ORIGINAL PAPER

ON D-PENCIL SURFACE BY USING DARBOUX FRAME IN MINKOWSKI 3-SPACE

MUHAMMED T. SARIAYDIN[1], TALAT KORPINAR[2]

Manuscript received: 27.01.2019; Accepted paper: 26.03.2019;
Published online: 30.06.2019.

Abstract. *In this paper, the* D *- Pencil surface is studied in Minkowski 3-space. By using Darboux Frame in Minkowski space, we give the necessary and sufficient condition for a pencil surface.* Then, *we obtain general conditions of each other different* G_1, G_2, G_3 *developable ruled surface with the line of curvature of the pencil surface. Finally, we construct the corresponding surfaces which possessing some representative curves as lines of curvature.*

Keywords: *The line of curvature, Darboux frame, Pencil Surface.*

1. INTRODUCTION

In geometry, Ruled surfaces have already been widely used in designing cars, ships, production of products and several additional areas for example movement analysis and simulation of a rigid body and model-based object recognition systems. Contemporary surface area modeling systems consist of ruled surfaces [1-4]. The geometry of ruled surfaces is vital for studying kinematical and positional mechanisms in Euclidean 3-space. This surface area is often found in the scientific study from days gone by. For example, Odehnal explored subdivision algorithms ruled surfaces and Chen defined the mu-basis for a rational ruled surface [5-20].

A developable surface may be formed by bending or rolling a planar surface without stretching out or tearing; in additional terms, it can become created or unrolled isometrically onto a plane. Developable surfaces are also known as singly curved surfaces since one of their principal curvatures is usually zero. Developable surfaces are broadly utilized with components that are not really responsive to extending. Applications consist of the development of ship hulls, ducts, shoes, clothing, and car parts such as upholstery, body windshields and panels [6].

In this paper, we study the new parametric representation of a the D-pencil surface by using Darboux frame in Minkowski 3-space. The first, we tersely summarize properties Darboux frame and Frenet frame which are parameterized by arc-length parameter s and the fundamental ideas on curves and surfaces. The second, we get general circumstances of one another different G_1, G_2 ve G_3 surface with the timelike line of curvature of the timelike pencil surface $P(s,t)$. Then, we similarly obtain general conditions with spacelike line of curvature of the timelike pencil surface and the spacelike line of curvature of the spacelike pencil surface. Furthermore, for a 3D parametric curve $\alpha(s)$, where s is the arc-length

[1] Selçuk University, Department of Mathematics, Konya, Turkey. E-mail: talatsariaydin@gmail.com.
[2] Muş Alparslan University, Department of Mathematics, Muş, Turkey. E-mail: talatkorpinar@gmail.com.

parameter, by using the Darboux frame, we deduce the necessary and sufficient condition for a surface pencil to contain $\alpha(s)$ as a line of curvature. Finally, some representative curves are chosen to construct the corresponding surfaces which possessing these curves as lines of curvature E_1^3.

2. MATERIALS AND METHODS

Let us consider Minkowski 3-space $E_1^3 = \{E_1^3, (+,+,-)\}$ and let the Lorentzian inner product of $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

A vector $X \in E_1^3$ is called a spacelike vector when $\langle X, X \rangle > 0$ or $X = 0$. It is called timelike and null vector in case of $\langle X, X \rangle < 0$ and $\langle X, X \rangle = 0$ for $X \neq 0$ respectively [21].

The vector product of vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in E_1^3 is defined by [22],

$$X \times Y = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2).$$

A surface in E_1^3 is called a timelike surface if the normal vector on the surface spacelike vector and is called spacelike surface if the normal vector on the timelike vector.

On the other hand, for given non-null and null curves fully lying on a surface in Minkowski 3-space E_1^3, let us begin to set up the Darboux equations for these curves according to Lorentzian casual characters of surfaces and the curves lying on it as follows:

i - If the surface W is an oriented spacelike surface, then the curve $\xi = \xi(s)$ lying on W is a spacelike curve. Thus, the equations which describe the Darboux frame of $\xi(s)$ is given by:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{g}(s) \\ \mathbf{n}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{g}(s) \\ \mathbf{n}(s) \end{pmatrix},$$

where a prime denotes differentiation with respect to s. For this frame the following is satisfied

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{g}, \mathbf{g} \rangle = -\langle \mathbf{n}, \mathbf{n} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{n} \rangle &= \langle \mathbf{n}, \mathbf{g} \rangle = -\langle \mathbf{g}, \mathbf{T} \rangle = 0. \end{aligned}$$

In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface W and $\mathbf{g}$ is a unit vector given by $\mathbf{g} = \mathbf{n} \times \mathbf{T}$

ii - Let W be an oriented timelike surface, then the curve $\xi = \xi(s)$ lying on W can be a spacelike or a timelike curve.

In case of $\xi(s)$ is a spacelike curve, the derivative formula of the Darboux frame of $\xi(s)$ takes the form:

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{g}(s)\\ \mathbf{n}(s)\end{pmatrix}=\begin{pmatrix}0 & \kappa_g & -\kappa_n\\ \kappa_g & 0 & \tau_g\\ \kappa_n & \tau_g & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{g}(s)\\ \mathbf{n}(s)\end{pmatrix},$$

where $\mathbf{T}$, $\mathbf{n}$, $\mathbf{g}$ satisfy the following properties:

$$\begin{aligned}\langle \mathbf{T},\mathbf{T}\rangle &= \langle \mathbf{n},\mathbf{n}\rangle = -\langle \mathbf{g},\mathbf{g}\rangle = 1,\\ \mathbf{T}\times\mathbf{g} &= -\mathbf{n},\ \mathbf{g}\times\mathbf{n} = -\mathbf{T},\ \mathbf{n}\times\mathbf{T} = \mathbf{g}.\end{aligned}$$

In case of $\xi(s)$ is a timelike curve, the derivative formula of the Darboux frame of $\xi(s)$ takes the form:

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{g}(s)\\ \mathbf{n}(s)\end{pmatrix}=\begin{pmatrix}0 & \kappa_g & \kappa_n\\ \kappa_g & 0 & -\tau_g\\ \kappa_n & \tau_g & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{g}(s)\\ \mathbf{n}(s)\end{pmatrix}.$$

where $\mathbf{T}$, $\mathbf{n}$, $\mathbf{g}$ satisfy the following properties:

$$\begin{aligned}-\langle \mathbf{T},\mathbf{T}\rangle &= \langle \mathbf{n},\mathbf{n}\rangle = \langle \mathbf{g},\mathbf{g}\rangle = 1,\\ \mathbf{T}\times\mathbf{n} &= -\mathbf{g},\ \mathbf{n}\times\mathbf{g} = \mathbf{T},\ \mathbf{g}\times\mathbf{T} = -\mathbf{n}.\end{aligned}$$

In (i), (ii), κ_g is geodesic curvature of the $\xi(s)$, κ_n and τ_g are respectively the normal curvature and the geodesic torsion of the curve $\xi(s)$ [23].

Definition 2.1. Let x and y be future pointing (or past pointing) timelike vectors in E_1^3. Then there is a unique real number $\phi \geq 0$ such that $\langle x, y\rangle_L = -|x||y| cos\phi$. This number is called the hyperbolic angle between the vectors *x* and *y* [21].

Definition 2.2. Let x and y be spacelike vectors in E_1^3. Then span a timelike vector subspace. Then there is a unique real number $\phi \geq 0$ such that $\langle x, y\rangle_L = |x||y| cosh\phi$ This number is called the central angle between the vectors *x* and *y* [21].

Definition 2.3. Let x and y be spacelike vectors in E_1^3. Then span a spacelike vector subspace. Then there is a unique real number $\phi \geq 0$ such that $\langle x, y\rangle_L = |x||y| cos\phi$ This number is called the spacelike angle between the vectors *x* and *y* [21].

Definition 2.4. Let x be a spacelike vector and y be timelike vector in E_1^3. Then span a spacelike vector subspace. Then there is a unique real number $\phi \geq 0$ such that

$\langle x,y\rangle_L = |x||y| sinh\phi$ This number is called the Lorentzian timelike angle between the vectors x and y [21].

A surface is regular if it admits a tangent plane at each point. The unit normal of the regular surface $W:(s,t)\to W(s,t)$ at each point is defined by

$$\mathbf{m}(s,t)=\frac{\mathsf{W}_s(s,t)\times\mathsf{W}_t(s,t)}{\|\mathsf{W}_s(s,t)\times\mathsf{W}_t(s,t)\|}, \tag{1}$$

where W_s and W_t are the partial derivatives of the parametric representation with respect to s and t , respectively.

Theorem 2.5. A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable surface [11].

3. D -PENCIL SURFACES BY USING DARBOUX FRAME

In this paper, our goal is to find the necessary and sufficient condition for the given curve $\xi(s)$ being a line of curvature on the pencil surface $\mathsf{W}(s,t)$. Given a spatial curve $\xi: s\to\xi(s)$, where s is the arc-length parameter. On the other hand, the parametric form of the surface $\mathsf{W}(s,t):[0,L]\times[0,T]\to\mathsf{E}_1^3$ possess $\xi(s)$ as a line of curvature is given by

$$\mathsf{W}(s,t)=\xi(s)+x(s,t)\mathbf{T}(s)+y(s,t)\mathbf{g}(s)+z(s,t)\mathbf{n}(s), \tag{2}$$

where $0\le s\le L, 0\le t\le T,$ $x(s,t)$, $y(s,t)$ and $z(s,t)$ are C^1 functions. The values of these functions indicate the extension-like, flexion-like, and retortion-like effects, by the point unit through the time t, starting from $\xi(s)$, respectively.

3.1. TIMELIKE GENERATOR CURVES ON TIMELIKE PENCIL SURFACE IN MINKOWSKI 3-SPACE E_1^3

Firstly, the normal $\mathbf{m}(s,t)$ can be computed by taking the cross product of the partial differentials, that is, based on the following derivation using the Darboux formula

$$\begin{aligned}\frac{\partial\mathsf{W}(s,t)}{\partial s} \quad = \quad & [1+y(s,t)\kappa_g(s)+z(s,t)\kappa_n(s)+\frac{\partial x(s,t)}{\partial s}]\mathbf{T}(s)\\ & +[x(s,t)\kappa_g(s)+z(s,t)\tau_g(s)+\frac{\partial y(s,t)}{\partial s}]\mathbf{g}(s)\\ & +[x(s,t)\kappa_n(s)-y(s,t)\tau_g(s)+\frac{\partial z(s,t)}{\partial s}]\mathbf{n}(s)\end{aligned}$$

and

$$\frac{\partial \mathrm{W}(s,t)}{\partial t}=\frac{\partial x(s,t)}{\partial t}\mathbf{T}(s)+\frac{\partial y(s,t)}{\partial t}\mathbf{g}(s)+\frac{\partial z(s,t)}{\partial t}\mathbf{n}(s).$$

The normal vector which occurred vector product of $\frac{\partial \mathrm{W}(s,t)}{\partial s}$ and $\frac{\partial \mathrm{W}(s,t)}{\partial t}$ can be expressed as

$$\mathbf{m}(s,t)=\eta_1(s,t)\mathbf{T}(s)+\eta_2(s,t)\mathbf{g}(s)+\eta_3(s,t)\mathbf{n}(s),$$

Where

$$\begin{aligned}\eta_1(s,t) \;=\; & -x(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t}-z(s,t)\tau_g(s)\frac{\partial z(s,t)}{\partial t}-\frac{\partial y(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t}\\ & +x(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t}-y(s,t)\tau_g(s)\frac{\partial y(s,t)}{\partial t}+\frac{\partial z(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t},\end{aligned}$$

$$\begin{aligned}\eta_2(s,t) \;=\; & -\frac{\partial z(s,t)}{\partial t}-y(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t}-z(s,t)\kappa_n(s)\frac{\partial z(s,t)}{\partial t}-\frac{\partial x(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t}\\ & +x(s,t)\kappa_n(s)\frac{\partial x(s,t)}{\partial t}-y(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t}+\frac{\partial z(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t},\end{aligned}$$

$$\begin{aligned}\eta_3(s,t) \;=\; & \frac{\partial y(s,t)}{\partial t}+y(s,t)\kappa_g(s)\frac{\partial y(s,t)}{\partial t}+z(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t}+\frac{\partial x(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t}\\ & -x(s,t)\kappa_g(s)\frac{\partial x(s,t)}{\partial t}-z(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t}-\frac{\partial y(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}.\end{aligned}$$

Following, we derive the necessary and sufficient condition for a surface pencil that possesses the given curve as a common line of curvature. Since the curve $\xi(s)$ is an isoparametric curve on the surface $\mathrm{W}(s,t)$. According to Theorem 2.5, the curve $\xi(s)$ is the line of curvature on the surface $\mathrm{W}(s,t)$ if and only if the normal $\mathbf{m}_1(s)$, $\mathbf{m}_2(s)$, $\mathbf{m}_3(s)$ to the curve and the normal $\mathbf{m}(s,t)$ to the surface $\mathrm{W}(s,t)$ are parallel to each other.

Case 1. Let

$$\mathrm{G}_1(s,t)=\xi(s)+t\mathbf{m}_1(s)\text{and}\mathbf{m}_1(s)=\mathbf{g}(s)\cos\gamma_1+\mathbf{n}(s)\sin\gamma_1$$

be a parametric surface possess $\xi(s)$ as a timelike generator curve in E_1^3. According to theorem 2.5, $\xi(s)$ is the line of curvature of the surface $\mathrm{W}(s,t)$ if and only if $\mathrm{G}_1(s,t)$ is developable and the normal vector $\mathbf{m}_1$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [24], the surface $\mathrm{G}_1(s,t)$ is developable if and only if $\left(\xi'(s),\mathbf{m}_1(s),\mathbf{m}_1'(s)\right)=0$. After simple computation, we have

$$\begin{aligned}\left(\xi'(s),\mathbf{m}_1(s),\mathbf{m}_1'(s)\right) &= 0\\ \Leftrightarrow\ & -\tau_g(s)\cos^2\gamma_1+\gamma_1'\cos_1^2\gamma_1-\tau_g(s)\sin_1^2\gamma_1+\gamma_1'\sin_1^2\gamma_1=0\\ \Leftrightarrow\ & \gamma_1'(s)=\tau_g(s).\end{aligned}$$

That is,

$$\gamma_1(s)=\int_{s_0}^{s}\tau_g(s)ds+\gamma_0(s_0),$$

where is s_0 the starting value of arc-length and $\gamma_0=\gamma(s_0)$. Hereafter, in this paper, we assume $s_0=0$. If $\mathbf{m}_1(s)$ is parallel to $\mathbf{m}(s)$, the curve $\xi(s)$ is the line of curvature of the surface $\mathsf{W}(s,t)$.

Corollary 3.1.1. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0,$$

$$\eta_1(s,t_0)=0,\eta_2(s,t_0)=\lambda_1(s)\cos\gamma_1,\eta_3(s,t_0)=\lambda_1(s)\sin\gamma_1,$$

where $0\le t_0\le T, 0\le s\le L, \lambda_1(s)\neq 0$.

Case 2. Let

$$\mathsf{G}_2(s,t)=\xi(s)+t\mathbf{m}_2(s)\text{and}\mathbf{m}_2(s)=\mathbf{T}(s)\cosh\gamma_2+\mathbf{g}(s)\sinh\gamma_2$$

be a parametric surface possess $\xi(s)$ as a timelike generator curve in E_1^3. According to theorem 2.5, $\xi(s)$ is the line of curvature of the surface $\mathsf{W}(s,t)$ if and only if $\mathsf{G}_2(s,t)$ is developable and the normal vector $\mathbf{m}_2$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [24], the surface $\mathsf{G}_2(s,t)$ is developable if and only if $\left(\xi'(s),\mathbf{m}_2(s),\mathbf{m}_2'(s)\right)=0$. After simple computation, we have

$$\begin{aligned}\left(\xi'(s),\mathbf{m}_2(s),\mathbf{m}_2'(s)\right) &= 0\\ \Leftrightarrow\ & \kappa_n\cosh\gamma_2\sinh\gamma_2-\tau_g(s)\sinh^2\gamma_2=0\end{aligned}$$

That is,

$$\gamma_2(s)=\operatorname{arctan}h\left(\frac{\kappa_n}{\tau_g}\right).$$

If $\mathbf{m}_2(s)$ is parallel to $\mathbf{m}(s)$, the curve $\xi(s)$ is the line of curvature of the surface $\mathsf{W}(s,t)$.

Corollary 3.1.2. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0,$$

$$\eta_1(s,t_0)=\lambda_2(s)\cosh\gamma_2,\ \eta_2(s,t_0)=\lambda_2(s)\sinh\gamma_2,\ \eta_3(s,t_0)=0,$$

where $0\le t_0\le T,\ 0\le s\le L,\ \lambda_2(s)\ne 0$.

Case 3. Let

$$\mathsf{G}_3(s,t)=\xi(s)+t\mathbf{m}_3(s)\text{and}\mathbf{m}_3(s)=\mathbf{T}(s)\cosh\gamma_3+\mathbf{n}(s)\sinh\gamma_3$$

be a parametric surface possess $\xi(s)$ as a timelike generator curve in E_1^3. According to theorem 2.5, $\xi(s)$ is the line of curvature of the surface $\mathsf{W}(s,t)$ if and only if $\mathsf{G}_3(s,t)$ is developable and the normal vector $\mathbf{m}_3$ is parallel to the normal vector $\mathbf{m}(s,t)$. However, from [24], the surface $\mathsf{G}_3(s,t)$ is developable if and only if $\left(\xi'(s),\mathbf{m}_3(s),\mathbf{m}_3'(s)\right)=0$. After simple computation, we have

$$\begin{aligned}\left(\xi'(s),\mathbf{m}_3(s),\mathbf{m}_3'(s)\right) &= 0\\ \Leftrightarrow\quad -\kappa_g\cosh\gamma_2\sinh\gamma_2-\tau_g(s)\sinh^2\gamma_2 &= 0\end{aligned}$$

That is,

$$\gamma_3(s)=\operatorname{arctan}h\left(-\frac{\kappa_g}{\tau_g}\right).$$

If $\mathbf{m}_3(s)$ is parallel to $\mathbf{m}(s)$, the curve $\xi(s)$ is the line of curvature of the surface $\mathsf{W}(s,t)$.

Corollary 3.1.3. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0,$$

$$\eta_1(s,t_0)=\lambda_3(s)\cosh\gamma_3,\ \eta_2(s,t_0)=0,\ \eta_3(s,t_0)=\lambda_3(s)\sinh\gamma_3,$$

where $0\le t_0\le T,\ 0\le s\le L,\ \lambda_3(s)\ne 0$.

Now, we study special cases of parametric representations of a timelike surface pencil. we also consider the case when the marching-scale functions $x(s,t)$, $y(s,t)$ and $z(s,t)$ can be decomposed into two factors:

$$x(s,t)=l(s)X(t),$$

$$y(s,t)=m(s)Y(t),$$

$$z(s,t)=n(s)Z(t),$$

where $0\leq t\leq T, 0\leq s\leq L$ and $l(s), m(s), n(s), X(t), Y(t)$ and $Z(t)$ are C^1 functions and $l(s), m(s)$ and $n(s)$ are not identically zero.

Thus, by using corollary 3.1.1, corollary 3.1.2 and corollary 3.1.3, we can get the following theorems, respectively.

Theorem 3.1.4. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$m(s)Y'(t_0) = \lambda_1(s)\sin\gamma_1,$$

$$-n(s)Z'(t_0) = \lambda_1(s)\cos\gamma_1,$$

where $0\leq t_0\leq T$, $\lambda_1(s)\neq 0$.

Theorem 3.1.5. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$Y'(t_0) = 0,$$

$$-n(s)Z'(t_0) = \lambda_2(s)\sinh\gamma_2,$$

where $0\leq t_0\leq T$, $\lambda_2(s)\neq 0$.

Theorem 3.1.6. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$m(s)Y'(t_0) = \lambda_3(s)\sinh\gamma_3,$$

$$Z'(t_0) = 0,$$

where $0\leq t_0\leq T$, $\lambda_3(s)\neq 0$.

3.2. SPACELIKE GENERATOR CURVES ON TIMELIKE PENCIL SURFACE IN MINKOWSKI 3-SPACE $\mathbb{E}_1^3$

Firstly, the normal $\mathbf{z}(s,t)$ of the surface $\mathsf{W}(s,t)$ can be computed by

$$\mathbf{z}(s,t)=\mu_1(s,t)\mathbf{T}(s)+\mu_2(s,t)\mathbf{g}(s)+\mu_3(s,t)\mathbf{n}(s),$$

where

$$\begin{aligned}\mu_1(s,t) \;=\; & -x(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t}-z(s,t)\tau_g(s)\frac{\partial z(s,t)}{\partial t}-\frac{\partial y(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t}\\ & -x(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t}+y(s,t)\tau_g(s)\frac{\partial y(s,t)}{\partial t}+\frac{\partial z(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t},\end{aligned}$$

$$\begin{aligned}\mu_2(s,t) \;=\; & -\frac{\partial z(s,t)}{\partial t}-y(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t}-z(s,t)\kappa_n(s)\frac{\partial z(s,t)}{\partial t}-\frac{\partial x(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t}\\ & -x(s,t)\kappa_n(s)\frac{\partial x(s,t)}{\partial t}+y(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t}+\frac{\partial z(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t},\end{aligned}$$

$$\begin{aligned}\mu_3(s,t) \;=\; & -\frac{\partial y(s,t)}{\partial t}-y(s,t)\kappa_g(s)\frac{\partial y(s,t)}{\partial t}-z(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t}-\frac{\partial x(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t}\\ & +x(s,t)\kappa_g(s)\frac{\partial x(s,t)}{\partial t}+z(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t}+\frac{\partial y(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}.\end{aligned}$$

Theorem 3.2.1. Let $\mathsf{F}_1(s,t)=\xi(s)+t\mathbf{c}_1(s)$ and $\mathbf{c}_1(s)=\mathbf{g}(s)\cosh\phi_1+\mathbf{n}(s)\sinh\phi_1$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in $\mathbb{E}_1^3$. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0,$$

$$\phi_1=-\int_{s_0}^{s}\tau_g(s)ds+\phi_0(s_0),$$

$$\mu_1(s,t_0)=0,\ \mu_2(s,t_0)=\omega_1(s)\cosh\phi_1,\ \mu_3(s,t_0)=\omega_1(s)\sinh\phi_1,$$

where $0\le t_0\le T,\ 0\le s\le L,\ \omega_1(s)\ne 0$.

Theorem 3.2.2. Let $\mathsf{F}_2(s,t)=\xi(s)+t\mathbf{c}_2(s)$ and $\mathbf{c}_2(s)=\mathbf{T}(s)\cosh\phi_2+\mathbf{g}(s)\sinh\phi_2$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in $\mathbb{E}_1^3$. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=0,$$

$$\phi_2 = \operatorname{arctan} h\left(\frac{\kappa_n}{\tau_g}\right),$$

$$\mu_1(s,t_0) = \omega_2(s)\cosh\phi_2, \mu_2(s,t_0) = \omega_2(s)\sinh\phi_2, \mu_3(s,t_0) = 0,$$

where $0 \le t_0 \le T, 0 \le s \le L, \omega_2(s) \ne 0$.

Theorem 3.2.3. Let $\mathsf{F}_3(s,t) = \xi(s) + t\mathbf{c}_3(s)$ and $\mathbf{c}_3(s) = \mathbf{T}(s)\cos\phi_3 + \mathbf{n}(s)\sin\phi_3$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in E_1^3. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0) = y(s,t_0) = z(s,t_0) = 0,$$

$$\phi_3 = \operatorname{arctan} h\left(-\frac{\kappa_g}{\tau_g}\right),$$

$$\mu_1(s,t_0) = \omega_3(s)\cos\phi_3, \mu_2(s,t_0) = 0, \mu_3(s,t_0) = \omega_3(s)\sin\phi_3,$$

where $0 \le t_0 \le T, 0 \le s \le L, \omega_3(s) \ne 0$.

By using theorem 3.2.1, theorem 3.2.2 and theorem 3.2.3, we study special cases of parametric representations of a timelike pencil surface. Then, we can get the following corollarys, respectively.

Corollary 3.2.4. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0) = Y(t_0) = Z(t_0) = 0,$$

$$-m(s)Y'(t_0) = \omega_1(s)\sinh\phi_1,$$

$$-n(s)Z'(t_0) = \omega_1(s)\cosh\phi_1,$$

where $0 \le t_0 \le T$, $\omega_1(s) \ne 0$.

Corollary 3.2.5. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0) = Y(t_0) = Z(t_0) = 0,$$

$$Y'(t_0) = 0,$$

$$-n(s)Z'(t_0) = \omega_2(s)\sinh\phi_2,$$

where $0 \le t_0 \le T$, $\omega_2(s) \ne 0$.

Corollary 3.2.6. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathsf{W}(s,t)$ is

$$X(t_0) = Y(t_0) = Z(t_0) = 0,$$

$$\begin{aligned} -m(s)Y'(t_0) &= \omega_3(s)\sin\phi_3, \\ Z'(t_0) &= 0, \end{aligned}$$

where $0 \le t_0 \le T$, $\omega_3(s) \ne 0$.

3.3. SPACELIKE GENERATOR CURVES ON SPACELIKE PENCIL SURFACE IN MINKOWSKI 3-SPACE E_1^3

Firstly, the normal $\mathbf{u}(s,t)$ of the surface $\mathsf{W}(s,t)$ can be computed by

$$\mathbf{u}(s,t) = \pi_1(s,t)\mathbf{T}(s) + \pi_2(s,t)\mathbf{g}(s) + \pi_3(s,t)\mathbf{n}(s),$$

where

$$\begin{aligned} \pi_1(s,t) = {} & x(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t} + z(s,t)\tau_g(s)\frac{\partial z(s,t)}{\partial t} + \frac{\partial y(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t} \\ & - x(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t} - y(s,t)\tau_g(s)\frac{\partial y(s,t)}{\partial t} - \frac{\partial z(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t}, \end{aligned}$$

$$\begin{aligned} \pi_2(s,t) = {} & -\frac{\partial z(s,t)}{\partial t} + y(s,t)\kappa_g(s)\frac{\partial z(s,t)}{\partial t} - z(s,t)\kappa_n(s)\frac{\partial z(s,t)}{\partial t} - \frac{\partial x(s,t)}{\partial s}\frac{\partial z(s,t)}{\partial t} \\ & + x(s,t)\kappa_n(s)\frac{\partial x(s,t)}{\partial t} + y(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t} + \frac{\partial z(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}, \end{aligned}$$

$$\begin{aligned} \pi_3(s,t) = {} & -\frac{\partial y(s,t)}{\partial t} + y(s,t)\kappa_g(s)\frac{\partial y(s,t)}{\partial t} - z(s,t)\kappa_n(s)\frac{\partial y(s,t)}{\partial t} - \frac{\partial x(s,t)}{\partial s}\frac{\partial y(s,t)}{\partial t} \\ & + x(s,t)\kappa_g(s)\frac{\partial x(s,t)}{\partial t} + z(s,t)\tau_g(s)\frac{\partial x(s,t)}{\partial t} + \frac{\partial y(s,t)}{\partial s}\frac{\partial x(s,t)}{\partial t}. \end{aligned}$$

Theorem 3.3.1. Let $\mathsf{H}_1(s,t) = \xi(s) + t\mathbf{d}_1(s)$ and $\mathbf{d}_1(s) = \mathbf{g}(s)\cosh\varphi_1 + \mathbf{n}(s)\sinh\varphi_1$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in E_1^3. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0) = y(s,t_0) = z(s,t_0) = 0,$$

$$\varphi_1 = -\int_{s_0}^{s} \tau_g(s)ds + \varphi_0(s_0),$$

$$\pi_1(s,t_0) = 0,\ \pi_2(s,t_0) = \upsilon_1(s)\cosh\varphi_1,\ \pi_3(s,t_0) = \upsilon_1(s)\sinh\varphi_1,$$

where $0 \le t_0 \le T,\ 0 \le s \le L,\ \upsilon_1(s) \ne 0$.

Theorem 3.3.2. Let $\mathsf{H}_2(s,t) = \xi(s) + t\mathbf{d}_2(s)$ and $\mathbf{d}_2(s) = \mathbf{T}(s)\cos\varphi_2 + \mathbf{g}(s)\sin\varphi_2$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in E_1^3. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0) = y(s,t_0) = z(s,t_0) = 0,$$

$$\varphi_2 = \arctan\left(-\frac{\kappa_n}{\tau_g}\right),$$

$$\pi_1(s,t_0) = \upsilon_2(s)\cos\varphi_2,\ \pi_2(s,t_0) = \upsilon_2(s)\sin\varphi_2,\ \pi_3(s,t_0) = 0,$$

where $0 \le t_0 \le T,\ 0 \le s \le L,\ \upsilon_2(s) \ne 0$.

Theorem 3.3.3. Let $\mathsf{H}_3(s,t) = \xi(s) + t\mathbf{d}_3(s)$ and $\mathbf{d}_3(s) = \mathbf{T}(s)\cosh\varphi_3 + \mathbf{n}(s)\sinh\varphi_3$ be a parametric surface possess $\xi(s)$ as a spacelike generator curve in E_1^3. The given spatial curve $\xi(s)$ is a line of curvature on the surface $\mathsf{W}(s,t)$ if and only if

$$x(s,t_0) = y(s,t_0) = z(s,t_0) = 0,$$

$$\varphi_3 = \arctan h\left(-\frac{\kappa_g}{\tau_g}\right),$$

$$\pi_1(s,t_0) = \upsilon_3(s)\cosh\varphi_3,\ \pi_2(s,t_0) = 0,\ \pi_3(s,t_0) = \upsilon_3(s)\sinh\varphi_3,$$

where $0 \le t_0 \le T,\ 0 \le s \le L,\ \upsilon_3(s) \ne 0$.

By using theorem 3.3.1, theorem 3.3.2 and theorem 3.3.3, we study special cases of parametric representations of a timelike pencil surface. Then, we can get the following corollarys, respectively.

Corollary 3.3.4. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathrm{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$-m(s)Y'(t_0) = \upsilon_1(s)\sinh\varphi_1,$$

$$-n(s)Z'(t_0) = \upsilon_1(s)\cosh\varphi_1,$$

where $0 \le t_0 \le T$, $\upsilon_1(s) \ne 0$.

Corollary 3.3.5. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathrm{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$Y'(t_0) = 0,$$

$$-n(s)Z'(t_0) = \upsilon_2(s)\sin\varphi_2,$$

where $0 \le t_0 \le T$, $\upsilon_2(s) \ne 0$.

Corollary 3.3.6. The necessary and sufficient condition of the curve $\xi(s)$ be being a line of curvature on the surface $\mathrm{W}(s,t)$ is

$$X(t_0)=Y(t_0)=Z(t_0)=0,$$

$$-m(s)Y'(t_0) = \upsilon_3(s)\sin\varphi_3,$$

$$Z'(t_0) = 0,$$

where $0 \le t_0 \le T$, $\upsilon_3(s) \ne 0$.

REFERENCES

[1] Akbulut, F., *Darboux Vectors of a Curve on the Surface*, Ege Üniversitesi Fen Fakültesi, Izmir, 1983.
[2] Akutagawa, A., Nishikawa S., *Tohoku Math. J.*, **42**, 67, 1990.
[3] Baş, S., Körpınar, T., *Bol. Soc. Paran. Mat.*, **2**(31), 9, 2013.
[4] Do Carmo, M.P., *Differential Geometry of Curves and Surfaces*, Englewood Cliffs, Prentice Hall, 1976.
[5] Chen, F., Zheng, J., Sederberg, T.W., *Comp. Aid. Des.*, **18**, 61, 2001.

[6] Frey W.H., Bindschadler D., *Computer-Aid Design of a Class of Developable Bezier Surfaces*, General Motors R&D Publication,1993.
[7] Guggenheimer H., *Differential Geometry*, Dover Books on Mathematics, 1977.
[8] Hilbert D., Vossen S.C., *Geometry and the Imagination (2nd ed.)*, New York, Chelsea, 1952.
[9] Izumiya S., Takeuchi N., *Turkish Journal of Mathematics*, **28**, 153, 2004.
[10] Kızıltu S., Yaylı Y., *Mathematica Aeterna*, **8**(2), 689, 2012.
[11] Kobayashi S., Katsumi N., *Foundations of Differential Geometry*, Wiley-Interscience, 1996.
[12] Körpınar T., Sarıaydın M.T., Asil V., Submitted.
[13] Körpınar, T., *Differential Equations and Dynamical Systems*, 1, 2018.
[14] Körpinar, T., *Advances in Applied Clifford Algebras*, **28**(4), 83, 2018.
[15] Körpinar, T., *Journal of Advanced Physics*, **7**(2), 257, 2018.
[16] Körpinar, T., *Advances in Applied Clifford Algebras*, **28**(1), 9, 2018.
[17] Kühnel W., *Curves- Surfaces- Manifolds, Differantial Geometry*, American Mathematical Society, 2003.
[18] Li C.Y., Wang R.H., Zhu C.G., *Computer-Aided Design*, **43**, 1110, 2011.
[19] Li H., Chen W., *Advances in Mathematics*, **3**(28), 211, 1999.
[20] Odehmal B., *Journal for Geometry and Graphics*, **1**(12), 1, 2008.
[21] O'Neill B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.
[22] Park F.C., Yu J., Chun C., Ravani B., *Journal of Mechanical Design*, **124**(4), 602, 2002.
[23] Saad M.K., Abdel-Aziz H. S., Weiss G., Solimman M., *Relations among Darboux Frames of Null Bertrand Curves in Pseudo-Euclidean Space*, *1st Int. WLGK11*, 2011.
[24] Wanga G.J., Tangb K., Taic C.L., *Computer-Aided Design*, **5**(36), 447, 2004.