ORIGINAL PAPER

ON DUAL BICOMPLEX NUMBERS AND THEIR SOME ALGEBRAIC PROPERTIES

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Abstract. *The object of this work is to contribute to the development of bicomplex numbers. For this purpose, in this study we firstly introduced bicomplex numbers with coefficients from complex Fibonacci sequence. And then, using Babadag's work [1], we examined the dual form of the newly defined numbers. Moreover, we gave some fundamental identities such as Cassini and Catalan identities provided by the elements in defined sequence.*

Keywords: *Bicomplex numbers, Dual numbers, Fibonacci sequence.*

1. INTRODUCTION

Real quaternions are defined by Hamilton, in 1843, as an expansion of the complex numbers. Many identities and inequalities related to quaternions with coefficients real and integers have been studied by different authors such as Halici, Horadam and Iyer [2-4]. As is known the set of real quaternions is defined as

$$\mathbb{H} = \{q = a_0 + a_1 i + a_2 j + a_3 ij | a_0, a_1, a_2, a_3 \in \mathbb{R}\} \tag{1}$$

and is denoted by the letter $\mathbb{H}$ in the memory of Hamilton. The set $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ and its bases elements are $\{1, i, j, k\}$. The bases elements obey the quaternionic multiplication rules. $\mathbb{H}$ is a non-commutative algebra, but an associative algebra on $\mathbb{R}$. In [5], Horadam gave the quaternion recurrence relations. In [3], the author introduced complex Fibonacci numbers and Fibonacci quaternions. And then after the two important studies by Horadam, quaternions that its coefficients selected from different sequences, were taken into consideration by some authors. For example, in [2, 6], Halici studied Fibonacci and complex Fibonacci quaternions. Also, some different generalizations of these studied quaternions were also made. In [7], Tan and others gave a different generalization of real quaternions. Also, they give the generating function and the Binet formula for these quaternions. In [8], Iakin gave a generalization of quaternions and investigated the quaternions whose components are quaternions. On the other hand, in 1892, Segre introduced bicomplex numbers, which were similar to quaternions in many algebraic properties [9]. Although both the number systems are also an expansion of complex numbers the set of bicomplex numbers form a commutative ring with zero divisions while quaternions form a non-commutative division ring. Bicomplex numbers and hyper-complex numbers are considered by some authors. In [10], Price introduced the multicomplex spaces and functions. Luna-Elizarraras studied the algebra of bicomplex numbers and described how to define elementary functions such as polynomials and exponential functions

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in bicomplex algebra [11, 12]. In [13], Rochon used bicomplex numbers for introduce bicomplex dynamics. And Nurkan defined a new type of Fibonacci and Lucas numbers which are called bicomplex Fibonacci and bicomplex Lucas numbers[14]. Moreover, in recent years, some papers related to bicomplex algebra has also become a subject of research in physics and mathematics (for example, see [13, 15, 16]). Similar to quaternions bicomplex numbers with different coefficients have been considered by many authors [1, 17-27]. Since our aim in this study is to contribute to the development of bicomplex numbers, we present the study in two parts. In the first one we give a brief history of the studies on quaternions and bicomplex numbers. In the second one we discuss dual Fibonacci bicomplex numbers.

2. DUAL FIBONACCI BICOMPLEX NUMBERS WITH COMPLEX COEFFICIENT

Since we will take into account the dual bicomplex numbers with coefficients from complex Fibonacci sequence, let us remind that the some necessary definitions and concepts in below. We begin by recalling the concept of the complex Fibonacci sequence. The elements of complex Fibonacci sequence can be written as , $\{C_n\}_{n\geq 0}$,

$$C_n = F_n + iF_{n+1}, \quad i^2 = -1 \tag{2}$$

which is called as the nth complex Fibonacci number [3]. Then, for $n \geq 0$, this sequence is follows:

$$\mathbb{F}_{\mathbb{C}} = \{0 + i, 1 + i, 1 + 2i, 2 + 3i, 3 + 5i, \dots, F_n + iF_{n+1}, \dots\}. \tag{3}$$

Hence, the elements C_n in $\mathbb{F}_{\mathbb{C}}$ satisfy the following recurrence relation

$$C_{n+1} = C_n + C_{n-1}. \tag{4}$$

Now, let's give some necessary preliminaries about bicomplex numbers before defining the sequence we will build with the help of the elements of the complex Fibonacci sequence.

As is known, the letter $\mathbb{BC}$ represents the set of real bicomplex numbers [11] and is defined as $(z_1, z_2) = z_1 + z_2 j$. This number may also be written as a real linear combination of the four units $1, \mathrm{i}, \mathrm{j}, \mathrm{ij}$. A geometric interpretation is afforded by the four-dimensional Euclidean space. An element in the set of real bicomplex numbers is a number of the form

$$b = b_0 + b_1 i + b_2 j + b_3 k \tag{5}$$

where $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ is an ordered bases in the $4-$dimensional real space. The multiplication of bicomplex numbers can be expressed in terms of the bases elements and these elements provide the following rules:

$$i^2 = j^2 = -1, \ (ij)^2 = 1 \text{ and } ij = ji = k, \ ik = ki = -j, \ jk = kj = -i. \tag{6}$$

For all $b_1, b_2, b_3 \in \mathbb{BC}$ and $c \in \mathbb{R}$, the multiplication operation in $\mathbb{BC}$ is distributive with respect to addition and commutes with scalar multiplication:

$$b_1(b_2 + b_3) = b_1 b_2 + b_1 b_3, \ (b_2 + b_3)b_1 = b_1 b_2 + b_1 b_3, \ b_1(cb_2) = (cb_1)b_2 = c(b_1 b_2).$$

Similarly, we can write the set $\mathbb{BC}_F$ of bicomplex numbers whose are the coefficients from the Fibonacci sequence. Hence, any element of this set can be written as follows:

$$BQ_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k, \quad n \geq 0. \tag{7}$$

The numbers that defined above were considered and worked by Nurkan and Guven in [14]. And then, in [19], we gave a generalization for bicomplex Fibonacci numbers.

In our current study, inspired by the references both [1, 14, 19] we define bicomplex numbers with complex Fibonacci coefficients and derive their some fundamental properties. To be appropriate, in the rest of this work, we use the letter B_n for denote the nth bicomplex number with coefficient from complex Fibonacci sequence. So, the element $\mathrm{B_n}$ is

$$B_n = C_n + C_{n+1}i + C_{n+2}j + C_{n+3}ij. \tag{8}$$

Where $\mathrm{C_n}$ is defined in (2) and the units i, j and $ij = k$ are imaginaries and hyperbolic units, respectively. Then, the set of numbers $\mathrm{B_n}$ can be represented by

$$\mathbb{BC}_{CF} = \{\, B_n |\, B_n = C_n + C_{n+1}i + C_{n+2}j + C_{n+3}ij\,,\ i^2 = j^2 = -1,\ (ij)^2 = 1\}. \tag{9}$$

If you pay attention, we have the following second order recurrence relationship among the elements of $\mathbb{BC}_{CF}$:

$$B_{n+2} = B_{n+1} + B_n = C_{n+2} + iC_{n+3} + jC_{n+4} + ijC_{n+5}. \tag{10}$$

A dual number, which was discovered by Clifford [20] and later developed by Study (1891), is defined in the form $A = a + b\varepsilon;\ a, b \in \mathbb{R}$, and ε is an nilpotent number i.e. $\varepsilon^2 = 0,\ \varepsilon \neq 0$.

Similarly, the definition of dual bicomplex numbers with real coefficients can be given as follows.

$$DB = a + a^*\varepsilon \tag{11}$$

where (a, a^*) is bicomplex pair and ε is dual unit. Since we want to define a new sequence related to Fibonacci bicomplex numbers and derive some algebraic relation on it, let us recall the set of dual Fibonacci and Lucas bicomplex numbers is considered in [1].

Dual Fibonacci and Lucas bicomplex numbers are given as

$$\widetilde{x_n} = (F_n + \varepsilon F_{n+1}) + (F_{n+1} + \varepsilon F_{n+2})i_1 + (F_{n+2} + \varepsilon F_{n+3})i_2 + (F_{n+3} + \varepsilon F_{n+4})i_3 \tag{12}$$

and

$$\widetilde{k_n} = (L_n + \varepsilon L_{n+1}) + (L_{n+1} + \varepsilon L_{n+2})i_1 + (L_{n+2} + \varepsilon L_{n+3})i_2 + (L_{n+3} + \varepsilon L_{n+4})i_3 \tag{13}$$

respectively. Accordingly, the set of dual Fibonacci bicomplex numbers can be represented by

$$X_D = \left\{\widetilde{x_n}:\ \left(\widetilde{F_n},\ \widetilde{F_{n+1}},\ \widetilde{F_{n+2}}, \widetilde{F_{n+3}}\right):\ \widetilde{F}_n = F_n + \varepsilon F_{n+1}\right\}. \tag{14}$$

Now, let us define dual Fibonacci bicomplex numbers with complex coefficient and use the notation DB_n for denote the elements of this set. Then, we have

$$DB_n = B_n + \varepsilon B_{n+1},$$

$$\widehat{\mathbb{BC}_{CF}} = \{ DB_n \mid DB_n = B_n + \varepsilon B_{n+1}, \varepsilon \neq 0, \varepsilon^2 = 0\} \tag{15}$$

where

$$B_n = C_n + C_{n+1}i + C_{n+2}j + C_{n+3}ij. \tag{16}$$

Using the equality $DC_n = C_n + \varepsilon C_{n+1}$, we get

$$\mathrm{DB_n} = \mathrm{DC_n} + \mathrm{DC_{n+1}i} + \mathrm{DC_{n+2}j} + \mathrm{DC_{n+3}ij}. \tag{17}$$

On the other hand, if we want to calculate the last equality, then we obtain

$$\mathrm{DB_n} = \{(2\mathrm{i}-1)\mathrm{F_{n+1}} + (2\mathrm{k}-\mathrm{j})\mathrm{F_{n+3}}\} + \varepsilon\{(2\mathrm{i}-1)\mathrm{F_{n+2}} + (2\mathrm{k}-\mathrm{j})\mathrm{F_{n+4}}\} \tag{18}$$

or

$$\mathrm{DB_n} = (2\mathrm{i}-1)(\mathrm{DF_{n+1}} + \mathrm{DF_{n+3}j}). \tag{19}$$

Thus, we have also the following second order recurrence relationship.

$$\mathrm{DB_n} + \mathrm{DB_{n+1}} = (\mathrm{B_n} + \mathrm{B_{n+1}}) + \varepsilon(\mathrm{B_{n+1}} + \mathrm{B_{n+2}}) = \mathrm{DB_{n+2}}. \tag{20}$$

It should be noted that many important properties involving these numbers can be derived with the help of the recurrence relation in (19).

For the purpose of later use we give the definition of conjugate in below.

Definition 1. According to the bases elements in $\widehat{\mathbb{BC}_{CF}}$ three different conjugate definition can be given as follows:

$$DB_n{}^{i} = DB_n - DB_{n+1}i + DB_{n+2}j - DB_{n+3}k. \tag{21}$$

$$DB_n{}^{j} = DB_n + DB_{n+1}i - DB_{n+2}j - DB_{n+3}k. \tag{22}$$

$$DB_n{}^{k} = DB_n - DB_{n+1}i - DB_{n+2}j + DB_{n+3}k. \tag{23}$$

These conjugates can be called as conjugates i, j and k, respectively. For any element DB_n, different conjugate and norm values can be calculated by the above definition.We give some relationships between the dual bicomplex numbers and Fibonacci numbers. Thus, for DB_n, we have

$$DB_n{}^{i} = -(2i+1)\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\}, \tag{24}$$

$$DB_n{}^{j} = (2i-1)\{(F_{n+1} - jF_{n+3}) + \varepsilon(F_{n+2} - jF_{n+4})\}, \tag{25}$$

$$DB_n{}^{k} = -(2i+1)\{(F_{n+1} - jF_{n+3}) + \varepsilon(F_{n+2} - jF_{n+4})\}. \tag{26}$$

We give the following two corollary without proof.

Corollary 1. For the elements DB_n the following equalities are satisfied.

$$DB_n + DB_n{}^{i} = -2\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} - 2kF_{n+4})\}, \tag{27}$$

$$DB_n + DB_n{}^{j} = -2(1 - 2i)(F_{n+1} + \varepsilon F_{n+2}), \tag{28}$$

$$DB_n + DB_n{}^{k} = -2\{(F_{n+1} + 2kF_{n+3}) + \varepsilon(F_{n+2} - jF_{n+4})\}. \tag{29}$$

Corollary 2. For the elements DB_n the following equalities are satisfied.

$$DB_n{}^{i} + DB_n{}^{j} = -2\{(F_{n+1} + 2kF_{n+3}) + \varepsilon(F_{n+2} - jF_{n+4})\} \tag{30}$$

$$DB_n{}^{i}\, DB_n{}^{j} = 5\{F_{n+1}{}^{2} + F_{n+3}{}^{2} + 2j\varepsilon(-1)^{n+1}\} \tag{31}$$

$$DB_n{}^{i} + DB_n{}^{k} = -2(1 + 2i)(F_{n+1} + \varepsilon F_{n+2}) \tag{32}$$

$$DB_n{}^{i}\, DB_n{}^{k} = (3 - 4i)(F_{n+1}{}^{2} + F_{n+3}{}^{2} - 2\varepsilon F_{n+3}F_{n+4}) \tag{33}$$

$$DB_n{}^{j} + DB_n{}^{k} = -2\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} - 2kF_{n+4})\} \tag{34}$$

$$DB_n{}^{j}\, DB_n{}^{k} = -5\{F_{2n+4} + 2jF_{n+1}F_{n+3} + 2\varepsilon(F_{n+1}F_{n+2} - 2jF_{n+2}F_{n+3})\} \tag{35}$$

$$DB_n + DB_n{}^{i} + DB_n{}^{j} + DB_n{}^{k} = -4(F_{n+1} + \varepsilon F_{n+2}) \tag{36}$$

The following corollary is related with some algebraic properties of the conjugates.

Corollary 3. For the elements of DB_n, we have

$$i)\quad \{(DB_n)(DB_m)\}^{i} = (DB_n{}^{i})(DB_m{}^{i}) = (DB_m{}^{i})(DB_n{}^{i}), \tag{37}$$

$$ii)\quad \{(DB_n)(DB_m)\}^{j} = (DB_n{}^{j})(DB_m{}^{j}) = (DB_m{}^{j})(DB_n{}^{j}), \tag{38}$$

$$iii)\quad \{(DB_n)(DB_m)\}^{ij} = (DB_n{}^{ij})(DB_m{}^{ij}) = (DB_m{}^{ij})(DB_n{}^{ij}). \tag{39}$$

Proof: First, let's calculate the product $(DB_n)(DB_m)$. Using the following equalities

$$DB_n = (2i - 1)(DF_{n+1} + D\,F_{n+3}\,j)$$

and

$$DB_m = (2i - 1)(DF_{m+1} + D\,F_{m+3}\,j)$$

we get

$$(2i - 1)^2\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\}\{(F_{m+1} + jF_{m+3}) + \varepsilon(F_{m+2} + jF_{m+4})\}$$

If we make the necessary arrangements and calculations we find

$$(-3 - 4i)\{(F_{n+1}F_{m+1} - F_{n+3}F_{m+3}) + j(F_{n+1}F_{m+3} + F_{n+3}F_{m+1}) + \varepsilon(A + jB)\}$$

where

$$A = F_{n+1}F_{m+2} + F_{n+2}F_{m+1} - F_{n+3}F_{m+4} - F_{n+4}F_{m+3},$$

$$B = F_{n+1}F_{m+4} + F_{n+2}F_{m+3} + F_{n+3}F_{m+2} + F_{n+4}F_{m+1}.$$

The conjugate expression of the last equation can be easily written. On the other hand, now let's calculate the right side of the equality in i). If we use the following equality

$$DB_n{}^i = -(2i+1)\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\},$$

then, we get the value $(DB_n{}^i)(DB_m{}^i)$. This value is as follows:

$$(2i+1)^2\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\}\{(F_{m+1} + jF_{m+3}) + \varepsilon(F_{m+2} + jF_{m+4})\}.$$

Later, if necessary algebraic operations are completed, then

$$(-3+4i)\{F_{n+1}F_{m+1} - F_{n+3}F_{m+3} + j(F_{n+1}F_{m+3} + F_{n+3}F_{m+1}) + \varepsilon(A + jB)\}$$

is obtained. Hence, we have the equality

$$\{(DB_n)(DB_m)\}^i = (DB_n{}^i)(DB_m{}^i).$$

Similarly, if we calculate $(DB_m{}^i)(DB_n{}^i)$, then we have

$$\{(DB_m)(DB_n)\}^i = (DB_n{}^i)(DB_m{}^i).$$

Thus, we have

$$\{(DB_n)(DB_m)\}^i = (DB_n{}^i)(DB_m{}^i) = (DB_m{}^i)(DB_n{}^i).$$

Other equations can be proved in a similar way.

Now, due to the definitions of different conjugates the following norm definitions can be given. Hence, for the number DB_n, these norms which can be called as i, j, k norm are follows.

$$N_i\, DB_n = DB_n\, DB_n{}^i, \tag{40}$$

$$N_j\, DB_n = DB_n\, DB_n{}^j, \tag{41}$$

$$N_k\, DB_n = DB_n\, DB_n{}^k. \tag{42}$$

In the next corollary, we calculate all the norms of $n-th$ element in $\widehat{\mathbb{BC}_{CF}}$.

Corollary 4. For the element DB_n we have

i) $N_i\, DB_n = -5\{(F_{2n+4} - 2jF_{n+1}F_{n+3}) + 2\varepsilon(F_{2n+5} - j\,(F_{2n+3} + 3F_{n+1}F_{n+2}))\}$, (43)

ii) $N_j\, DB_n = -(3+4i)\{F_{n+1}{}^2 + F_{n+3}{}^2 + 2\varepsilon(F_{2n+5} + 2F_{n+1}F_{n+2})\}$, (44)

iii) $N_k\, DB_n = -5\{F_{2n+4} - 2\varepsilon(F_{2n+5} + 2F_{n+1}F_{n+2})\}$. (45)

Proof: It is enough to see the correctness of the first claim given in the corollary. From the definition norm $N_i\, DB_n$ is

$$N_i\, DB_n = 5\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\}\{(F_{n+1} + jF_{n+3}) + \varepsilon(F_{n+2} + jF_{n+4})\},$$

$$N_i\, DB_n = 5\left({F_{n+1}}^2 - {F_{n+3}}^2 + 2jF_{n+1}F_{n+3}\right) + 5\varepsilon\{F_{n+1}F_{n+2} - F_{n+3}F_{n+4}$$

$$+F_{n+2}F_{n+1} - F_{n+4}F_{n+3} + j(F_{n+4}F_{n+1} + F_{n+3}F_{n+2} + F_{n+4}F_{n+1} + F_{n+2}F_{n+3})\}.$$

From the following equalities, that is

$${F_{n+1}}^2 - {F_{n+3}}^2 = -F_{2n+4}$$

$$F_{n+1}F_{n+2} - F_{n+3}F_{n+4} = -F_{2n+5}$$

we obtain that

$$N_i\, DB_n = -5\left\{(F_{2n+4} - 2jF_{n+1}F_{n+3}) + 2\varepsilon\left(F_{2n+5} - j\,(F_{n+1}F_{n+4} + F_{n+2}F_{n+3})\right)\right\},$$

$$N_i\, DB_n = -5\left\{(F_{2n+4} - 2jF_{n+1}F_{n+3}) + 2\varepsilon\left(F_{2n+5} - j\,(F_{2n+3} + 3F_{n+1}F_{n+2})\right)\right\}.$$

Thus, the proof is completed.

Now, let's give the Binet formula, which is one of the important identities for the second order recurrence relations which is used to obtain the desired number.

Theorem 1. (Binet's Formula) Let DB_n be the $n-th$ dual Fibonacci bicomplex numbers with complex coefficient. Then, for $n \geq 0$,

$$DB_n = \frac{1}{\sqrt{5}}(\alpha_d\alpha^n + \beta_d\beta^n) \tag{46}$$

where

$$\alpha_d = (2i-1)(1+\alpha^2 j)(\alpha + \varepsilon\alpha^2),\ \ \beta_d = (2i-1)(1+\beta^2 j)(\beta + \varepsilon\beta^2). \tag{47}$$

Proof: The general solution of recurrence relation $DB_{n+2} = DB_{n+1} + DB_n$ is

$$DB_n = a\alpha^n + b\beta^n.$$

Using the initial conditions imply that

$$DB_0 = a + b,\ \ DB_1 = a\alpha + b\beta\ .$$

Later solving this system, the values a and b

$$a = \frac{DB_1 - \beta DB_0}{\sqrt{5}} \text{ and } b = \frac{\alpha DB_0 - \mathrm{D}B_1}{\sqrt{5}}$$

are obtained. Substituting the values a and b in the equation $DB_n = a\alpha^n + b\beta^n$, then we have

$$DB_n = \frac{DB_1\alpha^n - \beta\alpha^n DB_0 + \alpha\beta^n DB_0 - DB_1\beta^n}{\sqrt{5}},$$

$$DB_n = \frac{1}{\sqrt{5}}\{(DB_1 - \beta DB_0)\alpha^n + (\alpha DB_0 - DB_1)\beta^n\},$$

$$DB_n = \frac{1}{\sqrt{5}}\{\alpha(2i-1)(1+\alpha^2 j)(1+\varepsilon\alpha)\alpha^n + \beta(2i-1)(1+\beta^2 j)(1+\varepsilon\beta)\beta^n\}$$

is obtained. Thus, we have

$$DB_n = \frac{1}{\sqrt{5}}(\alpha_d\alpha^n + \beta_d\beta^n)$$

this ends the proof.

The generating function is a function that corresponds to the Binet formula, which finds the desired elements of the sequence providing the recursive relation. Now, in the following theorem, let's give generating function for dual bicomplex numbers with coefficients from complex Fibonacci numbers.

Theorem 2. For the elements of $\widehat{\mathbb{BC}_{CF}}$ generating function is

$$G(t) = \frac{(F_3 i - F_1)\{(F_2 + F_4 j) + \varepsilon(F_3 + F_5 j)\}}{1-t-t^2}. \quad (48)$$

Proof: Let $G(t)$ be the generating function of dual bicomplex numbers with coefficients from complex Fibonacci numbers. That is

$$G(t) = DB_0 + DB_1 t + DB_2 t^2 + \cdots + DB_n t^n + \cdots.$$

We can make adjustments to the above generating function by using the recurrence relation involving these numbers. In other words, if we multiply $G(t)$ by $-t$ and $-t^2$, respectively,

$$-tG(t) = -(DB_0 t + DB_1 t^2 + DB_2 t^3 + \cdots + DB_n t^{n+1} + \cdots)$$

and

$$-t^2 G(t) = -(DB_0 t^2 + DB_1 t^3 + DB_2 t^4 + \cdots + DB_n t^{n+2} + \cdots).$$

and then, making the necessary operations to find $G(t)$, we get the following desired result.

$$G(t) = \frac{DB_0 + (DB_1 - DB_0)t}{1-t-t^2} = \frac{(-1+2i-3j+6k) + \varepsilon(-2+4i-5j+10k)}{1-t-t^2}$$

$$G(t) = \frac{(2i-1)\{(1+3j) + \varepsilon(2+5j)\}}{1-t-t^2} = \frac{(F_3 i - F_1)\{(F_2 + F_4 j) + \varepsilon(F_3 + F_5 j)\}}{1-t-t^2}$$

which follows the result.

In the next theorem, we give one of the very important relation on the Fibonacci-like numbers which is known as Cassini's Identity. Cassini identity helps to quantify how much deviation from geometry in each term of the sequence.

Theorem 3. For $n \geq 1$, we have

$$DB_{n-1}DB_{n+1} - {DB_n}^2 = (-1)^{n+1}\alpha_d\beta_d, \tag{49}$$

where

$$\alpha_d\beta_d = (3+4i)\big(2+\beta+j(\alpha+2)\big)(1+\varepsilon). \tag{50}$$

Proof: From the Binet formula, left side of the equation (49) is

$$\frac{1}{\sqrt{5}}\,(\alpha_d\alpha^{n-1}+\beta_d\beta^{n-1})\frac{1}{\sqrt{5}}\,(\alpha_d\alpha^{n+1}+\beta_d\beta^{n+1}) - \frac{1}{5}(\alpha_d\alpha^n+\beta_d\beta^n)^2.$$

$$= \frac{1}{5}\{(\alpha_d\alpha^{n-1}+\beta_d\beta^{n-1})(\alpha_d\alpha^{n+1}+\beta_d\beta^{n+1}) - (\alpha_d\alpha^n+\beta_d\beta^n)^2\}.$$

$$DB_{n-1}DB_{n+1} - {DB_n}^2 = \frac{1}{5}\{(\alpha_d\beta_d((-1)^{n-1}\beta^2+(-1)^{n-1}\alpha^2-2(-1)^n)\}.$$

After some algebraic manipulation, the desired result is obtained.

$$DB_{n-1}DB_{n+1} - {DB_n}^2 = (-1)^{n+1}(3+4i)\big(2+\beta+j(\alpha+2)\big)(1+\varepsilon).$$

Using Binet's formula, the Cassini's formula can be generalized. This generalization was established by E.C. Catalan(1814-1894). The following theorem gives the Catalan identity involving elements of $\widehat{\mathbb{BC}_{CF}}$.

Theorem 4. For the elements of $\widehat{\mathbb{BC}_{CF}}$, the following equality is satisfied.

$$DB_{n+k}DB_{n-k} - {DB_n}^2 = \frac{(-1)^n}{5}\alpha_D\beta_D\{(-1)^k-2\}\,. \tag{51}$$

Proof: Using the Binet formula, the left side equation (51) is equal to this:

$$\frac{1}{5}\,\{(\alpha_d\alpha^{n+k}+\beta_d\beta^{n+k}\,)(\alpha_d\alpha^{n-k}+\beta_d\beta^{n-k}\,) - (\alpha_d\alpha^n+\beta_d\beta^n)^2\}$$

$$= \frac{1}{5}\big({\alpha_d}^2\alpha^{2n}+\alpha_d\beta_d\alpha^{n+k}\beta^{n-k}+\beta_d\alpha_d\beta^{n+k}\alpha^{n-k}+{\beta_d}^2\beta^{2n}\big)$$

$$-\frac{1}{5}\big({\alpha_d}^2\alpha^{2n}+2\alpha_d\alpha^n\beta_d\beta^n+{\beta_d}^2\beta^{2n}\big).$$

If we make some necessary arrangements, then we get

$$DB_{n+k}DB_{n-k} - {DB_n}^2 = \frac{(-1)^n}{5}\alpha_d\beta_d\{(-1)^k-2\}.$$

So, the claim is true.

In the Theorem 4, if we take $k = 1$ we get Cassini identity related to dual bicomplex number whose coefficients are complex Fibonacci numbers.

Now let's give the following identity which is known as Honsberger identity or Convolution theorem in the literature.

Theorem 5. For the elements of $\widehat{\mathbb{BC}_{CF}}$ the following equality is satisfied.

$$DB_{k-1}DB_n + DB_kDB_{n+1} = \frac{1}{5}\{{\alpha_d}^2\alpha^{n+k-1}(1+\alpha^2) + {\beta_d}^2\beta^{n+k-1}(1+\beta^2)\} \tag{52}$$

where

$${\alpha_d}^2 = -(3+4i)(1+2\alpha^2 j - \alpha^4)(\alpha^2 + 2\alpha^3\varepsilon) \tag{53}$$

and

$${\beta_d}^2 = -(3+4i)(1-\beta^4+2\beta^2 j)(\beta^2+2\beta^3\varepsilon). \tag{54}$$

Proof: To prove this equality, we can use the Binet formula for each term on the left side of the equation. Then, $DB_{k-1}DB_n + DB_kDB_{n+1}$ is

$$\frac{1}{5}\left({\alpha_d}^2\alpha^{k-1+n} + \alpha_d\alpha^{k-1}\beta_d\beta^n + \beta_d\beta^{k-1}\alpha_d\alpha^n + {\beta_d}^2\beta^{k-1+n}\right)$$

$$+\frac{1}{5}\left({\alpha_d}^2\alpha^{k+n+1} + \alpha_d\alpha^k\beta_D\beta^{n+1} + \beta_d\beta^k\alpha_d\alpha^{n+1} + {\beta_d}^2\beta^{k+n+1}\right).$$

Hence, $DB_{k-1}DB_n + DB_kDB_{n+1}$ is

$$\frac{-1}{5}(3+4i)\{\alpha^{n+k-1}(1+2\alpha^2 j-\alpha^4)(1+2\alpha\varepsilon)(\alpha^2+\alpha^4).$$

$$+\beta^{n+k-1}(1-\beta^4+2\beta^2 j)(1+2\beta\varepsilon)(\beta^2+\beta^4)\}$$

So, we get

$$DB_{k-1}DB_n + DB_kDB_{n+1} = \frac{1}{5}\left\{{\alpha_d}^2\alpha^{n+k-1}(1+\alpha^2) + {\beta_d}^2\beta^{n+k-1}(1+\beta^2)\right\}.$$

Thus, the proof is completed.

Now, for DB_n numbers we give the d'Ocagne identity by the following theorem.

Theorem 6. The equality $DB_mDB_{n+1} - DB_nDB_{m+1}$ is

$$\frac{1}{5}\alpha_d\beta_d\ \{(-1)^m(\beta^{-m+n+1} + \alpha^{-m+n+1}) + (-1)^n(\beta^{-n+m+1} + \alpha^{-n+m+1})\}. \tag{55}$$

Proof: By the aid of Binet formula, the left side of the desired equation can be written as follows:

$$\frac{1}{5}\ \{(\alpha_d\alpha^m + \beta_d\beta^m)\ (\alpha_d\alpha^{n+1} + \beta_d\beta^{n+1}) - (\alpha_d\alpha^n + \beta_d\beta^n)\ (\alpha_d\alpha^{m+1} + \beta_d\beta^{m+1})\}.$$

After performing the necessary calculations we get

$$\frac{1}{5}\alpha_d\beta_d\ \{(-1)^m(\beta^{-m+n+1}+\alpha^{-m+n+1})+(-1)^n(\beta^{-n+m+1}+\alpha^{-n+m+1})\}$$

which is desired.

In the next theorem, we give an identity known as Vajda identity in the literature that this generalizes the Catalan's identity.

Theorem 7. For the elements DB_n the following equality is satisfied.

$$DB_{n+m}DB_{n+k}-DB_nDB_{n+m+k}=\frac{(-1)^{n+1}}{5}\alpha_D\beta_D(\alpha^k-\beta^k)(\alpha^m-\beta^m). \tag{56}$$

Proof: For the left side of the equation (56), we can use the Binet formula. Using the formula

$$DB_n=\frac{1}{\sqrt{5}}(\alpha_d\alpha^n+\beta_d\beta^n)$$

for $DB_{n+m}DB_{n+k}-DB_nDB_{n+m+k}$

$$\frac{1}{5}\alpha_d\beta_d\{(-1)^n\alpha^m\beta^k+(-1)^n\alpha^k\beta^m-(-1)^n\beta^{m+k}-(-1)^n\alpha^{m+k}\}.$$

can be written. So, we get

$$DB_{n+m}DB_{n+k}-DB_nDB_{n+m+k}=\frac{(-1)^n}{5}\alpha_d\beta_d\{\alpha^m(\beta^k-\alpha^k)+\beta^m(\alpha^k-\beta^k)\},$$

$$DB_{n+m}DB_{n+k}-DB_nDB_{n+m+k}=\frac{(-1)^{n+1}}{5}\alpha_d\beta_d(\alpha^k-\beta^k)(\alpha^m-\beta^m).$$

Hence, the proof is completed.

4. CONCLUSIONS

In the reference [1], the author investigated the dual Fibonacci and Lucas bicomplex numbers. Subsequently, in this paper we defined a new sequence by taking the complex Fibonacci numbers as the coefficients of the elements of the set in which the author works. Then, we investigated some important properties of the newly defined numbers and gave the Binet formula involving elements of this sequence. Also, for the elements of this set we gave some fundamental identities that have an important place in the literature such as Cassini's and Catalan's identities.

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