0(x,t) = -2B(B+\beta)k^2 csc\text{?}h^2(k\xi),$$

As the initial approximations of $u(x,t)$ and$v(x,t)$, to choose the auxiliary linear operator

$$L[\emptyset(t;q)] = \frac{\partial^\alpha \emptyset(t;q)}{\partial t^\alpha},$$

with the property $L[C] = 0$, where C is integral constant. Furthermore, we define a system of nonlinear operators as

$$N_1[\emptyset(t;q)] = \frac{\partial^\alpha \emptyset_1(t;q)}{\partial t^\alpha} + \emptyset_1(t;q)\big(\emptyset_1(t;q)\big)_x + \big(\emptyset_2(t;q)\big)_x + \beta\big(\emptyset_1(t;q)\big)_{xx},$$

$$N_2[\emptyset(t;q)] = \frac{\partial^\alpha \emptyset_2(t;q)}{\partial t^\alpha} + (\emptyset_1(t;q)\emptyset_2(t;q))_x + \alpha_1\big(\emptyset_1(t;q)\big)_{xxx} - \beta\big(\emptyset_2(t;q)\big)_{xx},$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\emptyset(t;q) - u_0(x,t)] = q\hbar N_1[\emptyset(t;q)],$$

$$(1-q)\mathcal{L}[\emptyset(t;q) - v_0(x,t)] = q\hbar N_2[\emptyset(t;q)],$$

Obviously, when q=0 and q=1,

$$\emptyset_1(t;0) = u_0(x,t), \quad \emptyset_1(t;1) = u(x,t),$$
$$\emptyset_2(t;0) = v_0(x,t), \quad \emptyset_2(t;1) = v(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset_1(t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m,$$
$$\emptyset_2(t;q) = v_0(x,t) + \sum_{m=1}^{+\infty} v_m(x,t)q^m,$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \emptyset_1(t;q)}{\partial q^m}|_{q=0},$$
$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \emptyset_2(t;q)}{\partial q^m}|_{q=0},$$

Define the vectors

$$u_n = \{u_0(x,t), u_1(x,t), \dots \dots \dots, u_n(x,t)\},$$
$$v_n = \{v_0(x,t), v_1(x,t), \dots \dots \dots, v_n(x,t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!

We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$
$$L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar R_m(\bar{v}_{m-1}),$$

Subject to initial conditions

$$u_0(x,t) = \omega - 2Bkcoth(k\xi)\,, \quad v_0(x,t) = -2B(B+\beta)k^2 csch^2(k\xi),$$
$$R_m(\bar{u}_{m-1}) = u_{m-1}^{\alpha} + u_{m-1}(u_{m-1})_x + \beta(u_{m-1})_{xx},$$
$$R_m(\bar{v}_{m-1}) = v_{m-1}^{\alpha} + (u_{m-1}v_{m-1})_x + \alpha_1(u_{m-1})_{xxx} - \beta(v_{m-1})_{xx},$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar j_t^{\alpha}[R_m(\bar{u}_{m-1})],$$
$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar j_t^{\alpha}[R_m(\bar{v}_{m-1})].$$

We start with initial *approximations* $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$ and can obtain $u_1(x,t)$ and $v_1(x,t)$ as follows:

$$u_1(x,t) = \hbar\Big[2Bk^2 cschh^2\big(k(x+x_0)\big)\Big(\omega - 2Bkcoth\big(k(x+x_0)\big)\Big) + 4B(B+\beta)k^3 cschh^2\big(k(x+x_0)\big)coth(k\xi) - 4B\beta k^3 cschh^2\big(k(x+x_0)\big)coth\big(k(x+x_0)\big)\Big]\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$v_1(x,t) = \hbar\Big[4\omega B(B+\beta)k^3 cschh^2\big(k(x+x_0)\big)coth\big(k(x+x_0)\big) - 4B^2(B+\beta)k^4 cschh^4\big(k(x+x_0)\big) + 2cschh^2\big(k(x+x_0)\big)coth^2\big(k(x+x_0)\big) + 4\alpha_1 Bk^4\Big(cschh^4\big(k(x+x_0)\big) + 2cschh^2\big(k(x+x_0)\big)coth^2\big(k(x+x_0)\big)\Big) + 4B\beta(B+\beta)k^4\Big(cschh^4\big(k(x+x_0)\big) + 2cschh^2\big(k(x+x_0)\big)coth^2\big(k(x+x_0)\big)\Big)\Big]\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$\vdots$$

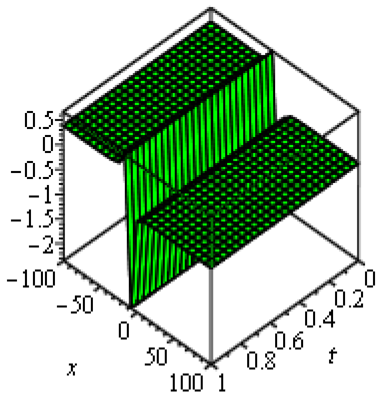
Then the solution expression can be written in the form

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n, \quad v(x,t) = \sum_{n=o}^{+\infty} b_n t^n,$$

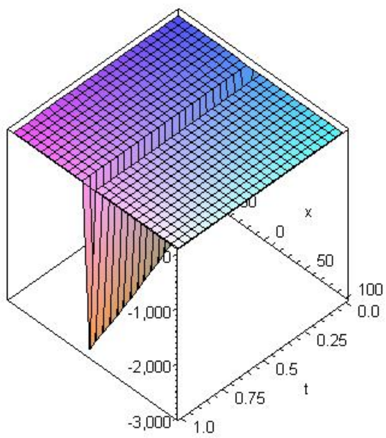
$u(x,t) = \omega - 2Bkcoth\big(k(x+x_0)\big) + \hbar\Big[2Bk^2 cschh^2\big(k(x+x_0)\big)\Big(\omega -$
2Bkcothkx+x0+4BB+βk3cschh2kx+x0cothkx+x0−4Bβk3cschh2kx+x0cothkx+x0tαΓα+1,

$v(x,t) =$
$-2B(B+\beta)k^2 cschh^2(k(x+x_0)) + \hbar\Big[4\omega B(B+\beta)k^3 cschh^2\big(k(x+x_0)\big)coth\big(k(x+$
x0−4B2B+βk4cschh4kx+x0+2cschh2kx+x0coth2kx+x0+4α1Bk4cschh4kx+x0+2cschh2kx+x0coth2kx+x0+4BβB+βk4cschh4kx+x0+2cschh2kx+x0coth2kx+x0tαΓα+1.

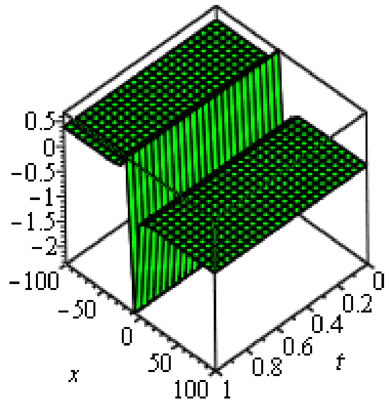
For $\hbar = -1$ graphical presentation of $u(x,t)$and $v(x,t)$ for different values of α is given below.



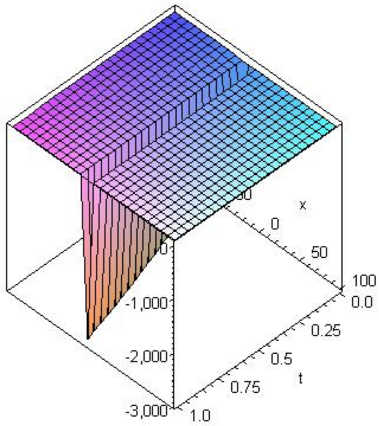
$u(x,t), \alpha = 0.25$



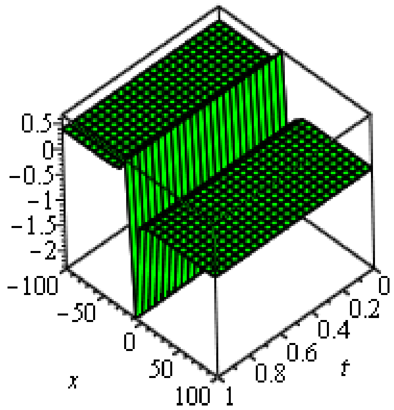
$v(x,t), \alpha = 0.25$



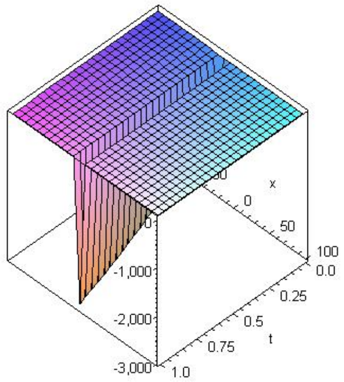
$u(x,t), \alpha = 0.5$



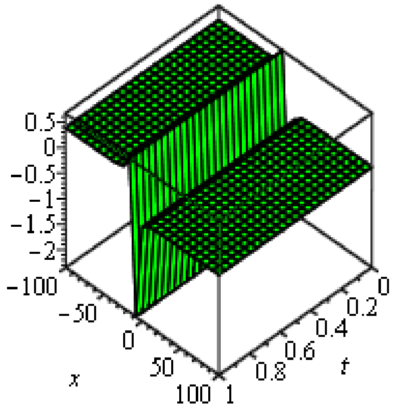
$v(x,t), \alpha = 0.5$



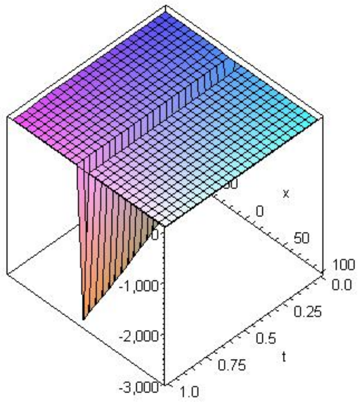
$u(x,t), \alpha = 0.75$



$v(x,t), \alpha = .75$



$u(x,t), \alpha = 1$



$v(x,t), \alpha = 1$

Problem 2: We consider the one-dimensional Cauchy Reaction-diffusion problems of fractional order

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + p(x,t)u(x,t), \quad (x,t) \in \Omega \subset R^2.$$

where u is the concentration, p is the reaction parameter and $D > 0$ is the diffusion coefficient, are subject to the initial and boundary conditions

$$u(x,0) = g(x), \quad x \in R.$$
$$u(0,t) = f_0(t), \quad \frac{\partial u}{\partial t}(0,t) = f_1(t), \quad t \in R.$$

when $p(x,t)$ =const, $p(x,t) = -1$. In this case, we put the problem in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) - u(x,t), \quad (x,t) \in \Omega \subset R^2.$$

With the initial and boundary conditions

$u(x,0) = e^x + x, \quad x \in R.$
$u(0,t) = 1$ and
$\frac{\partial u}{\partial t}(0,t) = e^{-t} - 1, \quad t \in R.$

Solution: We solve the problem when $D = 1$. Now we apply the HAM to solve Cauchy Reaction-diffusion Problem. The solution $u(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2, \dots \dots \dots..\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n \ ,$$

where a_n is coefficient. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = e^x + x \,,$$

As the initial approximations of $u(x,t)$ to choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha},$$

with the property

$$L[C] = 0,$$

where C is a integral constant. Furthermore, we define a system of nonlinear operators as

$$N[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi}{\partial x^2} + \phi(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)],$$

Obviously, when q=0 and q=1,

$$\phi(t;0) = u_0(x,t), \quad \phi(t;1) = u(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1,2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m,$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{u}_n = \{u_0(t), u_1(t), \dots \dots \dots, u_n(t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!,

We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$

Subject to initial condition $u_0(x,0) = 0$,

$$R_m(\bar{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - \frac{\partial^2 u_{m-1}}{\partial x^2} + u_{m-1},$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar j_t^\alpha[R_m(\bar{u}_{m-1})].$$

We now successfully obtain

$$u_1 = \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2\left[x\frac{t^\alpha}{\Gamma(\alpha+1)} + x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$u_3 = \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2\left[x\frac{t^\alpha}{\Gamma(\alpha+1)} + x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + (\hbar^2 x + \hbar^3 x)\frac{t^\alpha}{\Gamma(\alpha+1)} + (\hbar^2 x +$$

2ℏ3xt2αΓ2α+1

$$+\hbar^3 x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

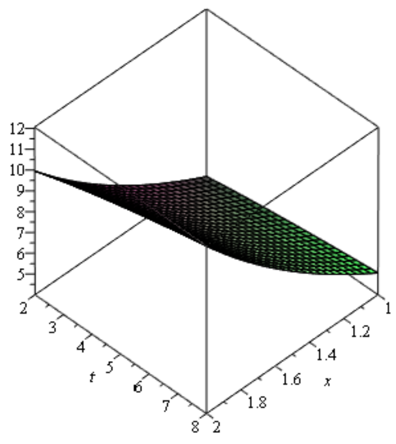
$$\vdots$$

Then the solution expression can be written in the form

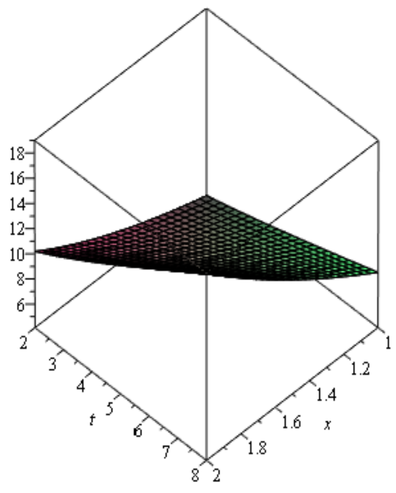
$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n \ ,$$

$$u(x,t) = e^x + x + \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 \left[x \frac{t^\alpha}{\Gamma(\alpha+1)} + x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] + \hbar x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 \left[x \frac{t^\alpha}{\Gamma(\alpha+1)} + \ x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] + (\hbar^2 x + \hbar^3 x) \frac{t^\alpha}{\Gamma(\alpha+1)} + (\hbar^2 x + 2\hbar^3 x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \hbar^3 x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots.$$

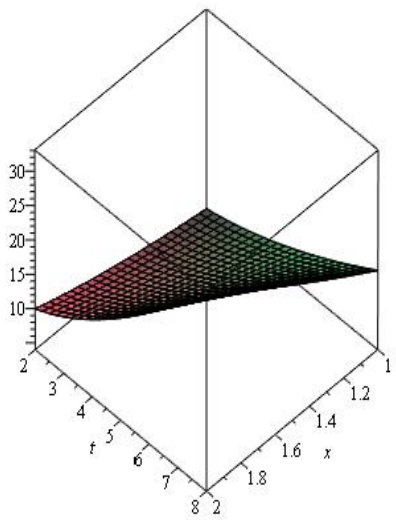
For $\hbar = -1$ graphical presentation of $u(x,t)$ for different values of α is given below.



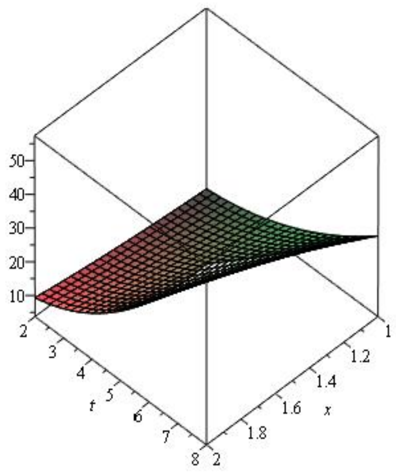
For α=0.25



For α=0.50



For α=0.75



For α=1

Problem 3: Consider the Cauchy Reaction-diffusion problem when $(x,t) = p(x)$, $p(x) = -1 - 4x^2$. In this case, we put the problem in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = D \frac{\partial^2 u}{\partial x^2}(x,t) + (-1 - 4x^2) u(x,t), \ \ (x,t) \in \Omega \subset R^2.$$

with the initial and boundary conditions

$$u(x,0) = e^{x^2}, \qquad x \in R.$$

$u(0,t) = e^t$ and
$\frac{\partial u}{\partial t}(0,t) = 0, \quad t \in R.$

Solution: Now we apply the HAM to solve Cauchy Reaction-diffusion Problem. The solution $u(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2,\dots\dots\dots..\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n \ ,$$

where a_n is coefficient. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = e^{x^2} \ ,$$

As the initial approximations of $u(x,t)$ to choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha},$$

with the property

$$L[C] = 0,$$

where C is a integral constant. Furthermore, we define a system of nonlinear operators as

$$N[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi}{\partial x^2} + \phi(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)],$$

Obviously, when q=0 and q=1,

$$\phi(t;0) = u_0(x,t), \quad \phi(t;1) = u(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1,2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m,$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{u}_n = \{u_0(t), u_1(t), \dots \dots \dots ., u_n(t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!,
We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$

Subject to initial condition $u_0(x,0) = 0$,

$$R_m(\bar{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - \frac{\partial^2 u_{m-1}}{\partial x^2} - (-1-4x^2)u_{m-1},$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h j_t^\alpha [R_m(\bar{u}_{m-1})].$$

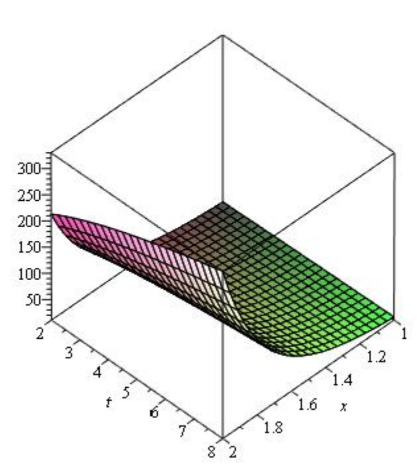
We now successfully obtain

$$u_1 = -\hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)},$$
$$u_2 = -\hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_3 = -\hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar^2 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^2 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^3 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{x^2} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$\vdots$$

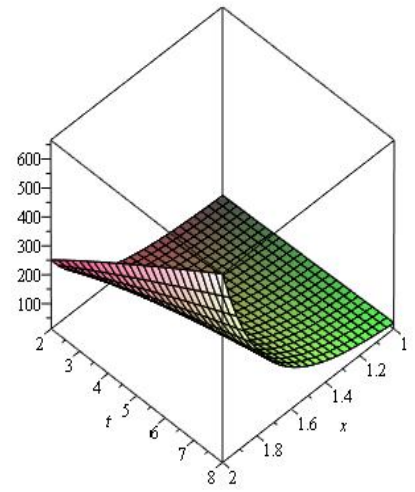
Then the solution expression can be written in the form

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n ,$$
$$u(x,t) = e^{x^2} - \hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} - \hbar^2 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar^2 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^2 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{x^2} \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^3 e^{x^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{x^2} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots.$$

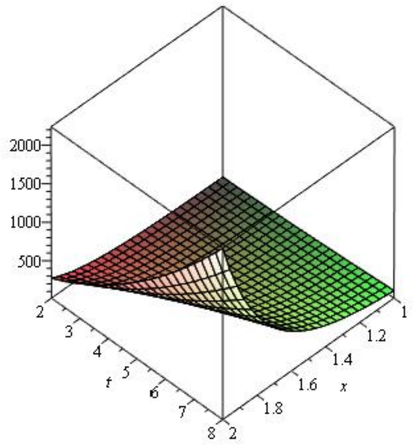
For $\hbar = -1$ graphical presentation of $u(x,t)$ for different values of α is given below.



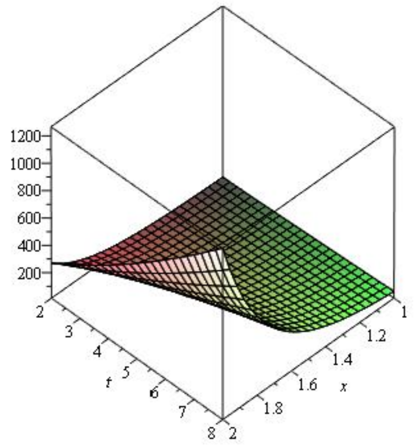
For α=0.25



For α=0.50



For α=1



For α=0.75

Problem 4: Consider the Cauchy Reaction-diffusion Problem when $p(x,t) = p(t)$ and $p(t) = 2t$. In this case, we put the problem in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) + 2tu(x,t), \;\; (x,t) \in \Omega \subset R^2.$$

with the initial and boundary conditions

$$u(x,0) = e^x, \quad x \in R.$$
$$u(0,t) = e^{(t+t^2)} \text{ and}$$
$$\frac{\partial u}{\partial t}(0,t) = e^{(t+t^2)}, \quad t \in R.$$

Solution: Now we apply the HAM to solve Cauchy Reaction-diffusion problem. The solution $u(x,t)$ can be expressed by a set of base functions.

$$\{t^n | n = 0,1,2, \dots \dots \dots . .\},$$

In the following forms

$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n \ ,$$

where a_n is coefficient. This provides us with the first rule of solution expression. Under the rule of solution expression and according to initial condition, it is straightforward to choose

$$u_0(x,t) = e^x \ ,$$

As the initial approximations of $u(x,t)$ to choose the auxiliary linear operator

$$L[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha},$$

with the property

$$L[C] = 0,$$

where C is a integral constant. Furthermore, we define a system of nonlinear operators as

$$N[\phi(t;q)] = \frac{\partial^\alpha \phi(t;q)}{\partial t^\alpha} - \frac{\partial^2 \phi}{\partial x^2} + \phi(t;q),$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)],$$

Obviously, when q=0 and q=1,

$$\phi(t;0) = u_0(x,t), \quad \phi(t;1) = u(x,t),$$

Therefore as the embedding parameter q increases from 0 to 1, The solution $\emptyset(t;q)$ varies from the initial guess to the solution for i=1,2. Expanding $\emptyset(t;q)$ in Taylor series with respect to q, one has:

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m,$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m}|_{q=0}$$

Define the vector

$$\bar{u}_n = \{u_0(t), u_1(t), \dots \dots \dots ., u_n(t)\}.$$

Differentiating the zero-order deformation equation m-times with respect to q, and finally dividing by m!.

We gain the mth order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\bar{u}_{m-1}),$$

Subject to initial condition $u_0(x,0) = 0$,

$$R_m(\bar{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - \frac{\partial^2 u_{m-1}}{\partial x^2} - 2tu_{m-1},$$

Now the solution of the mth-order deformation equation for m ≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar j_t^\alpha[R_m(\bar{u}_{m-1})].$$

We now successfully obtain

$$u_1 = -\hbar e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$u_2 = -\hbar e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar^2 e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar^2 e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar^2 e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 4\hbar^2 e^x \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)},$$

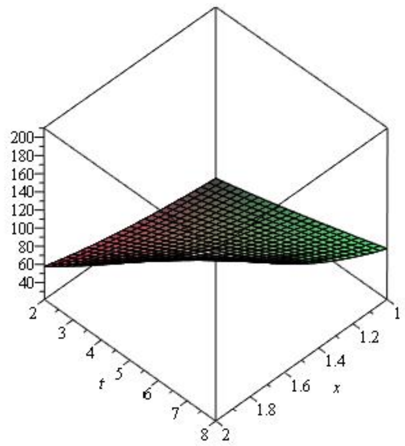
⋮

Then the solution expression can be written in the form

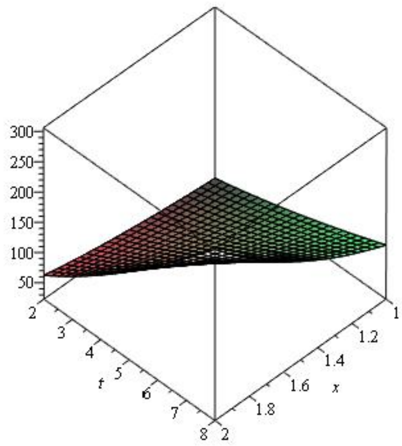
$$u(x,t) = \sum_{n=o}^{+\infty} a_n t^n ,$$

$$u(x,t) = e^x - \hbar e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar^2 e^x \frac{t^\alpha}{\Gamma(\alpha+1)} - 2\hbar^2 e^x \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \hbar^2 e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 4\hbar^2 e^x \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \cdots.$$

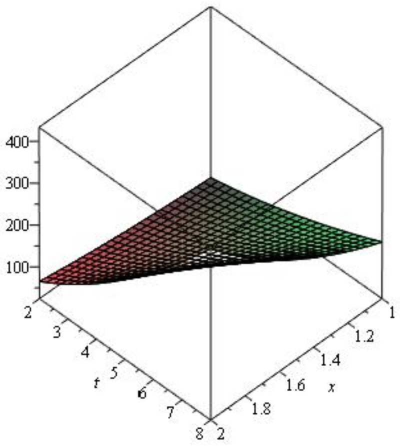
For $\hbar = -1$ graphical presentation of $u(x,t)$ for different values of α is given below.



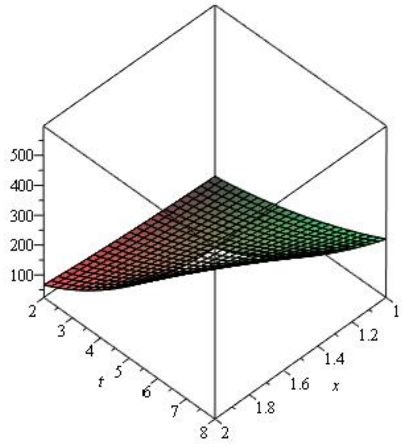
For α=0.25



For α=0.50



For α=0.75



For α=1

5. CONCLUSION

In this study, we used homotopy Analysis Method (HAM) to obtain analytical solutions of analytical solutions of nonlinear Whitham-Broer-Kaup partial differential equations and Cauchy Reaction-diffusion problems of fractional order. Numerical results coupled with the graphical representations reveal the complete compatibility of proposed algorithm for such problems. It is also observed that both the results obtained by proposed algorithms are in complete agreement with each other.

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