ORIGINAL PAPER

A NUMERICAL ANALYSING OF THE GEW EQUATION USING FINITE ELEMENT METHOD

SEYDI BATTAL GAZI KARAKOC[1]

Manuscript received: 11.02.2019; Accepted paper: 18.04.2019;
Published online: 30.06.2019.

Abstract. *This paper is devoted to generate new numerical solutions of the generalized equal width wave (GEW) equation with Subdomain finite element method based on quartic B-splines over finite elements. Accuracy and efficiency of the proposed method is demonstrated by employing propagation of single solitary wave.* L_2 *and* L_∞ *error norms are used to measure differences between the analytical and numerical solutions and also, three invariants* I_1*;* I_2 *and* I_3 *have been calculated to determine the conservation properties of the presented algorithm. Fourier stability analysis of the linearized scheme shows that it is unconditionally stable. Numerical experiments prove the correctness and durableness of the method which can be further used for solving such problems.*

Keywords: *GEW equation, Subdomain, Quartic B-spline, Finite element method, Solitary waves, Soliton.*

1. INTRODUCTION

The nonlinear physical phenomena are related to the nonlinear evolution equations (NLEEs), which are appeared in many areas of scientific and engineering fields such as hydrodynamics, plasma physics, optical fibers, fluid dynamics, nonlinear optics, quantum mechanics, solid state physics, mathematical biology and chemical kinematics, chemical physics, geochemistry etc [1]. To better understand these nonlinear phenomena, it is important to explore their exact solutions. But exact solutions of these equations are commonly not derivable, particularly when the nonlinear terms are contained. In so far as only limited classes of these equations are solved by analytical means, numerical solutions of these nonlinear partial differential equations are very operable to examine physical phenomena. In this paper, we investigate an important nonlinear wave equation, the generalized equal width wave equation, of the form

$$U_t + \varepsilon U^p U_x - \mu U_{xxt} = 0, \tag{1.1}$$

where p is a positive integer ε and μ are positive parameters, t is time, x is the space coordinate and $U(x,t)$ is the wave amplitude. Physical boundary conditions have need for $U \to 0$ as $|x| \to \infty$. Boundary

[1] Nevsehir Haci Baktas Veli University, Department of Maths. Faculty of Science and Art, 50300, Nevsehir, Turkey. E-mail: sbgk44@gmail.com

$$\begin{aligned} &U(a,t)=0, \quad U(b,t)=0, \\ &U_x(a,t)=0, \quad U_x(b,t)=0, \\ &U_{xx}(a,t)=0, \quad U_{xx}(b,t)=0, \quad t>0 \end{aligned} \tag{1.2}$$

and initial conditions are selected

$$U(x,0)=f(x) \quad a\le x\le b, \tag{1.3}$$

where $f(x)$ is a localized disturbance inside the considered interval and will be determined later. In the fluid problems as known, the quantity U indicates the wave amplitude of the water surface or a alike physical cardinality. In the plasma treatments, U is the negative of the electrostatic potential [2]. This equation has a lot of implementations in physical situations for example unidirectional waves propagating in a water channel, long waves in near-shore zones, and many others [3]. For $p=1$, Eq.(1.1) is reduced to the EW equation as an important equation in the study of non-linear dispersive waves since it defines a large number of important physical phenomena [4-7]. Another special case of the GEW is obtained when $p=2$. This corresponds to the modified equal width wave (MEW) equation [8-14]. In the literature, there are limited number of papers on the GEW equation. Hamdi et al. [15] derived exact solitary wave solutions of the GEW equation. Evans and Raslan [16] considered the GEW equation by using the collocation method based on quadratic B-splines to obtain the numerical solutions of the single solitary wave, interaction of solitary waves and birth of solitons. The GEW equation solved numerically by a cubic B-spline collocation method by Raslan [17]. The homogeneous balance method was used to construct exact travelling wave solutions of generalized equal width equation by Taghizadeh et al. [18]. The equation was solved numerically by a meshless method based on a global collocation with standard types of radial basis functions (RBFs) in [3]. Quintic B-spline collocation method with two different linearization techniques and a lumped Galerkin method based on cubic B-spline functions were employed to obtain the numerical solutions of the GEW equation by Karakoc and Zeybek, [2, 19] respectively. In this work, we have built Subdomain finite element method for the GEW equation using quartic B-spline functions.

The rest of the paper can be summarized briefly as follows: the approximate method for the solution of Eq.(1.1) is proposed in the next section. Section 3 contains a lineear stability analysis of the algorithm followed by Section 4 which belongs to numerical experiments of traveling single solitary wave with different initial and boundary conditions.

2. QUARTIC B-SPLINES AND ANALYSIS OF THE SUBDOMAIN METHOD

To analyze the numerical behavior of the Eq.(1.1), the solution domain is constrained on a closed interval $[a,b]$. Prenter [20] defined following quartic B-spline functions $\varphi_m(x)$ $(m=-2,-1,...,N,N+1)$ at the points x_m which generate a basis over the domain $[a,b]$ by

$$\varphi_m(x)=\frac{1}{h^4}\begin{cases}(x-x_{m-2})^4, & [x_{m-2},x_{m-1}]\\ (x-x_{m-2})^4-5(x-x_{m-1})^4, & [x_{m-1},x_m]\\ (x-x_{m-2})^4-5(x-x_{m-1})^4+10(x-x_m)^4, & [x_m,x_{m+1}]\\ (x_{m+3}-x)^4-5(x_{m+2}-x)^4, & [x_{m+1},x_{m+2}]\\ (x_{m+3}-x)^4, & [x_{m+2},x_{m+3}]\\ 0 & otherwise.\end{cases} \tag{2.1}$$

The numerical solution $U_N(x,t)$ is written in terms of the quartic B-splines as

$$U_N(x,t)=\sum_{j=m-2}^{N+1}\varphi_j(x)\delta_j(t), \tag{2.2}$$

in which parameters $\delta_j(t)$ are procured using boundary and weighted residual conditions. The nodal values U_m, U'_m, U''_m and U'''_m at the knots x_m are got from trial function (2.2) and quartic B-splines (2.1) in the following form

$$\begin{aligned}U_N(x_m,t)=U_m&=\delta_{m-2}+11\delta_{m-1}+11\delta_m+\delta_{m+1},\\ U'_m&=\tfrac{4}{h}(-\delta_{m-2}-3\delta_{m-1}+3\delta_m+\delta_{m+1}),\\ U''_m&=\tfrac{12}{h^2}(\delta_{m-2}-\delta_{m-1}-\delta_m+\delta_{m+1}),\\ U'''_m&=\tfrac{24}{h^3}(-\delta_{m-2}+3\delta_{m-1}-3\delta_m+\delta_{m+1}).\end{aligned} \tag{2.3}$$

Finite element method pertains to the family of weighted residual methods and one of these standard method is Subdomain method [21]. In this method, we separate the physical region into a number of non-overlapping subdomains. Each weight function is taken as unity over a specific subdomain and set equal to zero over other the other parts. For one-dimensional problems the weight function is,

$$W_m(x)=\begin{cases}1, & x\in[x_m,x_{m+1}],\\ 0, & otherwise,\end{cases}\qquad m=1,2,\ldots,n. \tag{2.4}$$

In weighted residual method

$$\int_a^b W_m R(x)dx=\int_{x_m}^{x_{m+1}}R(x)dx=0 \tag{2.5}$$

can be written. This implies that the average of the residual over each of n subdomains is forced to be zero [22]. When Subdomain finite element method is applied to Eq.(1.1) with weight function (2.4) we obtain the weak form of Eq.(1.1) as

$$\int_{x_m}^{x_{m+1}}1.(U_t+\varepsilon U^pU_x-\mu U_{xxt})dx=0. \tag{2.6}$$

Implementing the transformation $h\xi=x-x_m$ into weak form (2.6) and integrating (2.6) term by term with some manupulation by parts, brings along

$$\begin{aligned}&\tfrac{h}{5}\left(\dot{\delta}_{m-2}+26\dot{\delta}_{m-1}+66\dot{\delta}_{m}+26\dot{\delta}_{m+1}+\dot{\delta}_{m+2}\right)+Z_m(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2})-\\&\mu\tfrac{12}{h^2}(\dot{\delta}_{m-2}+2\dot{\delta}_{m-1}-6\dot{\delta}_{m}+2\dot{\delta}_{m+1}+\dot{\delta}_{m+2})=0,\end{aligned}\tag{2.7}$$

where the dot typifies differentiation with respect to t and

$$Z_m=\varepsilon(\delta_{m-2}+11\delta_{m-1}+11\delta_m+\delta_{m+1})^p.\tag{2.8}$$

Changing the time derivative $\dot{\delta}$ with the forward difference approximation $\dot{\delta}=\frac{\delta^{n+1}-\delta^n}{\Delta t}$ and the parameter δ by the Crank-Nicolson equality $\delta=\frac{1}{2}(\delta^n+\delta^{n+1})$, then system (2.7) turn into the following matrix system:

$$\begin{aligned}&\omega_{m1}\delta_{m-2}^{n+1}+\omega_{m2}\delta_{m-1}^{n+1}+\omega_{m3}\delta_{m}^{n+1}+\omega_{m4}\delta_{m+1}^{n+1}+\omega_{m5}\delta_{m+2}^{n+1}\\&=\omega_{m5}\delta_{m-2}^{n+1}+\omega_{m4}\delta_{m-1}^{n+1}+\omega_{m3}\delta_{m}^{n+1}+\omega_{m2}\delta_{m+1}^{n+1}+\omega_{12}\delta_{m+2}^{n+1}\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}&\omega_{m1}=1-EZ_m-M, &&\omega_{m2}=26-10EZ_m-2M,\\&\omega_{m3}=66+6M, &&\omega_{m4}=26+10EZ_m-2M,\\&\omega_{m5}=1+EZ_m-M, &&E=\tfrac{5\Delta t}{2h}\quad M=\tfrac{20\mu}{h^2},\quad m=0,1,...,N-1.\end{aligned}\tag{2.10}$$

The system (2.9) composes of N linear equations including $(N+4)$ unknown parameters. We need four additional limitations to find a unique solution of this system. Executing the boundary conditions (1.2) to the system (2.9) we remove the parameters $\delta_{-2},\delta_{-1},\delta_N$, and δ_{N+1} from the system (2.9) which then becomes a matrix equation for the N unknowns $d=(\delta_0,\delta_1,\ldots,\delta_{N-1})$ of the form $Md^{n+1}=Nd^n$. A lumped form of Z_m is computed as $\left(\frac{U_m+U_{m+1}}{2}\right)^p$ and

$$Z_m=\frac{\varepsilon}{2^p}(\delta_{m-2}+12\delta_{m-1}+22\delta_m+12\delta_{m+1}+\delta_{m+2})^p\ .\tag{2.11}$$

The resulting system is solved by a modified form of well known Thomas algorithm [23] and in this solution procedure an inner iteration is also implemented at each time step to decrease the non-linearity. In an effort to solve this system, it is necessary to obtain the initial parameters δ^0 by using the initial condition $U(x,0)=f(x)$ and following derivatives at the boundaries;

$$\begin{aligned}&U_N(x_m,0)=U(x_m,0), &&U_N'(a,0)=U_N'(b,0)=0,\\&U_N''(a,0)=U_N''(b,0)=0, &&U_N'''(a,0)=U_N'''(b,0)=0.\end{aligned}$$

So, a new $N\times N$ dimensional solvable matrix is obtained for δ_m^0 parameters.

$$\begin{bmatrix} 18 & 6 & & & & & \\ 11.5 & 11.5 & 1 & & & & \\ 1 & 11 & 11 & 1 & & & \\ & & & & & & \\ & & & 1 & 11 & 11 & 1 \\ & & & & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ . \\ \delta_{N-2}^0 \\ \delta_{N-1}^0 \end{bmatrix} = \begin{bmatrix} U(x_0,0) \\ U(x_1,0) \\ . \\ U(x_{N-2},0) \\ U(x_{N-1},0) \end{bmatrix}.$$

3. STABILITY ANALYSIS

In this section, to demonstrate the stability analysis of the numerical method, we have used Fourier method based on Von-Neumann theory and presume that the quantity U^p in the nonlinear term a local constant such as Z_m [24] $\cdot$ If we put the Fourier mode

$$\delta_j^n = g^n e^{ijkh}$$

into scheme (2.9) and the necessary operations are performed, following growth factor is obtained

$$g = \frac{a-ib}{a+ib},$$

where

$$a = 33+3M+(26-2M)\cos(kh)+(1-M)\cos(2kh),$$
$$b = 10EZ_m\sin(kh)+EZ_m\sin(2kh).$$

Since $|g|$ is 1, the von Neumann necessary criterion is provided so our linearized scheme is unconditionally stable.

4. NUMERICAL RESULTS AND DISCUSSION

The purpose of this section is to examine the deduced algorithm using different test problems concerned with the dispersion of single solitary waves. To see the difference between numerical solution and analytic solution, we have used the error norms defined as follows

$$L_2 = \left\|U^{exact} - U_N\right\|_2 = \sqrt{h\sum_{J=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2},$$

and

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty = \max_j\left|U_j^{exact} - (U_N)_j\right|.$$

The exact solution of GEW equation is given in [3, 19] as

$$U(x,t)=\sqrt[p]{\frac{c(p+1)(p+2)}{2\varepsilon}\operatorname{sec}h^2[\frac{p}{2\sqrt{\mu}}(x-ct-x_0)]}$$

which corresponds to a solitary wave of amplitude $\sqrt[p]{\frac{c(p+1)(p+2)}{2\varepsilon}}$, speed c , width $\frac{p}{2\sqrt{\mu}}$ and initially centered at x_0. The conservation properties of the GEW equation concerned with mass, momentum and energy are confirmed by finding the following three invariants [2, 16, 17].

$$I_1=\int_{-\infty}^{\infty}U(x,t)dx,\qquad I_2=\int_{-\infty}^{\infty}[U^2(x,t)+\mu U_x^2(x,t)]dx,\qquad I_3=\int_{-\infty}^{\infty}U^{p+2}(x,t)dx.$$

4.1. DISPERSION OF A SINGLE SOLITARY WAVE

For the numerical simulations of the movement of single solitary wave, three sets of parameters have been taken and discussed.

For the first set, we choose the parameters $p=2$, $c=0.5$, $h=0.1$, $\Delta t=0.2$, $\mu=1$, $\varepsilon=3$ and $x_0=30$ through the interval $[0,80]$ to compare with that of previous papers [2, 19]. These parameters stand for the motion of a single solitary wave with amplitude 1.0 and the algorithm is run to time $t=20$ over the solution region. A comparison with exact solution as well as the calculated values in [2, 19] has been done and listed in Table (1) at $t=20$. Fig. 1 depicts the numerical solutions at different time levels at $t=0,10,20$. The quantile of error at discoint times are plotted in Fig. 2.

Table 1. Values of the invariants and comparison of the error norms for single solitary wave with p=2, c=0.5, h=0.1, Δt=0.2, μ=1 and ε=3 at t=20.

Method	I_1	I_2	I_3	L_2	L_∞
Exact	3.1415961	2.6666667	1.3333333	0.0000000	0.0000000
Present $t=0$	3.1415963	2.6666625	1.3333283	0.0000000	0.0000000
$t=5$	3.1374227	2.6611036	1.3277747	0.0074320	0.0052489
$t=10$	3.1332986	2.6556033	1.3222912	0.0190058	0.0130097
$t=15$	3.1292124	2.6501590	1.3168748	0.0346782	0.0232969
$t=20$	3.1251634	2.6447698	1.3115241	0.0544214	0.0360834
Cubic Gal. [2]	3.1589605	2.6902580	1.3570299	0.0380303	0.0262900
Quintic Coll. First Scheme [19]	3.1253043	2.6445829	1.3113394	0.0513210	0.0341675
Quintic Coll. Second Scheme [19]	3.1416722	2.6669051	1.3335718	0.0167509	0.0102639

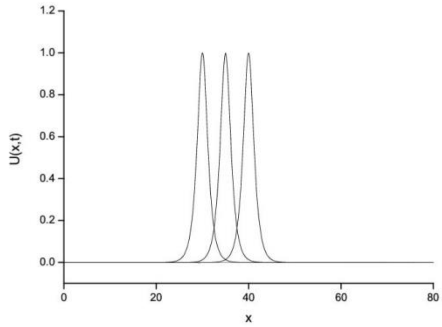

Figure 1. Motion of single solitary wave for p=2, c=0.5, h=0.1, Δt=0.2, μ=1, ε=3 over the interval [0,80] at t=0,10,20.

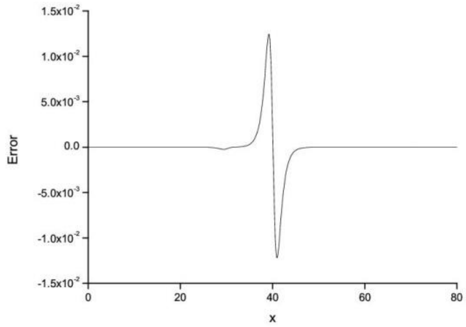

Figure 2. Error graph for p=2, c=0.5, h=0.1, Δt=0.2, μ=1, ε=3 at t=20.

For our second experiment, if the parameters are taken $p=3, c=0.3, h=0.1,$ $\Delta t=0.2,$ $\varepsilon=3,$ $\mu=1,$ $x_0=30$ with interval $[0,80]$ the solitary wave has amplitude $1.0.$ The experiment is carried out for times up to $t=20.$ Table (2) represents a comparison of the values of the invariants and error norms obtained by the present method with those obtained in [2,19] at $t=20.$ Fig. (3) displays the motion of the solitary wave at time leves $t=0,10,20.$ The aberration of error at discrete times are depicted in Fig. (4).

Table 2. Values of the invariants and comparison of the error norms for single solitary wave with p=3, c=0.3, h=0.1, Δt=0.2, μ=1 and ε=3 at t=20.

Method	I_1	I_2	I_3	L_2	L_∞
Present $t=0$	2.8043580	2.4639124	0.9855618	0.0000000	0.0000000
$t=5$	2.8009794	2.4589310	0.9805878	0.0057707	0.0045666
$t=10$	2.7976460	2.4540202	0.9756990	0.0154171	0.0111684
$t=15$	2.7973553	2.4491772	0.9708921	0.0288633	0.0212993
$t=20$	2.7911063	2.4444001	0.9661645	0.0460246	0.0333820
Cubic Gal. [2]	2.8187398	2.4852249	0.0070200	0.0165563	0.0137045
Quintic Coll. First Scheme [19]	2.8043570	2.4639086	0.9855602	0.0080147	0.0053823
Quintic Coll. Second Scheme [19]	2.8042943	2.4637495	0.9854011	0.0070855	0.0048047

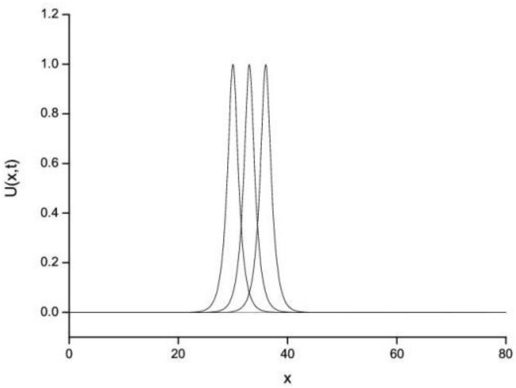

Figure 3. Motion of single solitary wave for p=3, c=0.3, h=0.1, Δt=0.2, μ=1, ε=3 over the interval [0,80] at t=0,10,20.

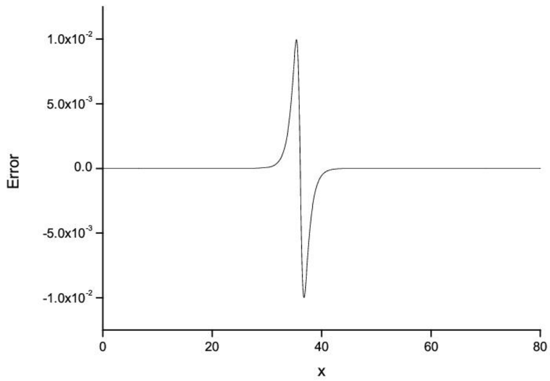

Figure 4. Error graph for p=3, c=0.3, h=0.1, Δt=0.2, μ=1, ε=3 at t=20.

For the final treatment, we select the parameters $p=4$, $c=0.2$, $h=0.1$, $\Delta t=0.2$, $\varepsilon=3$, $\mu=1$, $x_0=30$ over the interval $[0,80]$ to enable comparison with those of earlier papers [2,19]. So solitary wave has amplitude 1.0 and the simulations are performed to time $t=20$ to get the error norms L_2 and L_∞ and the numerical invariants I_1, I_2 and I_3. Table (3) displays a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other methods [2,19]. The behaviours of solutions for values $p=4, c=0.2, h=0.1, \Delta t=0.2$ at times $t=0,10$ and 20 are shown in Figure (5). Error distributions at time $t=20$ are drawn graphically in Figure (6).

Table 3. Values of the invariants and comparison of the error norms for single solitary wave with p=4, c=0.2, h=0.1, ∆t=0.2, μ=1 and ε=3 at t=20.

Method	I_1	I_2	I_3	L_2	L_∞
Present $t=0$	2.6220516	2.3561965	0.7853952	0.0000000	0.0000000
$t=5$	2.6195341	2.3522596	0.7814646	0.0050342	0.0041635
$t=10$	2.6170567	2.3483875	0.7776117	0.0130014	0.0103774
$t=15$	2.6146144	2.3445730	0.7738285	0.0238590	0.0185860
$t=20$	2.6122055	2.3408135	0.7701119	0.0375343	0.0287549
Cubic Gal. [2]	2.6327833	2.3730032	0.8023383	0.0089061	0.0082199
Quintic Coll. First Scheme [19]	2.6220508	2.3561901	0.7853939	0.0042169	0.0029795
Quintic Coll. Second Scheme [19]	2.6219284	2.3559327	0.7851364	0.0033908	0.0024703

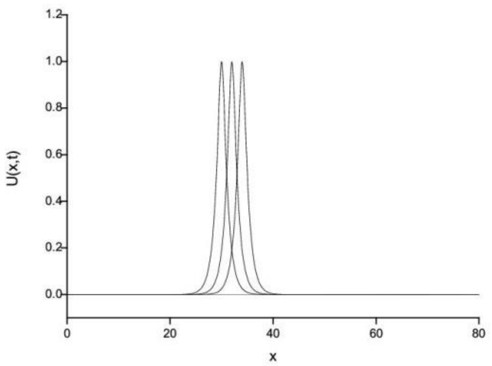

Figure 5. Motion of single solitary wave for p=4, c=0.2, h=0.1, ∆t=0.2, μ=1, ε=3 over the interval [0,80] at t=0,10,20.

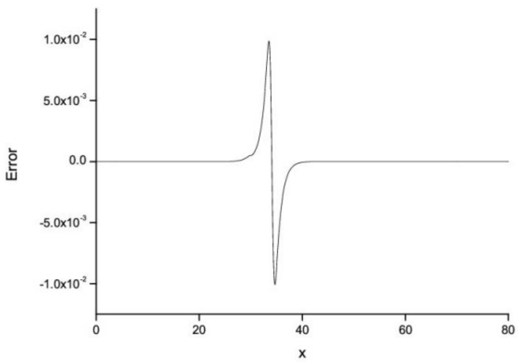

Figure 6. Error graph for p=4, c=0.2, h=0.1, ∆t=0.2, μ=1, ε=3 at t=20.

CONCLUSION

In this study, solitary-wave solutions of the GEW equation have been successfully obtained by using Subdomain method based on quartic B-splines. To prove the performance and accuracy of the numerical method L_2, L_∞ error norms and the invariant quantities I_1, I_2 and I_3 have been calculated. The obtained numerical results predicate that our error norms are as small as required and they are smaller than existing numerical calculations or too close to the result in literature. Therefore, our numerical technique is suitable for getting numerical solutions of partial differential equations.

REFERENCES

[1] Esfahani, A., *Physics Letters A*, **374**, 3635, 2010.
[2] Karakoc, S. B. G. and Zeybek, H., *Stat., Optim. Inf. Comput.*, **4**, 30, 2016.
[3] Panahipour, H., *Communications in Numerical Analysis*, **2012**, 1, 2012.
[4] Gardner, L. R. T., Gardner, G. A., Ayoub, F. A. and Amein, N. K., *Commun. Numer. Meth. En.*,**13**, 583, 1197.
[5] Zaki, S. I., *Comp. Methods in Appl. Mech. and Eng.*, **189**, 587, 2000.
[6] Esen, A., *Applied Mathematics and Computation*, **168**, 270, 2005.
[7] Dag, I. and Saka, B., *Mathematical and Computational Applications*, **9**, 381, 2004.
[8] Karakoc, S. B. G. and Geyikli, T., *International Journal of Differential Equations*, **2012**, 1, 2012.
[9] Geyikli, T. and Karakoc, S. B. G., , *Bull. Belg. Math. Soc. Simon Stevin*, **19, 215,** 2012.
[10] Geyikli, T. and Karakoc, S. B. G., *Appl. Math.*, **2**, 739, 2011.
[11] Geyikli, T. and Karakoc, S. B. G., *Computational Mathematics and Mathematical Physics*, **55**, 3, 2015.
[12] Karakoc, S. B. G., *Numerical solutions of the modified equal width wave equation with finite elements method*, PhD thesis, Inonu University, Malatya, Turkey, 2011.
[13] Esen, A., *International Journal of Computer Mathematics*, **83, 449,** 2006.
[14] Saka, B., *Mathematical and Computer Modelling*, **45**, 1096, 2007.
[15] Hamdi, S., Enright, W. H., Schiesser, W. E. and Gottlieb, J. J., *Proceedings of the International Conference on Computational Science and Its Applications, in: LNCS*, **2668**, 725, 2003.
[16] Evans, D. J. and Raslan, K. R., *Int. J. Comput. Math.* **82**, 4, 2005.
[17] Raslan, K. R., *Int. J. Simulation and Process Modelling*, **2**, 37, 2006.
[18] Taghizadeh, N., Mirzazadeh, M. and Farahrooz, F., Math. *Sci. Lett.* **2**, 99, 2013.
[19] Zeybek, H. and Karakoc, S. B. G., *Electronic Transactions on Numerical Analysis*, **46**, 71, 2017.
[20] Prenter P. M., *Splines and Variational Methods*. John Wiley & Sons, New York, 245, 1975.
[21] Lindgren, L. E., *From Weighted Residual Methods to Finite Element Methods*, 2009.

[22] Salih, A., *Weighted Residual Methods,* Department of Aerospace Engineering Indian
[23] Institute of Space Science and Technology, Thiruvananthapuram - December 2016.
[24] Von Rosenberg, D.U., *Methods for solution of partial differential equation*, American
[25] Elsevier Publishing Inc., New York, 113, 1969.
[26] Karakoc, S. B. G., Yagmurlu, N. M. and Ucar, Y., *Boundary Value Problems*, **2013**, 1, 2013.