ORIGINAL PAPER

THE TIMELIKE BEZIER SPLINE IN MINKOWSKI 3-SPACE

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Abstract. *The purpose of this study is to develop a Bezier spline in Minkowski 3 space called by the Timelike Bezier spline. In this paper firstly, we investigate the Frenet frame, curvatures and derivative formulations at the starting and end points of the Timelike Bezier Spline. Moreover, we obtain the derivative formulas of the Bishop frame and the curvatures according to the Bishop frame at starting and end points of the Timelike Bezier spline in Minkowski 3-space. Consequently we give some examples for this concept.*

Keywords: *Timelike Bezier spline, Frenet frame, Bishop frame, Minkowski 3-space*

1. INTRODUCTION

The term CAGD (Computer Aided Geometric Design) was coined by R.Barnhill and R. Riesenfeld [1] in 1974. Several kinds of splines have been developed in CAGD. One of them is Bezier splines. The original development of Bezier splines took place in the automobile industry during the period 1958-1960 by two Frenchmen, Pierre Bezier [2-4] at Renault and Paul de Casteljau [5] at Citröen. A large background of the history of Bezier splines can be found in the reference [6]. The Bezier splines are polynomial curves which have a popular and particular mathematical representation. Their popularity is due to the fact that they possess a number of mathematical properties which facilitate their manipulation and analysis, and yet no mathematical knowledge is required in order to use the curves [7]. In addition, it is also possible to increase the degree of the polynomial spline; for example, a curve segment can be split into two segments without changing the shape of the curve, or the degree of the curve can be formally increased also without changing its shape. That is to say, Bezier splines are in a form that is easy for a person to control, see in [6]. Some basic preliminaries and many papers about the Bezier spline can be found in the books [7-11]. Furthermore, the curvature, torsion and principal form of the Bezier spline have been study by Incesu and Gürsoy in [12]. The shapes of Plane and cubic Bezier spline have been published by Georgiev in [13, 14]. In 1907 the mathematican Hermann Minkowski explored Minkowski Space, introduced by his former student Albert Einstein in 1905 and based on the previous work Lorentz and Poincare. Minkowski realized that the special theory of relativity could best be understood in a four-dimensional space possessing a Minkowski metric having the form $g(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2 - x_3y_3$, where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}_1^3$. Some basic concepts about Minkowski space can be found in the references [15-17]. Recently, the curves have been begun to study in the Minkowski space. Differently from Euclidean space, in the

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Minkowski space the curves have different properties because of the Minkowski metric and inner product. Georgiev firstly presented an original paper related spacelike Bezier curves in the three-dimensional Minkowski space in 2008. In his work, he defined the spacelike Bezier spline and gave some sufficient conditions for these kinds of curves. Furthermore, he obtained the curvature and torsion of the spacelike Bezier spline, see in [18]. His other work was about the spacelike Bezier surfaces in the three-dimensional Minkowski space in 2009, [19]. Chalmoviansky, Pokorna and Barbora have studied quadratic and planar cubic spacelike Bezier splines in the three-dimensional Minkowski space. The authors obtained the spacelike conditions, the set of admissible points of contact, the boundary map and area of admissible solutions for quadratic and planar cubic Bezier splines in their works [20,21]. Also Ugail, Marquaez, Yılmaz in [22] published a study named with "On Bezier Surfaces in the three dimensional Minkowski space" in 2011. They obtained conditions of timelike and spacelike Bezier surfaces. Then they solved the Plateau-Bezier problem by obtaining conditions on the control points to be external of the Dirichlet functional and compared the area functionals for the minimal Bezier surface in Euclidean and Minkowski space. Furthermore, the equivalence problem for vectors in the two dimensional Minkowski spacetime and its application to Bezier splines was considered by Ören [23].

The purpose of our paper is to investigate the timelike Bezier splines in Minkowski space which is not studied before. The outline of this paper is as follows. In Section1, we first give basic notations of Minkowski space and Bezier splines in Euclidean space. In section 2, we define the timelike Bezier splines in type I and II. Then we obtain some conditions for timelike Bezir spline and introduce the timelike hodograph curve of the timelike Bezier splines. In section 3, we calculate the Serret-Frenet frame, derivative equations, curvature and torsion of the timelike Bezier splines at the starting point t=0. Then we investigate the Bishop frame and give the relation for Bishop [24] and Frenet frame at the starting point. In section 4, we work on the end point of the Bezier spline and give some calculations similarly in section 3. In Section 5, a numeric example is given finally in section 6, we conclude the work presented in this paper.

2. MATERIALS AND METHODS

The Type I Bezier splines are examined low-grade Bezier curves in $\mathbb{R}^2$. A Bezier spline with two control point $\mathbf{b_0}, \mathbf{b_1} \in \mathbb{R}^2$ is called a linear Bezier spline and denote by $\mathrm{b}(t) = (1-t)\mathbf{b_0} + t\,\mathbf{b_1}$. The spline is defined on the interval $[0,1]$, so the starting point of the curve is $\mathrm{B}(0) = b_0$ and finishing point is $\mathrm{B}(1) = b_1$, that is , the Bezier spline interpolates the first and last control points. If we suppose three control points $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2} \in \mathbb{R}^2$, then the Bezier spline is called quadratic Bezier spline in type I and denote by $\mathrm{b}(t) = (1-t)^2\,\mathbf{b_0} + 2(1-t)\,\mathrm{t}\,\mathbf{b_1} + t^2\mathbf{b_2}$ for $t \in [0,1]$. The triangle $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}$ obtained by joining the control points with line segments, in their prescribed order, is called ‘‘ control polygon ’’ or ‘‘convex hull’’. Suppose four control points $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} \in \mathbb{R}^2$, then the Bezier spline is called "cubic Bezier spline in type II" and denoted by $\mathrm{b}(t) = (1-t)^3\,\mathbf{b_0} + 3(1-t)^2\,\mathrm{t}\,\mathbf{b_1} + 3(1-t)t^2\mathbf{b_2} + t^3\mathbf{b_3}\quad t \in [0,1]$. Cubic Bezier splines provide a greater range of shapes than quadratic Bezier splines, since they can exhibit loops, sharp corners and inflections. Given n+1 control points $\mathbf{b_0}, \mathbf{b_1}, ..., \mathbf{b_n} \in \mathbb{R}^2$ the Bezier spline of degree n is defined to be $\mathrm{b}^{\mathrm{n}}(t)=\sum_{i=0}^{\mathrm{n}}\mathbf{b_i}.\mathrm{B}_i^{\mathrm{n}}(t)$, where $\mathrm{B}_i^n(t)=\binom{\mathrm{n}}{i}t^{\mathrm{i}}(1-t)^{\mathrm{n}-i}$ are called the Berstein polynomials or the Berstein basis functions of degree n. Suppose that the control points are taken in three dimensional, then the Bezier spline is called a spatial Bezier spline or Bezier spline in type II, [7-10].

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ provide with the Lorentzian inner product g given by $\mathrm{g}(\mathbf{x},\mathbf{y})=x_1y_1+x_2y_2-x_3y_3$, where $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{y}=(y_1,y_2,y_3)\in\mathbb{R}_1^3$. A vector $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}_1^3$ is called spacelike if $\mathrm{g}(\mathbf{x},\mathbf{x})>0$ or $\mathbf{x}=0$; timelike if $\mathrm{g}(\mathbf{x},\mathbf{x})<0$; lightlike if $\mathrm{g}(\mathbf{x},\mathbf{x})=0$ and $\mathbf{x}\neq 0$. Also a timelike vector $\mathbf{x}$ is called future pointing timelike vector if $\langle\mathbf{x},e\rangle<0$ with $e=(0,0,1)$. Similarly a timelike vector $\mathbf{x}$ is called past pointing timelike vector if $\langle\mathbf{x},e\rangle>0$.The vector $\mathbf{x}$ and $\mathbf{y}$ are ortogonal if and only if $\mathrm{g}(\mathbf{x},\mathbf{y})=0$. The norm of a vector $\mathbf{x}$ on Minkowski space $\mathbb{R}_1^3$ is defined by $\|\mathbf{x}\|_{IL}=\sqrt{|\mathrm{g}(\mathbf{x},\mathbf{x})|}$ .If the vector $\mathbf{x}$ is timelike because of $\mathrm{g}(\mathbf{x},\mathbf{x})<0$, the form will be $\|\mathbf{x}\|_L=\sqrt{-\mathrm{g}(\mathbf{x},\mathbf{x})}$. Also if the timelike vectors $\mathbf{x}$ and $\mathbf{y}$ is given,then the inner product can be shown as $g(\mathbf{x},\mathbf{y})=-\|\mathbf{x}\|\|\mathbf{y}\|\cosh\theta$, here θ is angle between these vectors. The cross product on $\mathbb{R}_1^3$ is given by $\mathbf{x}\wedge_{\mathrm{IL}}\mathbf{y}=(x_3y_2-x_2y_3,x_1y_3-x_3y_1,x_1y_2-x_2y_1)$ for $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}_1^3$. If the angle between $\mathbf{x}$ **a**nd $\mathbf{y}$ vector is θ, the cross product can be defined by $\|\mathbf{x}\wedge_{IL}\mathbf{y}\|=\|\mathbf{x}\|\|\mathbf{y}\|\sinh\theta$ for the timelike vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}_1^3$. We denote the coordinate functions on Minkowski space by

$$P_i:\mathbb{R}_1^3\to\mathbb{R},\ \mathbf{x}=(x_1,x_2,x_3)\to P_i(\mathbf{x})=x_i\quad i=1,2,3.$$

The Lorentzian and hyperbolic sphere of radius 1 in $\mathbb{R}_1^3$ are defined by

$$S_1^2=\left\{\mathbf{x}\in\mathbb{R}_1^3\middle|\langle\mathbf{x},\mathbf{x}\rangle=1\right\}\quad\text{and}\quad H_0^2=\left\{\mathbf{x}\in\mathbb{R}_1^3\middle|\langle\mathbf{x},\mathbf{x}\rangle=-1\right\},$$

respectively. Let α be curve in $\mathbb{R}_1^3$. We say that α is timelike curve (resp. spacelike, lightlike) at t if the tangent vector $\alpha'(t)$ is a timelike (resp. spacelike, lightlike) vector. We denote by $\{T,N,B\}$ the moving Serret-Frenet from along the curve α. Then $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are the tangent, the principal normal and the binormal vector of the curve α, respectively. If the α curve is time-like curve, then $\mathbf{T}$ is timelike vector, $\mathbf{N}$ and $\mathbf{B}$ are spacelike vectors which satisfy

$$\mathbf{T}\wedge_{IL}\mathbf{N}=-\mathbf{B},\quad \mathbf{N}\wedge_{IL}\mathbf{B}=\mathbf{T},\quad \mathbf{B}\wedge_{IL}\mathbf{T}=-\mathbf{N}. \tag{1}$$

The derivative of Serret-Frenet frame equations for a timelike curve is $\mathbf{T}'=\kappa v_1\mathbf{N}$, $\mathbf{N}'=\kappa v_1\mathbf{T}+\tau v_1\mathbf{B}$, $\mathbf{B}'=-\tau v_1\mathbf{N}$ where v_1 is the velocity of the curve. The curvature and the torsion functions for timelike curve is respectively,

$$\kappa=\frac{\left\|\frac{db^n(t)}{dt}\wedge_{II}\frac{d^2b^n(t)}{dt^2}\right\|_{II}}{\left\|\frac{db^n(t)}{dt}\right\|_{II}^3} \text{ and } \tau=\frac{\left(\frac{db^n(t)}{dt},\frac{d^2b^n(t)}{dt},\frac{d^3b^n(t)}{dt}\right)}{\left\|\frac{db^n(t)}{dt}\wedge_{II}\frac{d^2b^n(t)}{dt}\right\|^2}$$, [15,16,17].

The Bishop frames was initiated by L.Bishop in 1975 in order to create some new frames which have different advantages from Serret-Frenet frames. L.Bishop introduced the Bishop frame $\{T,N_1,N_2\}$ and the derivation equations of the Bishop frame as $\mathbf{T}'=k_1\mathbf{N}_1+k_2\mathbf{N}_2$, $\mathbf{N}_1'=-k_1\mathbf{T}$, $\mathbf{B}'=-k_2\mathbf{T}$, here $k_1=\kappa\cos\xi$ and $k_2=\kappa\sin\xi$ are the curvatures of the Bishop frame. Then, it is easy to see $\kappa=\sqrt{k_1^2+k_2^2}$, [24]. In our paper we will use the Bishop frame of the timelike curve in Minkowski three-space. In [25] the authors introduced the Bishop frame $\{T,N_1,N_2\}$ of the timelike curve and obtained the derivation equations as $\mathbf{T}'=k_1\mathbf{N}_1+k_2\mathbf{N}_2$, $\mathbf{N}_1'=k_1\mathbf{T}$, $\mathbf{B}'=k_2\mathbf{T}$ where $k_1=\kappa\cos\xi$ and $k_2=\kappa\sin\xi$ are the curvatures of the Bishop frame. Then, it is easy to see $\kappa=\sqrt{k_1^2+k_2^2}$, [25].

3. RESULTS AND DISCUSSION

Definition 3.1. A Bezier spline $\mathrm{b}(t)=(1-t)\mathbf{b_0}+t\,\mathbf{b_1}$ for $\mathbf{b_0},\mathbf{b_1}\in\mathbb{R}_1^2$ and $t\in[0,1]$ is called "a linear timelike Bezier spine", if the tangent vector s $b'(t)$ is timelike vector for all $t\in[0,1]$ in Minkowski space $\mathbb{R}_1^2$.

Definition 3.2. A timelike Bezier spline with $\mathbf{b_0},\mathbf{b_1},\mathbf{b_2}\in\mathbb{R}_1^2$

$$\mathrm{b}(t)=(1-t)^2\,\mathbf{b_0}+2(1-t)t\,\mathbf{b_1}+\mathrm{t}^2\mathbf{b_2}$$

is called ''quadratic timelike Bezier spline in type I'' for all t.

Definition 3.3. A timelike Bezier spline $\mathbf{b_0},\mathbf{b_1},\mathbf{b_2},\mathbf{b_3}\in\mathbb{R}_1^2$

$$\mathrm{b}(\mathrm{t})=(1-t)^3\,\mathbf{b_0}+3(1-t)^2\,\mathrm{t}\,\mathbf{b_1}+3(1-t)\mathrm{t}^2\mathbf{b_2}+t^3\mathbf{b_3}$$

for $\mathrm{t}\in[0,1]$ is called "a cubic timelike Bezier spline in type I". If the tangent vectors $b'(t)$ is future pointing timelike or past pointing timelike, then the timelike Bezier spline $b(t)$ is called "a future pointing timelike Bezier spline" or "a past pointing timelike Bezier spline", respectively.

Definition 3.4. Let $\mathbf{b_0},\mathbf{b_1},...,\mathbf{b_n}\in\mathbb{R}_1^3$ be points in the same cone and $\mathrm{b}^{\mathrm{n}}(t):[0,1]\to\mathbb{R}_1^3$ be Bezier curve given by parametric equation $\mathrm{b}^n(t)=\sum_{\mathrm{i}=0}^{n}\mathbf{b_i}.\mathrm{B}_{\mathrm{i}}^n(t)$. For $\forall\mathrm{t}\in[0,1]$ the tangent vectors $\dfrac{db^n(t)}{dt}$ of the Bezier curve $b^n(t)$ are timelike vectors then the the curve $b^n(t)$ is called timelike Bezier spline.

Let determine the r.th degree derivative of the timelike Bezier spline $b^n(t)$ as

$$\frac{\mathrm{d}^r b^n}{\mathrm{d}t^r}(\mathrm{t}) = \frac{n!}{(n-r)!}\sum_{\mathrm{i}=0}^{n-r} \mathbf{\Delta^r b_i}.\mathrm{B}_{\mathrm{i}}^{\mathrm{n-r}}(\mathrm{t}) \tag{2}$$

and the $\mathbf{\Delta^r b_i}$ vector is in the form of $\mathbf{\Delta^r b_i} = \sum_{\mathrm{i}=0}^{\mathrm{r}} \binom{\mathrm{r}}{\mathrm{j}} (-1)^{\mathrm{r-j}} \mathrm{b}_{\mathrm{i+j}}$ and where for control points $\mathbf{\Delta^r b_j = \Delta^{r-1} b_{j+1} - \Delta^{r-1} b_j}$. $\mathbf{b_i} \in \mathbb{R}_1^3$, $\mathrm{i} = 0,1,...,n$, the timelike Bezier spline is in the convex hull. The function $s = \int_0^l \left\| \dfrac{db^n}{dt} \right\| dt$ is called the arc length function of the regular timelike Bezier spline $b^n(t)$.

Theorem 3.5. Let $b^n(t)$ is timelike Bezier spline. The tangent vector at the begining point $\mathbf{b_0}$ and the tangent vector at the end point $\mathbf{\Delta' b_{n-1}}$ are timelike vectors.

Proof: Since $b^n(t)$ is timelike Bezier spline $g\left(\dfrac{db^n(t)}{dt}, \dfrac{db^n(t)}{dt}\right) < 0$ for $\forall t \in [0,1]$. Then $\Delta \mathrm{b}_0$ is timelike vector. At the starting point $t = 0$ from the Eq. (1), $\left.\dfrac{db^n(t)}{dt}\right|_{t=0} = n.\Delta b_0$. Thus $g(n.\Delta \mathrm{b}_0, n.\Delta \mathrm{b}_0) < 0$ $\mathrm{n} \in \mathbb{R}$ and $\Delta \mathrm{b}_0$ is timelike vector. Similarly at the end point t=1, the first derivative of $b^n(t)$ is $\left.\dfrac{\mathrm{db^n}(t)}{\mathrm{dt}}\right|_{\mathrm{t}=1} = \mathrm{n}.\Delta \mathrm{b}_{\mathrm{n}-1}$. Then

$$g(n\Delta b_{n-1}, n\Delta b_{n-1}) = n^2 g(\Delta b_{n-1}, \Delta b_{n-1}) < 0.$$

So we obtain that $\Delta \mathrm{b}_{\mathrm{n}-1}$ tangent vector at the end point is timelike vector.

Theorem 3.6. Let $b^n(t)$ be Bezier spline. If all vectors $\mathbf{\Delta' b_0 = b_1 - b_0}$, $\mathbf{\Delta' b_1 = b_2 - b_1}$, **...** , $\mathbf{\Delta' b_n = b_n - b_{n-1}}$ are timelike vectors, they satisfy the condition

$$\mathrm{P}_3^2(\Delta' b_\mathrm{i}) > \mathrm{P}_1^2(\Delta' b_\mathrm{i}) + \mathrm{P}_2^2(\Delta' b_\mathrm{i}) \quad \text{for} \quad \mathrm{i} = 0,1,...,\mathrm{n}-1.$$

Proof: If $\Delta' b_\mathrm{i}$ are timelike vectors, then they satisfy $g(\Delta' b_\mathrm{i}, \Delta' b_\mathrm{i}) < 0$. Thus

$$g\Big(\big(\mathrm{P}_1(\Delta' \mathrm{b}_\mathrm{i}), \mathrm{P}_2(\Delta' \mathrm{b}_\mathrm{i}), \mathrm{P}_3(\Delta' b_\mathrm{i})\big), \big(\mathrm{P}_1(\Delta' b_\mathrm{i}), \mathrm{P}_2(\Delta' b_\mathrm{i}), \mathrm{P}_3(\Delta' b_\mathrm{i})\big)\Big) < 0$$

$$\mathrm{P}_1^2\left(\Delta' b_{\mathrm{i}}\right)+\mathrm{P}_2^2\left(\Delta' b_{\mathrm{i}}\right)-\mathrm{P}_3^2\left(\Delta' b_{\mathrm{i}}\right)<0$$
$$\mathrm{P}_3^2\left(\Delta' b_{\mathrm{i}}\right)>\mathrm{P}_1^2\left(\Delta' b_{\mathrm{i}}\right)+\mathrm{P}_2^2\left(\Delta' b_{\mathrm{i}}\right) \quad \text{for} \quad \mathrm{i}=0,1,\ldots,\mathrm{n}-1 \ .$$

Theorem 3.7. Let $b^n(t)$ be a Bezier spline. If all vectors of convex polygon are timelike then $b^n(t)$ is timelike.

Proof: The timelike vectors $\Delta' b_{\mathrm{i}}=\left(\mathrm{P}_1\left(\Delta' b_i\right),\mathrm{P}_2\left(\Delta' b_i\right),\mathrm{P}_3\left(\Delta' b_i\right)\right)$ and

$$\Delta' b_j=\left(\mathrm{P}_1\left(\Delta' b_j\right),\mathrm{P}_2\left(\Delta' b_j\right),\mathrm{P}_3\left(\Delta' b_j\right)\right)$$

for $i\neq j$ satisfy the conditions

$\mathrm{P}_3^2\left(\Delta' b_{\mathrm{i}}\right)>\mathrm{P}_1^2\left(\Delta' b_{\mathrm{i}}\right)+\mathrm{P}_2^2\left(\Delta' b_{\mathrm{i}}\right)$ and $\mathrm{P}_3^2\left(\Delta' b_{\mathrm{i}}\right)>\mathrm{P}_1^2\left(\Delta' b_{\mathrm{i}}\right)+\mathrm{P}_2^2\left(\Delta' b_{\mathrm{i}}\right)$.

$$\left|\mathrm{P}_3\left(\Delta' b_{\mathrm{i}}\right)\right|.\left|\mathrm{P}_3\left(\Delta' b_j\right)\right|>\sqrt{\left(\mathrm{P}_1^2\left(\Delta' b_{\mathrm{i}}\right)+\mathrm{P}_2^2\left(\Delta' b_i\right)\right).\left(\mathrm{P}_1^2\left(\Delta' b_j\right)+\mathrm{P}_2^2\left(\Delta' b_j\right)\right)}$$

$$>\sqrt{\left(\mathrm{P}_1\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_1\left(\Delta' b_j\right)+\mathrm{P}_2\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_2\left(\Delta' b_j\right)\right)^2+\left(\mathrm{P}_1\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_2\left(\Delta' b_j\right)\text{-}\mathrm{P}_2\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_1\left(\Delta' b_j\right)\right)^2}$$

$$>\left|\mathrm{P}_1\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_1\left(\Delta' b_j\right)+\mathrm{P}_2\left(\Delta' b_{\mathrm{i}}\right).\mathrm{P}_2\left(\Delta' b_j\right)\right|$$

From here it follows that $g\left(\dfrac{db^n(t)}{dt},\dfrac{db^n(t)}{dt}\right)<0$ for any $t\in[0,1]$.

Definition 3.8. Let $\mathbf{b_0},\mathbf{b_1},\ldots,\mathbf{b_{n\text{-}1}}\in\mathbb{R}_1^3$ and $\mathrm{b}^{\mathrm{n}}(\mathrm{t})$ is timelike Bezier spline. The first order derivative curve of $\mathrm{b}^{\mathrm{n}}(\mathrm{t})$ is called hodograf. If all $\Delta\mathbf{b}_{\mathrm{i}}$ are timelike vectors in the same cone then the hodograf curve of $\mathrm{b}^{\mathrm{n}}(\mathrm{t})$ is timelike curve in the same cone. We say this curve timelike hodograf curve. The equation of hodograf curve is $\dfrac{\mathrm{d}}{\mathrm{dt}}\mathrm{b}^{\mathrm{n}}(\mathrm{t})=\mathrm{n}\sum\limits_{\mathrm{j}=0}^{\mathrm{n}-1}\Delta\mathbf{b}_{\mathrm{j}}.\mathrm{B}_{\mathrm{j}}^{\mathrm{n}-1}(\mathrm{t})$, $\Delta\mathbf{b}_{\mathrm{j}}\in\mathbb{R}_1^3$ where $\Delta\mathbf{b}_{\mathrm{j}}=\mathbf{b}_{\mathrm{j+1}}-\mathbf{b}_{\mathrm{j}}$.

Let $\mathrm{b}^{\mathrm{n}}(\mathrm{t})$ be an arbitrary timelike Bezier spline and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}\big|_{\mathrm{t}=0}$ be the Serret-Frenet frame of $\mathrm{b}^{\mathrm{n}}(\mathrm{t})$ at the beginning point $t=0$, where $\mathbf{T}$ is timelike , $\mathbf{N}$ and $\mathbf{B}$ is spacelike. Then $g(\mathbf{T},\mathbf{T})=-1$, $g(\mathbf{N},\mathbf{N})=1$, $g(\mathbf{B},\mathbf{B})=1$ and $g(\mathbf{T},\mathbf{N})=0$, $g(\mathbf{T},\mathbf{B})=0$, $g(\mathbf{N},\mathbf{B})=0$. The speed function of timelike Bezier spline $b^n(t)$ is defined by

$$v=\left\|\frac{db^n(t)}{dt}\right\|_{IL}=\sqrt{-g\left(\frac{db^n(t)}{dt},\frac{db^n(t)}{dt}\right)}.$$

Moreover the arc length function of timelike Bezier spline $b^n(t)$ is defined by

$$s=\int_{t_0}^{t_1}\sqrt{-g\left(\frac{db^n(t)}{dt},\frac{db^n(t)}{dt}\right)}dt .$$

In the next section we obtain the Serret-Frenet frame of $b^n(t)$ timelike Bezier spline at the beginning point t=0 and at the end point t=1.

Theorem 3.9. Let $b^n(t)$ be timelike Bezier spline and $\mathbf{b}_i\in E_1^3$ are control points. If $\Delta\mathbf{b}_i$ are in the same cone then at the starting point t=0, the Serret Frenet frame of $b^n(t)$ is

$$\mathbf{T}\big|_{t=0}=\frac{\Delta\mathbf{b}_0}{\left\|\Delta\mathbf{b}_0\right\|_{IL}}=\frac{\Delta\mathbf{b}_0}{\sqrt{-g\left(\Delta\mathbf{b}_0,\Delta\mathbf{b}_0\right)}},\qquad \mathbf{N}\big|_{t=0}=-\frac{\Delta\mathbf{b}_0}{\left\|\Delta\mathbf{b}_0\right\|_{IL}}\coth\theta-\frac{\Delta\mathbf{b}_1}{\left\|\Delta\mathbf{b}_1\right\|_{IL}}\operatorname{csc}h\theta$$

$$\mathbf{B}\big|_{t=0}=\frac{\Delta\mathbf{b}_0\wedge_L\Delta\mathbf{b}_1}{\left\|\Delta\mathbf{b}_0\wedge_L\Delta\mathbf{b}_1\right\|}=\frac{\Delta\mathbf{b}_0\wedge_L\Delta\mathbf{b}_1}{\left\|\Delta\mathbf{b}_0\right\|.\left\|\Delta\mathbf{b}_1\right\|\sinh\theta}.$$

where θ is the angle between $\Delta\mathbf{b}_0$ and $\Delta\mathbf{b}_1$ vectors.

Proof: Since at the beginning point $t=0$ the vectors $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b}_1$ are timelike vectors and $\left\|\Delta\mathbf{b_0}\right\|_{IL}=\sqrt{-g\left(\Delta\mathbf{b_0},\Delta\mathbf{b_0}\right)}$, the tangent vector is $\mathbf{T}\big|_{t=0}=\frac{\Delta\mathbf{b}_0}{\left\|\Delta\mathbf{b}_0\right\|_{L}}=\frac{\Delta\mathbf{b}_0}{\sqrt{-g\left(\Delta\mathbf{b}_0,\Delta\mathbf{b}_0\right)}}$. Now, calculate the binormal and principal normal vector of arbitrary timelike Bezier spline $b^n(t)$ at the starting point. From the Serret –Frenet frame the binormal vector field is calculated by

$$B\big|_{t=0}=\left.\frac{\dfrac{db^n(t)}{dt}\wedge_L\dfrac{d^2b^n(t)}{dt^2}}{\left\|\dfrac{db^n(t)}{dt}\wedge_L\dfrac{d^2b^n(t)}{dt^2}\right\|_{IL}}\right|_{t=0}=\frac{n\Delta\mathbf{b}_0\wedge_{IL}\left[n.(n-1)\left\{\Delta\mathbf{b}_0-\Delta\mathbf{b}_0\right\}\right]}{\left\|n\Delta\mathbf{b}_0\wedge_{IL}\left[n.(n-1)\left\{\Delta\mathbf{b}_1-\Delta\mathbf{b}_0\right\}\right]\right\|}=\frac{\Delta\mathbf{b}_0\wedge_{IL}\Delta\mathbf{b}_1}{\left\|\Delta\mathbf{b}_0\right\|\left\|\Delta\mathbf{b}_1\right\|\sinh\theta}$$

and the principal normal is obtained by

$$\begin{aligned}N\,|_{t=0}&=-B\,|_{t=0}\wedge_{IL}T\,|_{t=0}\\&=-\left(\frac{-g\left(\Delta\mathbf{b}_0,\Delta\mathbf{b}_0\right).\Delta\mathbf{b}_1+g\left(\Delta\mathbf{b}_1,\Delta\mathbf{b}_0\right).\Delta\mathbf{b}_0}{\left\|\Delta\mathbf{b}_0\right\|_{IL}.\left\|\Delta\mathbf{b}_1\right\|_{IL}\sinh\theta.\left\|\Delta\mathbf{b}_0\right\|_{IL}}\right)=\frac{\Delta\mathbf{b}_0}{\left\|\Delta\mathbf{b}_0\right\|}\coth\theta-\frac{\Delta\mathbf{b}_1}{\left\|\Delta\mathbf{b}_1\right\|}\operatorname{csc}h\theta\end{aligned}$$

Theorem 3.10. Let $b^n(t)$ be n degree timelike Bezier spline and the control points $b_0, b_1, ..., b_n$ and Δb_i of $b^n(t)$ are timelike vectors. Then at the starting point t=0 the curvature and the torsion function of $b^n(t)$ are respectively,

$$\kappa|_{t=0} = \frac{n-1}{n}\frac{\|\Delta \mathbf{b}_1\|}{\|\Delta \mathbf{b}_0\|^2}.\sinh\theta \quad \text{ve} \qquad \tau|_{t=0} = -\frac{n-2}{n}\frac{\det(\Delta \mathbf{b}_0, \Delta \mathbf{b}_1, \Delta \mathbf{b}_2)}{\|\Delta \mathbf{b}_0 \wedge_{IL} \Delta \mathbf{b}_1\|^2}. \tag{3}$$

Proof: The curvature of $b^n(t)$ at the starting point t=0 is

$$\kappa|_{t=0} = \left.\frac{\left\|\dfrac{db^n(t)}{dt} \wedge_{IL} \dfrac{d^2b^n(t)}{dt^2}\right\|_{IL}}{\left\|\dfrac{db^n(t)}{dt}\right\|_{IL}^3}\right|_{t=0} = \frac{n-1}{n}\frac{\|\Delta \mathbf{b}_0 \wedge_{IL} (\Delta \mathbf{b}_1 - \Delta \mathbf{b}_0)\|_{IL}}{\|\Delta \mathbf{b}_0\|_{IL}^3} = \frac{n-1}{n}\frac{\|\Delta \mathbf{b}_1\|}{\|\Delta \mathbf{b}_0\|^2}.\sinh\theta$$

and the torsion of $b^n(t)$ at the starting point t=0 is

$$\tau|_{t=0} = \left.\frac{\left(\dfrac{db^n(t)}{dt} \quad \dfrac{d^2b^n(t)}{dt^2} \quad \dfrac{d^3b^n(t)}{dt^3}\right)}{\left\|\dfrac{db^n(t)}{dt} \wedge_{IL} \dfrac{d^2b^n(t)}{dt^2}\right\|^2}\right|_{t=0}$$

$$= \frac{n-2}{n}\frac{g\left(\Delta \mathbf{b}_0 \wedge_{IL} (\Delta \mathbf{b}_1 - \Delta \mathbf{b}_0), (\Delta \mathbf{b}_2 - 2\Delta \mathbf{b}_1 + \Delta \mathbf{b}_0)\right)}{\|\Delta \mathbf{b}_0 \wedge_{IL} (\Delta \mathbf{b}_1 - \Delta \mathbf{b}_0)\|_{IL}^2} = -\frac{n-2}{n}\frac{\det(\Delta \mathbf{b}_0, \Delta \mathbf{b}_1, \Delta \mathbf{b}_2)}{\|\Delta \mathbf{b}_0 \wedge_{IL} \Delta \mathbf{b}_1\|_{IL}^2}.$$

Theorem 3.11. Let $b^n(t)$ be timelike Bezier spline and $b_i, \Delta b_i \in E_1^3$ are timelike vectors. Then at the starting point t=0 the Serret Frenet derivative equations are:

$$\mathbf{T}'|_{t=0} = (n-1)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}.\sinh\theta.\mathbf{N}|_{t=0}$$

$$\mathbf{N}'|_{t=0} = (n-1)\frac{\|\Delta \mathbf{b}_1\|}{\|\Delta \mathbf{b}_0\|}.\sinh\theta.\mathbf{T}|_{t=0} - (n-2)\|\Delta \mathbf{b}_0\|\frac{\det(\Delta \mathbf{b}_0, \Delta \mathbf{b}_1, \Delta \mathbf{b}_2)}{\|\Delta \mathbf{b}_0 \wedge \Delta \mathbf{b}_1\|_{IL}^2}.\mathbf{B}|_{t=0}$$

$$\mathbf{B}'|_{t=0} = (n-2)\|\Delta b_0\|\frac{\det(\Delta b_0, \Delta \mathrm{b}_1, \Delta \mathrm{b}_2)}{\|\Delta b_0 \wedge \Delta \mathrm{b}_1\|_{IL}^2}.\mathbf{N}|_{t=0}$$

where θ is the angle between $\Delta \mathbf{b}_0$ and $\Delta \mathbf{b}_1$ vectors.

Proof: Using Eq. (1) and Eq.(3), the Serret-Frenet derivative equations for timelike curve is computed by

$$\mathbf{T}'|_{t=0}=\kappa|_{t=0}\ v_1.\mathbf{N}|_{t=0}=\frac{n-1}{n}\frac{\|\Delta\mathbf{b}_1\|}{\|\Delta\mathbf{b}_0\|^2}.\sinh\theta.\ n\|\Delta\mathbf{b}_0\|\mathbf{N}|_{t=0}=(n-1)\frac{\|\Delta b_1\|}{\|\Delta b_0\|}.\sinh\theta.\mathbf{N}|_{t=0}$$

$$\mathbf{N}'|_{t=0}=\kappa v_1\mathbf{T}|_{t=0}+\tau v_1\mathbf{B}|_{t=0}$$

$$=(n-1)\frac{\|\Delta\mathbf{b}_1\|}{\|\Delta\mathbf{b}_0\|}.\sinh\theta.\mathbf{T}|_{t=0}-(n-2)\|\Delta\mathbf{b}_0\|\frac{\det(\Delta\mathbf{b}_0,\Delta\mathbf{b}_1,\Delta\mathbf{b}_2)}{\|\Delta\mathbf{b}_0\wedge\Delta\mathbf{b}_1\|_{IL}^2}.\mathbf{B}|_{t=0}$$

$$\mathbf{B}'|_{t=0}=-\tau v_1\mathbf{N}|_{t=0}=-\left(-\frac{n-2}{n}\frac{(\Delta\mathbf{b}_0,\Delta\mathbf{b}_1,\Delta\mathbf{b}_2)}{\|\Delta\mathbf{b}_0\wedge_{IL}\Delta\mathbf{b}_1\|_{IL}^2}\right).n\|\Delta\mathbf{b}_0\|.\mathbf{N}|_{t=0}$$

$$=(n-2)\|\Delta b_0\|\frac{\det(\Delta b_0,\Delta \mathrm{b}_1,\Delta \mathrm{b}_2)}{\|\Delta b_0\wedge\Delta \mathrm{b}_1\|_{IL}^2}.\mathbf{N}|_{t=0}$$

where $v_1=n\|\mathbf{b}_1-\mathbf{b}_0\|=n\|\Delta\mathbf{b}_0\|$.

Theorem 3.12. Let $b^n(t)$ be timelike Bezier spline. Then at the starting point t=0 the curvatures of the Bishop frame for $b^n(t)$ are

$$k_1\big|_{t=0}=\kappa\cos\phi=\frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\cos\phi\,,$$

$$k_2\big|_{t=0}=\kappa\sin\phi=\frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\sin\phi \tag{4}$$

where θ is the angle between $\Delta\mathbf{b}_0$ and $\Delta\mathbf{b}_1$ vectors.

Proof: Since the Bishop frame $\{T|_{t=0},N_1|_{t=0},N_2|_{t=0}\}$ at the begining point $t=0$ is an orthonormal, the Minkowski inner products of the timelike Bishop elements are defined as

$$<T|_{t=0},T|_{t=0}>=-1\quad <N_1|_{t=0},N_1|_{t=0}>=1\quad <N_2|_{t=0},N_2|_{t=0}>=1$$

$$<T|_{t=0},N_1|_{t=0}>=0\quad <T|_{t=0},N_2|_{t=0}>=0\quad <N_1|_{t=0},N_2|_{t=0}>=0\,.$$

because of T is a timelike vector, N_1 and N_2 are spacelike vectors. Now, let us examine the curvatures of the Bishop frame and their derivative equations at the starting and ending point of the timelike Bezier spline $b^n(t)$. The curvature and torsion function at the start point $t=0$ of the timelike Bezier spline $b^n(t)$ can be given with the equations

$$k_1\big|_{t=0} = \kappa\cos\phi = \frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\cos\phi\,,\ k_2\big|_{t=0} = \kappa\sin\phi = \frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\sin\phi$$

where θ is the angle between $\Delta\mathbf{b}_0$ and $\Delta\mathbf{b}_1$ vectors. We see that here $\tan\phi = \frac{k_2}{k_1}\big|_{t=0}$ and $\phi = \arctan\phi = \left(\frac{k_2}{k_1}\big|_{t=0}\right)$. Also the angle ϕ is calculated from the equation

$$\phi' = \tau\bigg|_{t=0} = -\frac{n-2}{n}\frac{\det(\Delta b_0, \Delta b_1, \Delta b_2)}{\|\Delta b_0 \wedge b_1\|^2}\,.$$

Theorem 3.13. Let $b^n(t)$ be timelike Bezier spline. Then at the starting point t=0 the derivation formulas of the Bishop frame for $b^n(t)$ are

$$\mathbf{T}'\,|_{t=0} = (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\left[\cos\psi.\mathbf{N_1}\,|_{t=0} + \sin\psi.\mathbf{N_2}\,|_{t=0}\right]$$

$$\mathbf{N}_1'\,|_{t=0} = (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\cos\psi.\mathbf{T}\,|_{t=0}$$

$$\mathbf{N}_2'\,|_{t=0} = (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\sin\psi.\mathbf{T}\,|_{t=0}\,.$$

Proof: Using Eq. (4) and the constant number v_1 are substituted in the derivation equations for the Bishop frame of the timelike spline, the derivation formulas of the Bishop frame for the timelike Bezier spline at the starting point are obtained as:

$$\mathbf{T}'\,|_{t=0} = k_1\,|_{t=0}\ v_1.\mathbf{N_1}\,|_{t=0} + k_2\,|_{t=0}\ v_1.\mathbf{N_2}\,|_{t=0}$$

$$= (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\cos\psi.\mathbf{N_1}\,|_{t=0} + (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\sin\psi.\mathbf{N_2}\,|_{t=0}$$

$$= (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\left[\cos\psi.\mathbf{N_1} + \sin\psi.\mathbf{N_2}\right]$$

$$\mathbf{N}_1'\,|_{t=0} = k_1\,|_{t=0}\ v_1.\mathbf{T}\,|_{t=0} = \frac{(n-1)}{n}\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|^2}\sinh\theta\cos\psi.n\|\ \Delta b_0\|\ .\mathbf{T}\,|_{t=0}$$

$$= (n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\cos\psi.\mathbf{T}\,|_{t=0}$$

$$\mathbf{N}_2'\,|_{t=0} = k_2\,|_{t=0}\; v_1.\mathbf{T}\,|_{t=0} = \frac{(n-1)}{n}\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|^{\,2}}\sinh\theta\sin\psi.n\|\ \Delta b_0\|\ .\mathbf{T}\,|_{t=0}$$

$$=(n-1)\frac{\|\ \Delta b_1\|}{\|\ \Delta b_0\|}\sinh\theta\sin\psi.\mathbf{T}\,|_{t=0}$$

where $v_1 = n.\|\ \mathbf{b}_1 - \mathbf{b}_0\ \| = n.\left\|\Delta\mathbf{b}_0\right\|$.

Theorem 3.14. Let $b^n(t)$ be timelike Bezier spline and $\mathbf{b}_i \in E_1^3$ are control points. If $\Delta\mathbf{b}_{\mathrm{i}}$ are in the same cone then at the end point t=1 the Serret Frenet frame $\left\{\mathbf{T}\,|_{t=1}, \mathbf{N}\,|_{t=1}, \mathbf{B}\,|_{t=1}\right\}$ of $b^n(t)$ is

$$\mathbf{T}\big|_{t=1} = \frac{\Delta\mathbf{b}_{n-1}}{\left\|\Delta\mathbf{b}_{n-1}\right\|_{IL}} = \frac{\Delta\mathbf{b}_{n-1}}{\sqrt{-g\left(\Delta\mathbf{b}_{n-1}, \Delta\mathbf{b}_{n-1}\right)}}$$

$$\mathbf{N}\,|_{t=1} = \frac{\Delta\mathbf{b}_{n-2}}{\left\|\Delta\mathbf{b}_{n-2}\right\|}\csc h\varphi - \frac{\Delta\mathbf{b}_{n-1}}{\left\|\Delta\mathbf{b}_{n-1}\right\|}\coth\varphi$$

$$\mathbf{B}\,|_{t=1} = -\frac{\Delta\mathbf{b}_{n-1} \wedge_{IL} \Delta\mathbf{b}_{n-2}}{\left\|\Delta\mathbf{b}_{n-1}\right\|.\left\|\Delta\mathbf{b}_{n-2}\right\|\sinh\varphi}$$

where φ is the angle between Δb_{n-2} ve Δb_{n-1} vectors.

Proof: Let $b^n(t)$ be timelike Bezier spline and $\Delta\mathbf{b}_i \in \mathbb{R}_1^3$ are timelike vectors. Since at the end point $t=1$ the vector Δb_{n-2} and $\Delta\mathbf{b}_{n-1}$ are timelike vectors, the properties

$$\left\|\Delta\mathbf{b}_{n-1}\right\|_{IL}^2 = -g\left(\Delta\mathbf{b}_{n-1}, \Delta\mathbf{b}_{n-1}\right)$$

$$g(\Delta b_{n-2}, \Delta b_{n-1}) = -\left\|b_{n-2}\right\|\left\|b_{n-1}\right\|\cosh\varphi$$

$$\left\|\Delta b_{n-2} \wedge \Delta b_{n-1}\right\| = \left\|b_{n-2}\right\|\left\|b_{n-1}\right\|\sinh\varphi$$

are satisfied, where φ is the angle between Δb_{n-2} ve Δb_{n-1} vectors. Then the tangent vector is

$$T\big|_{t=1} = \frac{\dfrac{db^n(t)}{dt}}{\left\|\dfrac{db^n(t)}{dt}\right\|_{IL}} = \frac{\Delta b_{n-1}}{\sqrt{-g\left(\Delta b_{n-1}, \Delta b_{n-1}\right)}}$$

The binormal vector of arbitrary timelike Bezier spline $b^n(t)$ at the end point t=1 is calculated as

$$B\big|_{t=1} = \left.\frac{\dfrac{db^n(t)}{dt}\wedge_{IL}\dfrac{d^2b^n(t)}{dt^2}}{\left\|\dfrac{db^n(t)}{dt}\wedge_{IL}\dfrac{d^2b^n(t)}{dt^2}\right\|_{IL}}\right|_{t=1} = \frac{-\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}}{\left\|\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}\right\|} = -\frac{\Delta b_{n-1}\wedge\Delta b_{n-2}}{\left\|\Delta b_{n-1}\right\|.\left\|\Delta b_{n-2}\right\|\sinh\varphi}$$

and the normal vector is

$$N\,|_{t=1} = -B\,|_{t=1}\wedge_{IL} T\,|_{t=1} = \left(\frac{-g\left(\Delta b_{n-1},\Delta b_{n-1}\right).\Delta b_{n-2} + g\left(\Delta b_{n-2},\Delta b_{n-1}\right)\Delta b_{n-1}}{\left\|\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}\right\|_{IL}.\left\|\Delta b_{n-1}\right\|_{IL}}\right)$$

$$= \frac{\Delta b_{n-2}}{\left\|\Delta b_{n-2}\right\|}\operatorname{csc}h\varphi - \frac{\Delta b_{n-1}}{\left\|\Delta b_{n-1}\right\|}\coth\varphi$$

from the Serret-Frenet derivative in Eq (1).

Theorem 3.15. Let $b^n(t)$ be n degree timelike Bezier spline and the control points $b_0, b_1, ..., b_n$ and Δb_i of $b^n(t)$ are timelike vectors. Then at the end point $t=1$ the curvature and the torsion function of $b^n(t)$ are respectively,

$$\kappa\,|_{t=1} = \frac{n-1}{n}\frac{\left\|\Delta b_{n-2}\right\|}{\left\|\Delta b_{n-1}\right\|_{IL}^2}.\sinh\psi\ , \qquad \tau\,|_{t=1} = \frac{n-2}{n}\frac{\det\left(\Delta b_{n-1},\Delta b_{n-2},\Delta b_{n-3}\right)}{\left\|\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}\right\|^2}\ . \tag{5}$$

Proof: The curvature of $b^n(t)$ at the starting point t=1 is

$$\kappa\big|_{t=1} = \left.\frac{\left\|\dfrac{db^n(t)}{dt}\wedge\dfrac{d^2b^n(t)}{dt^2}\right\|}{\left\|\dfrac{db^n(t)}{dt}\right\|^3}\right|_{t=1} = \frac{(n-1)}{n}\frac{\left\|\Delta b_{n-1}\wedge_{IL}\left[\Delta b_{n-1}-\Delta b_{n-2}\right]\right\|}{\left\|\Delta b_{n-1}\right\|^3} = \frac{n-1}{n}\frac{\left\|\Delta b_{n-2}\right\|}{\left\|\Delta b_{n-1}\right\|^2}\sinh\varphi$$

and the torsion of $b^n(t)$ at the end point t=1 is

$$\tau\,|_{t=1} = \left.\frac{\left(\dfrac{db^n(t)}{dt}\dfrac{d^2b^n(t)}{dt^2}\dfrac{d^3b^n(t)}{dt^3}\right)}{\left\|\dfrac{db^n(t)}{dt}\wedge_{IL}\dfrac{d^2b^n(t)}{dt^2}\right\|^2}\right|_{t=1} = -\frac{n-2}{n}\frac{g\left(\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2},\ \Delta b_{n-3}\right)}{\left\|\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}\right\|_{IL}^2}$$

$$= \frac{n-2}{n}\frac{\det\left(\Delta b_{n-1},\Delta b_{n-2},\ \Delta b_{n-3}\right)}{\left\|\Delta b_{n-1}\wedge_{IL}\Delta b_{n-2}\right\|_{IL}^2}.$$

Theorem 3.16. Let $b^n(t)$ be timelike Bezier spline and $b_i, \Delta b_i \in E_1^3$ are timelike vectors. Then at the end point t=1 the Serret Frenet derivative equations are:

$$\mathbf{T}'|_{t=1} = (n-1)\frac{\| \Delta b_{n-2} \|}{\| \Delta b_{n-1} \|}\sinh\varphi.\mathbf{N}|_{t=1}$$

$$\mathbf{N}'|_{t=1} = (n-1)\frac{\| \Delta b_{n-2} \|}{\| \Delta b_{n-1} \|}\sinh\varphi.\mathbf{T}|_{t=1} + (n-2)\| \Delta b_{n-1} \| \frac{\det(\Delta b_{n-1}, \Delta \mathrm{b}_{\text{n-2}}, \Delta b_{n-3})}{\left\| \Delta b_{n-1} \wedge_{IL} \Delta \mathrm{b}_{\text{n-2}} \right\|_{IL}^2}.\mathbf{B}|_{t=1}$$

$$\mathbf{B}'|_{t=1} = -(n-2)\frac{\det(\Delta b_{n-1}, \Delta \mathrm{b}_{\text{n-2}}, \Delta \mathrm{b}_{\text{n-3}})}{\left\| \Delta b_{n-1} \wedge \Delta \mathrm{b}_{\text{n-2}} \right\|_{IL}^2}\| \Delta b_{n-1} \| .\mathbf{N}|_{t=1}$$

Proof: Using Eq.(1) and Eq.(5), the Serret-Frenet derivative formula can be proven.

Theorem 3.17. Let $b^n(t)$ be timelike Bezier spline. Then at the end point t=1 the curvatures of the Bishop frame for $b^n(t)$ are

$$k_1\big|_{t=0} = \kappa\cos\phi = \frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\cos\phi\,,$$

$$k_2\big|_{t=0} = \kappa\sin\phi = \frac{n-1}{n}\frac{|\Delta b_1|}{|\Delta b_0|^2}\sinh\theta\sin\phi \qquad (6)$$

where θ is the angle between $\Delta\mathbf{b}_0$ and $\Delta\mathbf{b}_1$ vectors and $\phi = \arctan\left(\frac{k_2|_{t=1}}{k_1|_{t=1}}\right)$.

Proof: Since the Bishop frame $\{T|_{t=1}, N_1|_{t=1}, N_2|_{t=1}\}$ of the timelike Bezier spline at the end point $t=1$ is an orthonormal, the Minkowski inner products of the Bishop elements are defined as

$$<T|_{t=1}, T|_{t=1}> = -1\,,\quad <N_1|_{t=1}, N_1|_{t=1}> = 1,\quad <N_2|_{t=1}, N_2|_{t=1}> = 1$$

$$<T|_{t=1}, N_1|_{t=1}> = 0,\quad <T|_{t=1}, N_2|_{t=1}> = 0, <N_1|_{t=1}, N_2|_{t=1}> = 0\,.$$

because of $T|_{t=1}$ is a timelike vector, $N_1|_{t=1}$ and $N_2|_{t=1}$ are spacelike vectors. Now, let us examine the curvatures of the Bishop frame and their derivative equations at the starting and ending point of the timelike Bezier spline $b^n(t)$. The curvature and torsion of the timelike Bishop angle at the end point $t=1$ of the Timelike Bezier spline $b^n(t)$ are obtained as Eq.(6)

We see that here $\tan\phi = \dfrac{k_2\,|_{t=1}}{k_1\,|_{t=1}}$ and $\phi = \arctan\left(\dfrac{k_2\,|_{t=1}}{k_1\,|_{t=1}}\right)$. Also the angle ϕ is calculated from the equation $\phi' = \tau\,|_{t=1} = \dfrac{(n-2)}{n}\dfrac{(\Delta b_{n-1}\,\Delta b_{n-2}\,\Delta b_{n-3})}{\left\|\Delta b_{n-1}\wedge\Delta b_{n-2}\right\|^2}$.

Theorem 3.18. The derivation formulas of the Bishop frame for the timelike Bezier spline at the end point are given by following equations

$$\mathbf{T}'\,|_{t=1} = (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\left[\cos\phi.\mathbf{N_1}\,|_{t=1} + \sin\phi.\mathbf{N_2}\,|_{t=1}\right]$$

$$\mathbf{N}_1'\,|_{t=1} = (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\cos\phi.\mathbf{T}\,|_{t=1}$$

$$\mathbf{N}_2'\,|_{t=1} = (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\sin\phi.\mathbf{T}\,|_{t=1}$$

where φ is the angle between Δb_{n-2}ve Δb_{n-1} vectors.

Proof: Using Eq. (6) and the constant number v_2 are substituted in the derivation equations of the timelike curve, the derivation formulas of the Bishop frame for the timelike Bezier spline at the end point are obtained as following:

$$\mathbf{T}'\,|_{t=1} = k_1\,|_{t=1}\,v_2.\mathbf{N_1}\,|_{t=1} + k_2\,|_{t=1}\,v_2.\mathbf{N_2}\,|_{t=1} = (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\left[\cos\phi.\mathbf{N_1}\,|_{t=1} + \sin\phi.\mathbf{N_2}\,|_{t=1}\right],$$

$$\mathbf{N}_1'\,|_{t=1} = k_1\,|_{t=1}\,v_2.\mathbf{T}\,|_{t=1} = \frac{(n-1)}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}\sinh\varphi\cos\phi.n\|\Delta b_{n-1}\|.\mathbf{T}\,|_{t=1}$$

$$= (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\cos\phi.\mathbf{T}\,|_{t=1}$$

$$\mathbf{N}_2'\,|_{t=1} = k_2\,|_{t=1}\,v_2.\mathbf{T}\,|_{t=1} = \frac{(n-1)}{n}\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|^2}\sinh\varphi\sin\phi.n\|\Delta b_{n-1}\|.\mathbf{T}\,|_{t=1}$$

$$= (n-1)\frac{\|\Delta b_{n-2}\|}{\|\Delta b_{n-1}\|}\sinh\varphi\sin\phi.\mathbf{T}\,|_{t=1} \qquad \text{where } v_2 = n\|b_n - b_{n-1}\| = n\left\|\Delta b_{n-1}\right\|.$$

AN APPLICATION

Let $b^n(t)$ be a cubic Bezier spline with control points $b_0=(2,1,)$, $b_1=(4,2,6)$ $b_2=(3,3,)$, $b_3=(2,3,11)$. Then vectors of the convex polygon $\Delta' b_0=b_1-b_0=(2,1,3)$,

$\Delta' b_1=b_2-b_1=(-1,1,2)$, $\Delta' b_2=b_3-b_2=(-1,0,3)$ are the timelike vectors because of $g(\Delta b_i,\Delta b_i)<0$. Now we will calculate frenet frame and curvatures of the timelike Bezier curve $b^n(t)$. The timelike Bezier spline is

$$b^n(t)=\sum_{i=0}^{3}B_i^n(t)=\begin{pmatrix} 2(1-t)^3+12(1-t)^2t+9(1-t)t^2+2t^3, \\ (1-t)^3+6(1-t)^2t+9(1-t)t^2+3t^3, \\ 3(1-t)^3+18(1-t)^2t+24(1-t)t^2+3t^3 \end{pmatrix}.$$

The first, second third derivations of $b^n(t)$ at $t=0$ are

$$\left.\frac{db^n(t)}{dt}\right|_{t=0}=n\,\Delta' b_0=(6,3,9),\quad \left.\frac{d^2b^n(t)}{dt^2}\right|_{t=0}=n.(n-1)\ \left(\Delta' b_1-\Delta' b_0\right)=(-18,0,-6)$$

$$\left.\frac{d^3b^n(t)}{dt^3}\right|_{t=0}=n.(n-1).(n-2)\ \left(\Delta' b_2-2\Delta' b_1+\Delta' b_0\right)=(18,-6,12)$$

where the norm of $\Delta' b_0$ is $\left\|\Delta' b_0\right\|_{IL}=\sqrt{-g\left(\Delta' b_0,\Delta' b_0\right)}=2$. The tangent vector of the timelike Bezier spline at the beginning point is $T|_{t=0}=\left(1,\frac{1}{2},\frac{3}{2}\right)$. We can see that the tangent vector T become timelike because of $g(T,T)=\sqrt{-\left(1^2+\left(\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2\right)}=-1$.The angle θ between the vectors $\Delta' b_0$ and $\Delta' b_1$ is $\cosh\theta=\frac{7}{2\sqrt{2}}$. The vectoral product is $\Delta b_0\wedge_{IL}\Delta b_1=(1,7,3)$. Furthermore the norm of vectoral product is $\|\Delta b_0\wedge_{IL}\Delta b\|=\sqrt{41}$. On the other hand we can calculate the norm of vectoral product as $\|\Delta b_0\wedge_{IL}\Delta b\|=\|\Delta b_0\|.\|\Delta b_1\|\sinh\theta$. Therefore we can obtain $\sinh\theta=\frac{\sqrt{41}}{2\sqrt{2}}$. The binormal vector of the timelike Bezier spline $B|_{t=0}=\frac{1}{\sqrt{41}}(1,7,3)$ is spacelike vector because of $g(B,B)=1$.

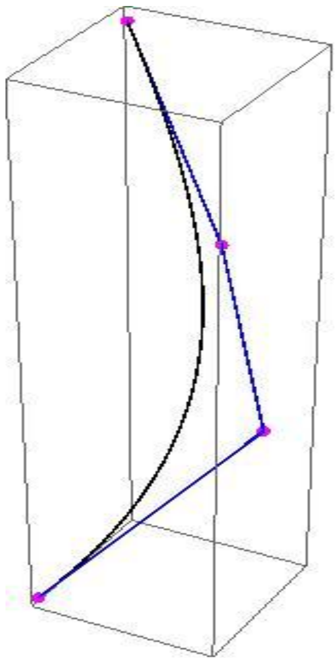

Figure 1. The timelike Bezier Spline

The normal vector is

$$N = -B \wedge_{IL} T = \frac{1}{2\sqrt{41}}(18, 2, 13) \tag{7}$$

The other solution of the principle normal is

$$N = \frac{\Delta b_0}{\|\Delta b_0\|}\coth\theta - \frac{\Delta b_1}{\|\Delta b_1\|}\csc h\theta = \frac{1}{2\sqrt{41}}(18, 3, 13) \tag{8}$$

As we see that the Eq. (7) and Eq. (8) are equal. Both equation can be satisfied for **N**. Thus we obtain the Serret-Frenet frame of $b^n(t)$ at $t = 0$ as $T|_{t=0} = \left(1, \frac{1}{2}, \frac{3}{2}\right)$, $N|_{t=0} = \frac{1}{2\sqrt{41}}(18, 2, 13)$, $B|_{t=0} = \frac{1}{\sqrt{41}}(1, 7, 3)$. Furthermore the curvature and torsion of $b^n(t)$ are $\kappa|_{t=0} = \frac{\sqrt{41}}{2}_{t=0}$ and $\tau|_{t=0} = \frac{-10}{123}$, respectively. Consequently if $v_1 = \left\|\frac{db^n(t)}{dt}\right\| = 6$, Serret-Frenet derivation matrix of the timelike Bezier spline from is obtained as

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{41}}{2} & 0 \\ \frac{\sqrt{41}}{2} & 0 & -\frac{20}{41} \\ 0 & \frac{20}{41} & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

4. CONCLUSIONS

The Bezier spline method is one of the most important mathematical representation of the curves in computer graphic and CAGD. In this paper, the Bezier splines with the timelike convex polygon are defined in Minkowski three spaces and called as the timelike Bezier splines.

Particularly, the Serret-Frenet frames, curvature and torsion of the timelike Bezier splines at the starting and end points are obtained. Then the derivative formulas of the Serret-Frenet frame are calculated. Moreover, another much more advantageous frame called Bishop frame are investigated at the starting and ending points. In the result an example is given for application.

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