ORIGINAL PAPER

SOME EXACT SOLUTIONS OF (3+1)-DIMENSIONAL SHALLOW WATER WAVE EQUATION AND (2+1)-DIMENSIONAL FIFTH-ORDER NONLINEAR EQUATION

UNAL IC[1]

Manuscript received: 02.03.2019; Accepted paper: 07.05.2019;
Published online: 30.06.2019.

Abstract. *In this paper, we implemented an improved tanh function Method for some exact solutions of (3+1)-dimensional Shallow water wave equation and (2+1)-dimensional fifth-order nonlinear equation.*
Keywords: *(2+1)-dimensional fifth-order nonlinear equation, (3+1)-dimensional shallow water wave equation, exact solutions, improved tanh function method.*

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics [1, 2]. There are some analytical methods for solving these equations in the literature [3-8]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. These methods are given in [9-16].

We used the improved tanh function method to find the exact solutions of the (3 + 1) – dimensional shallow water wave equation (SWWE) and (2+1)-dimensional fifth-order nonlinear equation in this study. This method is presented by Chen and Zhang [9].

2. ANALYSIS OF METHOD

Let's introduce the method briefly. Consider a general partial differential equation of two variables,

$$\varphi(v, v_t, v_x, v_{xx}, \dots) = 0, \tag{1}$$

and transform Eq. (1) with

$$v(x,t) = v(\phi),\ \phi = k(x - \beta t)$$

here k and β are constants. With this conversion, we obtain a nonlinear ordinary differential equation for $v(\phi)$,

[1] Firat University, Faculty of Education, Elazig, Turkey. E-mail: unalic@firat.edu.tr.

$$\varphi'(v', v'', v''', \dots) = 0. \tag{2}$$

We can express the solution of equation (2) as below,

$$v(\phi) = \sum_{i=0}^{M} a_i F^i(\phi), \tag{3}$$

here is M a positive integer and is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation.

If we write these solutions in equation (2), we obtain a system of algebraic equations for $F(\emptyset), F^2(\emptyset), \dots, F^i(\emptyset)$ after, if the coefficients of $F(\emptyset), F^2(\emptyset), \dots, F^i(\emptyset)$ are equal to zero, we can find the $k, \beta, a_0, a_1, \dots, a_n$ constants.

The basic step of the method is to make full use of the Riccati equation satisfying the tanh function and to use $F(\emptyset),$ solutions. The Riccati equation required in this method is given below

$$F'(\phi) = A + BF(\phi) + CF^2(\phi). \tag{4}$$

Here, $F'(\emptyset) = \frac{dF(\emptyset)}{d\emptyset}$ and A, B and C are constants. The authors expressed the solutions as the following [9].

Example 1. We consider the (3+1)-dimensional shallow water wave equation,

$$v_{xzt} + v_{xxxyz} - 2v_{xx}v_{yz} - 2v_y v_{xxz} - 4v_x v_{xyz} - 4v_{xz}v_{xy} = 0. \tag{5}$$

Using the traveling wave solution $v(x,t,y,z) = v(\emptyset)$, $\emptyset = k(x + \alpha y + \beta z - wt)$ for equation (5), we get

$$-\beta w v''' + k^2\alpha\beta v^{(5)} - 6k\alpha\beta(v'')^2 - 6k\alpha\beta v'v'''0, \tag{6}$$

balancing $(v'')^2$, $v'v'''$ with $v^{(5)}$ gives $M = 1$. The solution is as follows:

$$v = a_0 + a_1 F(\emptyset). \tag{7}$$

(7) is substituted in equation (6). We obtain a set of equations for $k, w, \alpha, \beta, a_0, a_1$. The obtained systems of algebraic equations are as follows:

$$-Aa_1B^2w\beta - 2A^2a_1Cw\beta - 12A^2a_1^2B^2k\alpha\beta - 12A^3a_1^2Ck\alpha\beta + Aa_1B^4k^2\alpha\beta + 22A^2B^2a_1Ck^2\alpha\beta + 16A^3a_1C^2k^2\alpha\beta = 0,$$

$$-a_1B^3w\beta - 8Aa_1BCw\beta - 24Aa_1^2B^3k\alpha\beta - 84A^2a_1^2BCk\alpha\beta + a_1B^5k^2\alpha\beta + 52AB^3a_1Ck^2\alpha\beta + 136A^2a_1BC^2k^2\alpha\beta = 0,$$

$$-7a_1B^2Cw\beta - 8Aa_1C^2w\beta - 12a_1^2B^4k\alpha\beta - 156Aa_1^2B^2Ck\alpha\beta - 84A^2a_1^2C^2k\alpha\beta + 31B^4a_1Ck^2\alpha\beta + 292Aa_1B^2C^2k^2\alpha\beta + 136A^2a_1C^3k^2\alpha\beta = 0,$$

$$-12a_1BC^2w\beta - 84a_1^2B^3Ck\alpha\beta - 264Aa_1^2BC^2k\alpha\beta + 180a_1B^3C^2k^2\alpha\beta + 480Aa_1BC^3k^2\alpha\beta = 0,$$

$$-6a_1C^3w\beta - 192a_1^2B^2C^2k\alpha\beta - 132Aa_1^2C^3k\alpha\beta + 390a_1B^2C^3k^2\alpha\beta + 240Aa_1C^4k^2\alpha\beta = 0,$$

$$-180a_1^2BC^3k\alpha\beta + 360a_1BC^4k^2\alpha\beta = 0,$$
$$-60a_1^2C^4k\alpha\beta + 120a_1C^5k^2\alpha\beta = 0. \tag{8}$$

If the system is solved, the coefficients are found as

$$C \neq 0, w = w, \beta = \beta,\ a_0 = 0, k = \frac{a_1}{2C}, a_1(2Aa_1 - B^2k) \neq 0,$$

$$\alpha = -\frac{2Cw}{a_1(2Aa_1 - B^2k)}, w\beta \neq 0, \tag{9}$$

with the help of the Mathematica program. After these operations, the solutions of equation (5) are found as follows:

Solutions 1

$$v_1 = a_1\left[coth\left(-a_1x - \frac{w}{a_1}y - a_1\beta z + a_1wt\right) \pm cosech\left(-a_1x - \frac{w}{a_1}y - a_1\beta z + a_1wt\right)\right]$$

$$v_2 = a_1\left[tanh\left(-a_1x - \frac{w}{a_1}y - a_1\beta z + a_1wt\right) \pm isech\left(-a_1x - \frac{w}{a_1}y - a_1\beta z + a_1wt\right)\right] \tag{10}$$

Solutions 2

$$v_3 = a_1\left[sec\left(a_1x - \frac{w}{a_1}y + a_1\beta z - a_1wt\right) + tan\left(a_1x - \frac{w}{a_1}y + a_1\beta z - a_1wt\right)\right]$$

$$v_4 = a_1\left[cosec\left(a_1x - \frac{w}{a_1}y + a_1\beta z - a_1wt\right) - cot\left(a_1x - \frac{w}{a_1}y + a_1\beta z - a_1wt\right)\right]$$

$$v_5 = a_1\left[cosec\left(-a_1x + \frac{w}{a_1}y - a_1\beta z + a_1wt\right) + cot\left(-a_1x + \frac{w}{a_1}y - a_1\beta z + a_1wt\right)\right]$$

$$v_6 = a_1\left[sec\left(-a_1x + \frac{w}{a_1}y - a_1\beta z + a_1wt\right) - tan\left(-a_1x + \frac{w}{a_1}y - a_1\beta z + a_1wt\right)\right] \tag{11}$$

Solutions 3

$$v_7 = a_1tanh\left(-\frac{a_1}{2}x - \frac{w}{2a_1}y - \frac{a_1}{2}\beta z + \frac{a_1}{2}wt\right)$$

$$v_8 = a_1coth\left(-\frac{a_1}{2}x - \frac{w}{2a_1}y - \frac{a_1}{2}\beta z + \frac{a_1}{2}wt\right) \tag{12}$$

Solutions 4

$$v_9 = a_1tan\left(\frac{a_1}{2}x - \frac{w}{2a_1}y + \frac{a_1}{2}\beta z - \frac{a_1}{2}wt\right) \tag{13}$$

Solutions 5

$$v_{10} = a_1cot\left(-\frac{a_1}{2}x + \frac{w}{2a_1}y - \frac{a_1}{2}\beta z + \frac{a_1}{2}wt\right) \tag{14}$$

Example 2. We consider the (2+1)-dimensional fifth-order nonlinear equation,

$$v_{ttt} - v_{tyyyy} - v_{txx} - 4v_{yy}v_{yt} - 4v_y v_{yyt} = 0, \tag{15}$$

Using the traveling wave solution $v(x,y,t) = v(\emptyset)$, $\emptyset = k(x + \alpha y - \beta t)$ for equation (15), we get

$$-\beta^3 v''' + k^2\alpha^4\beta v^{(5)} + 4k\alpha^3\beta(v'')^2 + 4k\alpha^3\beta v'v''' = 0, \tag{16}$$

balancing $(v'')^2$, $v'v'''$ with $v^{(5)}$ then gives $M = 1$. The solution is as follows:

$$v = a_0 + a_1 F(\emptyset), \tag{17}$$

(17) is substituted in equation (16). We obtain a set of equations for $k, \alpha, \beta, a_0, a_1$. The obtained systems of algebraic equations are as follows:

$$Aa_1B^2\beta + 2A^2a_1C\beta + 8A^2a_1^2B^2k\alpha^3\beta + 8A^3a_1^2Ck\alpha^3\beta + Aa_1B^4k^2\alpha^4\beta + 22A^2B^2a_1Ck^2\alpha^4\beta + 16A^3a_1C^2\alpha^4\alpha\beta - Aa_1B^2\beta^3 - 2A^2a_1C\beta^3 = 0,$$

$$a_1B^3\beta + 8Aa_1BC\beta + 16Aa_1^2B^3k\alpha^3\beta + 56A^2a_1^2BCk\alpha^3\beta + a_1B^5k^2\alpha^4\beta + 52AB^3a_1Ck^2\alpha^4\beta + 136A^2a_1BC^2k^2\alpha^4\beta - a_1B^3\beta^3 - 8Aa_1BC\beta^3 = 0,$$

$$7a_1B^2C\beta + 8Aa_1C^2\beta + 8a_1^2B^4k\alpha^3\beta + 104Aa_1^2B^2Ck\alpha^3\beta + 56A^2a_1^2C^2k\alpha^3\beta + 31B^4a_1Ck^2\alpha^4\beta + 292Aa_1B^2C^2k^2\alpha^4\beta + 136A^2a_1C^3k^2\alpha^4\beta - 7a_1B^2C\beta^3 - 8Aa_1C^2\beta^3 = 0,$$

$$12a_1BC^2\beta + 56a_1^2B^3Ck\alpha^3\beta + 176Aa_1^2BC^2k\alpha^3\beta + 180a_1B^3C^2k^2\alpha^4\beta + 480Aa_1BC^3k^2\alpha^4\beta - 12a_1BC^2\beta^3 = 0,$$

$$6a_1C^3w\beta + 128a_1^2B^2C^2k\alpha^3\beta + 88Aa_1^2C^3k\alpha^3\beta + 390a_1B^2C^3k^2\alpha^4\beta + 240Aa_1C^4k^2\alpha^4\beta - 6a_1C^3\beta^3 = 0,$$

$$120a_1^2BC^3k\alpha^3\beta + 360a_1BC^4k^2\alpha^4\beta = 0,$$

$$40a_1^2C^4k\alpha^3\beta + 120a_1C^5k^2\alpha^4\beta = 0. \tag{18}$$

If the system is solved, the coefficients are found as

$$Ck \neq 0,\ \beta = \sqrt{1 + B^2k^2\alpha^4 + \tfrac{4}{3}Aa_1k\alpha^3}, a_0 = 0\ ,\ \alpha = -\frac{a_1}{3Ck}\ , a_1 \neq 0. \tag{19}$$

with the help of the Mathematica program. After these operations, the solutions of equation (15) are found as follows,

Solutions 1

$$v_1 = a_1\left[coth\left(kx + \frac{2a_1}{3}y - \left(\frac{k}{9}\sqrt{81 + \frac{16a_1^4}{k^2}}\right)t\right) \pm cosech\left(kx + \frac{2a_1}{3}y - \left(\frac{k}{9}\sqrt{81 + \frac{16a_1^4}{k^2}}\right)t\right)\right]$$

$$v_2 = a_1\left[tanh\left(kx + \frac{2a_1}{3}y - \left(\frac{k}{9}\sqrt{81 + \frac{16a_1^4}{k^2}}\right)t\right) \pm isech\left(kx + \frac{2a_1}{3}y - \left(\frac{k}{9}\sqrt{81 + \frac{16a_1^4}{k^2}}\right)t\right)\right] \tag{20}$$

Solutions 2

$$v_3 = a_1\left[sec\left(kx - \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right) + tan\left(kx - \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right)\right]$$

$$v_4 = a_1\left[cosec\left(kx - \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right) - cot\left(kx - \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right)\right]$$

$$v_5 = a_1\left[cosec\left(kx + \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right) + cot\left(kx + \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right)\right]$$

$$v_6 = a_1\left[sec\left(kx + \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right) - tan\left(kx + \frac{2a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{16a_1^4}{k^2}})t\right)\right] \quad (21)$$

Solutions 3

$$v_7 = a_1 tanh\left(kx + \frac{a_1}{3}y - (\frac{k}{9}\sqrt{81 + \frac{4a_1^4}{k^2}})t\right)$$

$$v_8 = a_1 coth\left(kx + \frac{a_1}{3}y - (\frac{k}{9}\sqrt{81 + \frac{4a_1^4}{k^2}})t\right) \quad (22)$$

Solutions 4

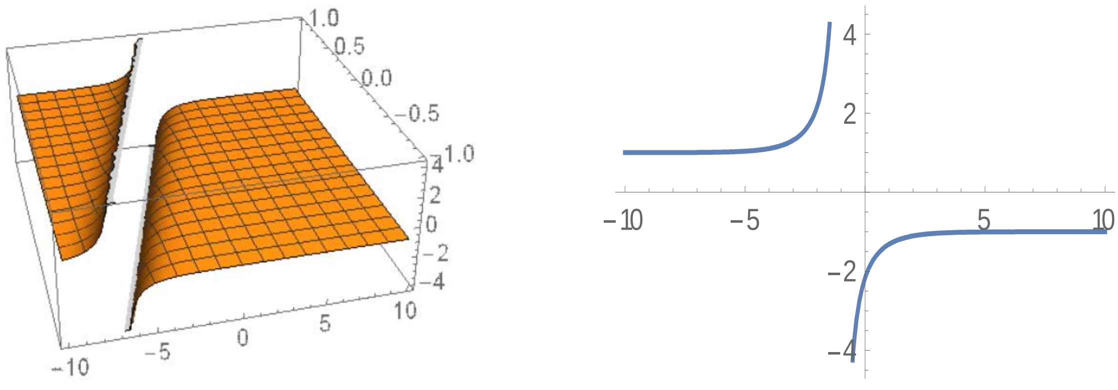
$$v_9 = a_1 tan\left(kx - \frac{a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{4a_1^4}{k^2}})t\right) \quad (23)$$

Solutions 5

$$v_{10} = a_1 cot\left(kx + \frac{a_1}{3}y - (\frac{k}{9}\sqrt{81 - \frac{4a_1^4}{k^2}})t\right) \quad (24)$$

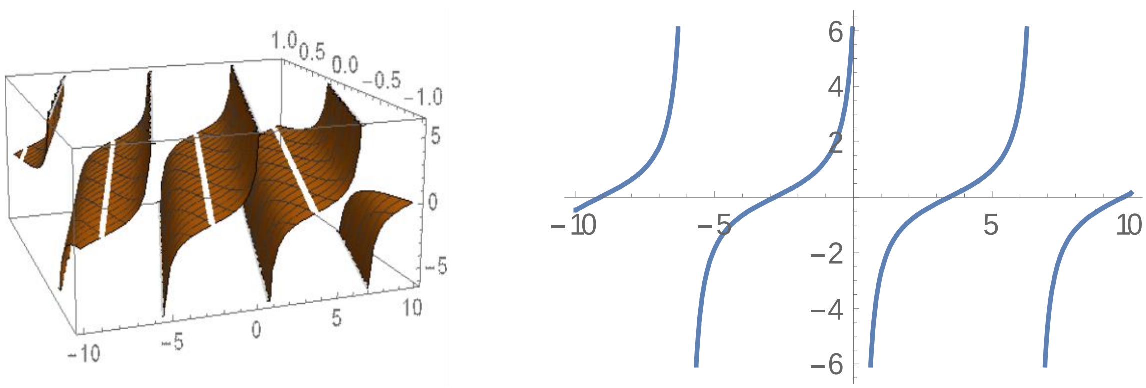
3. GRAPHS AND NUMERICAL EXPLANATIONS OF SOME SOLUTIONS

The (3+1)-dimensional shallow water wave equation: The shapes of Eqs.(10)-(12) and (16) are represented in Figs. 1-3 within the interval $-10 \le x \le 10, -1 \le t \le 1$. The (2+1)-dimensional fifth-order nonlinear equation: The shapes of Eqs.(27),(32) and (33) are represented in Figs. 4-6 respectively, within the interval $-10 \le x \le 10, -1 \le t \le 1$.



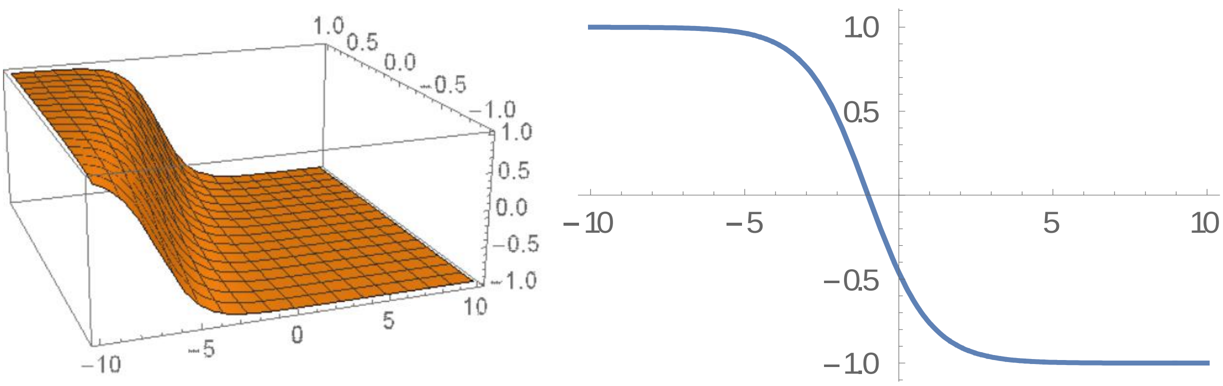
i) **ii)**

Figure 1. i) The 3D surfaces of Eq. (10) for $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$. **ii) The 2D surfaces of Eq. (10) for** $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$ **and** $t = 1$.



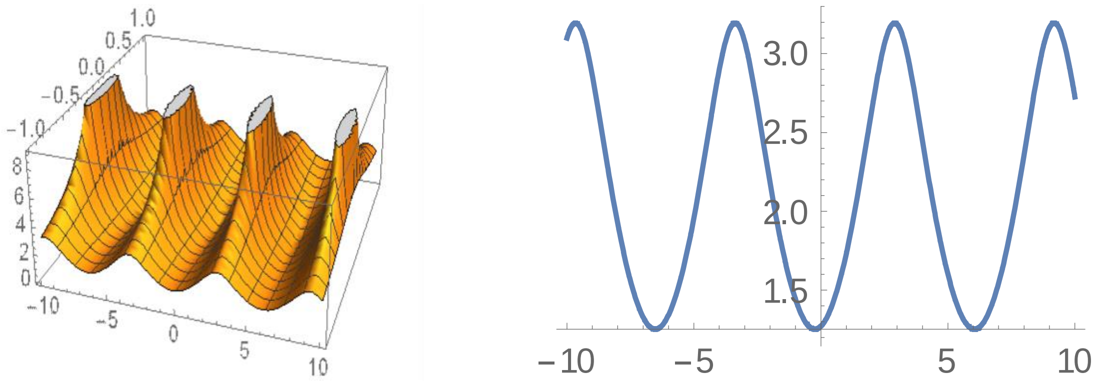
i) **ii)**

Figure 2. i) The 3D surfaces of Eq. (12) for $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$. **ii) The 2D surfaces of Eq. (12) for** $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$ **and** $t = 1$.



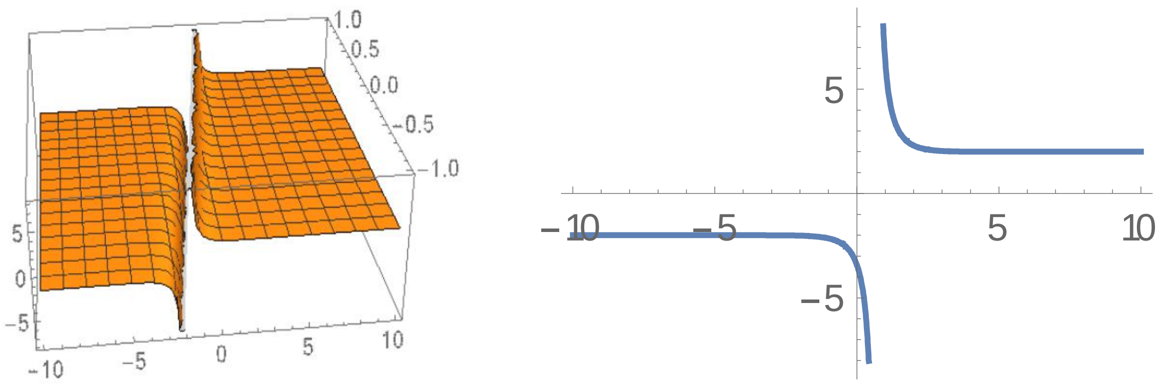
i) **ii)**

Figure 3. i) The 3D surfaces of Eq. (16) for $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$. **ii) The 2D surfaces of Eq. (16) for** $a_1 = 1, w = 3, \beta = 1, y = 1, z = 1$ **and** $t = 1$.



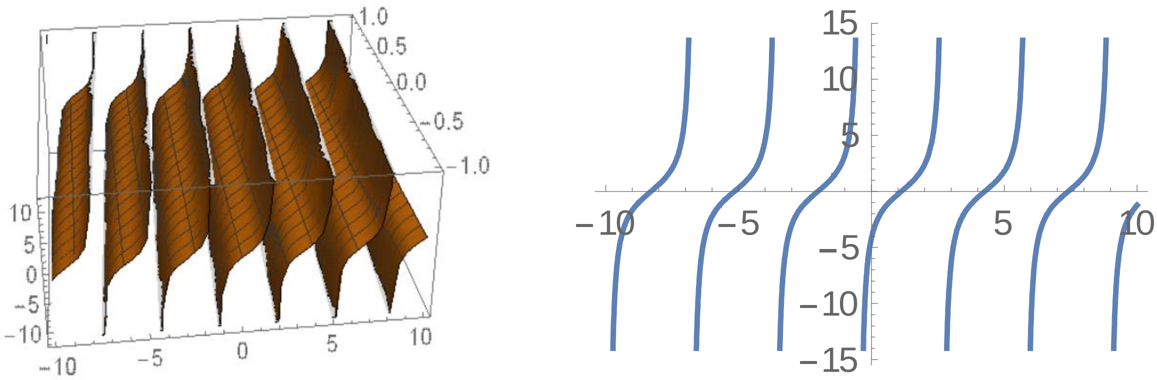
i) ii)

Figure 4. i) The 3D surfaces of Abs(Eq. (27)) for $a_1 = 2, y = 1, k = 1$. ii) The 2D surfaces of Abs(Eq. (27)) for $a_1 = 2, y = 1, k = 1$ and $t = 1$.



i) ii)

Figure 5. i) The 3D surfaces of Eq. (32) for $a_1 = 2, y = 1, k = 1$. ii) The 2D surfaces of Eq. (32) for $a_1 = 2, y = 1, k = 1$ and $t = 1$.



i) ii)

Figure 6. i) The 3D surfaces of Eq. (33) for $a_1 = 2, y = 1, k = 1$. ii) The 2D surfaces of Eq. (33) for $a_1 = 2, y = 1, k = 1$ and $t = 1$.

4. CONCLUSION

We used the improved tanh function method to find the exact solutions of the (3+1)-dimensional shallow water wave equation (SWWE) and (2+1)-dimensional fifth-order nonlinear equation. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

REFERENCES

[1] Debtnath, L., *Nonlinear Partial Differential Equations for Scientist and Engineers*, Birkhauser, Boston, MA, 1997.
[2] Wazwaz, A.M., *Partial Differential Equations: Methods and Applications*, Balkema, Rotterdam, 2002
[3] Shang, Y., *Applied Mathematics and Computation*, **187**, 1286, 2007.
[4] Abourabia, A.M., El Horbaty, M.M., *Chaos Solitons Fractals*, **29**, 354, 2006.
[5] Malfliet, W., *American Journal of Physics*, **60**, 650, 1992.
[6] Cariello, F., Tabor, M., *Physica*, D **39**, 77, 1989.
[7] Fan, E., *Physics Letters A*, **265**, 353, 2000.
[8] Clarkson, P.A., *Journal of Physics A: Mathematical and General*, **22**, 2355, 1989.
[9] Chen, H., Zhang, H., *Chaos Soliton Fract*, **19**, 71, 2004.
[10] Fu, Z., Liu, S., Zhao, Q., *Physics Letters A*, **290**, 72, 2001.
[11] Chen, Y., Wang, Q., Li, B., *Verlag der Zeitschrift f¨ur Naturforschung, T¨ubingen A*, **59**, 529, 2004.
[12] Chen, Y., Yan, Z., *Chaos Soliton Fract*, **29**, 948, 2006.
[13] Wang, M., Li, X., Zhang, J., *Physics Letters A*, **372**, 417, 2008.
[14] Li, L., Li, E., Wang, M., *Applied Mathematics-A J. Chin. U*, **25**, 454, 2010.
[15] Manafian, J., *Optik*, **127**, 4222, 2016.
[16] He, J.H., Wu, X.H., *Chaos Solitons Fractals*, **30**, 700, 2006.