ORIGINAL PAPER

SOME RESULTS AND A CONJECTURE ON CERTAIN SUBCLASSES OF GRAPHS ACCORDING TO THE RELATIONS AMONG CERTAIN ENERGIES, DEGREES AND CONJUGATE DEGREES OF GRAPHS

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Abstract. *Let G be a simple graph of order n with degree sequence $(d) = (d_1, d_2, \dots, d_n)$ and conjugate degree sequence $(d^*) = (d_1^*, d_2^*, \dots, d_n^*)$. In [1, 2] it was proven that $\mathcal{E}(G) \le \sum_{i=1}^{n} \sqrt{d_i}$ and $\sum_{i=1}^{n} \sqrt{d_i^*} \le LEL(G) \le IE(G) \le \sum_{i=1}^{n} \sqrt{d_i}$, where $\mathcal{E}(G), LEL(G)$ and $IE(G)$ are the energy, the Laplacian-energy-like invariant and the incidence energy of G, respectively, and in [2] it was concluded that the class of all connected simple graphs of order n can be dividend into four subclasses according to the position of $\mathcal{E}(G)$ in the order relations above. Then, they proposed a problem about characterizing all graphs in each subclass. In this paper, we attack this problem. First, we count the number of graphs of order n in each of four subclasses for every $1 \le n \le 8$ using a Sage code. Second, we present a conjecture on the ratio of the number of graphs in each subclass to the number of all graphs of order n as n approaches the infinity. Finally, as a first partial solution to the problem, we determine subclasses to which a path, a complete graph and a cycle graph of order $n \ge 1$ belong.*

Keywords: *energy, Laplacian energy, Laplacian-energy-like, incidence energy, degree sequence, conjugate degree sequence.*

1. INTRODUCTION

Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let d_i be the degree of vertex v_i for $i = 1,2, \dots, n$ such that $d_1 \ge d_2 \ge \dots \ge d_n$. The degree sequence of G is the finite sequence $(d) = (d_1, d_2, \dots, d_n)$ and the conjugate degree sequence of G is the sequence $(d^*) = (d_1^*, d_2^*, \dots, d_n^*)$, where $d^* = |j: d_j \ge i|$, see [3]. The adjacency matrix $A(G)$ of G is the matrix whose ij-entry is 1 if v_i and v_j are adjacent and 0 otherwise. Its eigenvalues are denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. The energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was defined by I. Gutman [4] in 1978. Since then there has been a great interest on this concept and its relatives, see the papers [2, 5-7] and the references cited therein.

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The Laplacian of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of $d_1, d_2, \dots, d_n$. It is clear that the Laplacian matrix has nonnegative eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The Laplacian energy $LE(G)$ of G is

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

$LE(G)$ is defined by I. Gutman and B. Zhou [8] in 2006. The recent results on this concept can be found in [9-10].

The Laplacian-energy-like invariant $LEL(G)$ of G is defined as

$$LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}.$$

It was defined by Liu and Liu [11] in 2008. For details on LEL see [2] and the references cited therein.

The incidence energy $IE(G)$ of G is the sum of all singular values of the incidence matrix of G. One can consult the text [3] for the definition of the incidence matrix of a graph. This type of graph energy was introduced by M. Jooyandeh et al. [12] in 2009. Then, Gutman and et al. [13] showed that if $q_1, q_2, \dots, q_n$ are the eigenvalues of the signless Laplacian matrix $Q(G)$ of G, i.e., $Q(G) = D(G) + A(G)$, then

$$IE(G) = \sum_{i=1}^{n} \sqrt{q_i}\,.$$

For the recent results, see [14] and the references therein.

There has been a great interest recently on these graph energies in the literature. In particular, many results which compare certain types of graph energies and graph invariants were published. Indeed, Akbari et al. [1] proved that

$$LEL(G) \leq IE(G).$$

Motivated by this result, Das et al. [2] showed that

$$\pi^*(G) \leq LEL(G), \qquad IE(G) \leq \pi(G) \;\; and \;\; E(G) \leq \pi(G),$$

and if G is a threshold graph, then $\pi^*(G) = LEL(G)$, where $\pi^*(G) = \sum_{i=1}^{n} \sqrt{d_i^*}$ and $\pi(G) = \sum_{i=1}^{n} \sqrt{d_i}$. In the same paper, Das et al. [2] presented the following open problem.

Problem 1.1. [Problem 5.4 [2]]
(1) Characterize all (connected) graphs for which $E(G) \leq \pi^*(G)$
(2) Characterize all (connected) graphs for which $\pi^*(G) < E(G) \leq LEL(G)$
(3) Characterize all (connected) graphs for which $LEL(G) < E(G) \leq IE(G)$
(4) Characterize all (connected) graphs for which $IE(G) < E(G) \leq \pi(G)$.

In the paper of Das et al. [2] each of left-hand inequalities is originally "less than or equal to." Since $E(G)$ might be equal to one of these graph invariants or graph energies for some graphs, we have to replace " $\leq$ " with " $<$ " in left-hand inequalities in (2)-(4). For example, for each odd integer n, $E(C_n) = IE(C_n)$, where C_n is the cycle graph with n vertices.

For simplicity, we can propose new notations for certain subclasses of graphs investigated in Problem 1.1. Let $\mathcal{G}_n$ denote the class of all connected simple graphs of order n. We also denote the classes of the graphs satisfying the conditions given in (1), (2), (3) and (4) by $\mathcal{G}_n^1, \mathcal{G}_n^2, \mathcal{G}_n^3$ and $\mathcal{G}_n^4$, respectively. Then we can restate Problem 1.1 "Characterize all graphs in each of $\mathcal{G}_n^1, \mathcal{G}_n^2, \mathcal{G}_n^3$ and $\mathcal{G}_n^4$."

In this paper, we attack Problem 1.1. First, using a Sage code [15], we find the number of graphs in each class of $\mathcal{G}_n^1, \mathcal{G}_n^2, \mathcal{G}_n^3$ and $\mathcal{G}_n^4$ separately. Second, in the light of the ratios of the numbers of graphs in these subclasses to the number of graphs in $\mathcal{G}_n$ for each $1 \leq n \leq 8$, we present a conjecture about the density of graphs of these subclasses in $\mathcal{G}_n$ as n approaches the infinity. Finally, as a first step in solving Problem 1.1, we show that a path is in $\mathcal{G}_n^4$, a complete graph is in $\mathcal{G}_n^1$ and a cycle graph is in one of the classes $\mathcal{G}_n^2$, $\mathcal{G}_n^3$ and $\mathcal{G}_n^4$ according to $n = 4k$, $n = 2k + 1$ and $n = 4k + 2$, respectively, for a positive integer k.

2. THE NUMBER OF GRAPHS IN EACH SUBCLASSES AND A CONJECTURE ON AVERAGE NUMBERS OF GRAPHS

Sage [15] has a graph library which includes all connected simple graphs of order $n \leq 8$. Using a Sage code based on the graph library, we count the number of graphs in all subclasses $\mathcal{G}_n^1, \mathcal{G}_n^2, \mathcal{G}_n^3$ and $\mathcal{G}_n^4$, see Table 2.1.

Table 2.1. The number of graphs in classes $\mathcal{G}_n, \mathcal{G}_n^1, \mathcal{G}_n^2, \mathcal{G}_n^3$ and $\mathcal{G}_n^4$.

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$\|\mathcal{G}_n\|$	1	1	2	6	21	112	853	11117
$\|\mathcal{G}_n^1\|$	1	0	0	4	12	58	440	5586
$\|\mathcal{G}_n^2\|$	0	0	0	0	4	39	381	5463
$\|\mathcal{G}_n^3\|$	0	0	1	1	4	12	28	59
$\|\mathcal{G}_n^4\|$	0	1	1	1	1	3	4	9

Then, it is natural to consider some certain ratios of the numbers of graphs in these four subclasses to the number of graphs in $\mathcal{G}_n$, see Table 2.2.

Table 2.2. Ratios of the numbers of graphs in subclasses to the number of graphs in Gn.

n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$\frac{\|\mathcal{G}_n^1\|}{\|\mathcal{G}_n\|}$	1.00000	0.00000	0.00000	0.66667	0.57143	0.51786	0.51583	0.50247
$\frac{\|\mathcal{G}_n^2\|}{\|\mathcal{G}_n\|}$	0.00000	0.00000	0.00000	0.00000	0.19048	0.34821	0.44666	0.49141
$\frac{\|\mathcal{G}_n^3\|}{\|\mathcal{G}_n\|}$	0.00000	0.00000	0.50000	0.16667	0.19048	0.10714	0.03283	0.00531
$\frac{\|\mathcal{G}_n^4\|}{\|\mathcal{G}_n\|}$	0.00000	1.00000	0.50000	0.16667	0.04762	0.02679	0.00469	0.00081

In the light of Table 2.2, we can present the following conjecture.

Conjecture 2.1. Let $n \geq 1$. Then, we have

$$lim_{n\to\infty}\frac{|\mathcal{G}_n^1|}{|\mathcal{G}_n|} = lim_{n\to\infty}\frac{|\mathcal{G}_n^2|}{|\mathcal{G}_n|} = \frac{1}{2} \quad and \quad lim_{n\to\infty}\frac{|\mathcal{G}_n^3|}{|\mathcal{G}_n|} = lim_{n\to\infty}\frac{|\mathcal{G}_n^4|}{|\mathcal{G}_n|} = 0.$$

Now, we call the limits in Conjecture 2.1 the average numbers of graphs in four subclasses. Hence, we can restate the conjecture as "The average numbers of graphs in $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ is $\frac{1}{2}$, while the average numbers for $\mathcal{G}_n^3$ and $\mathcal{G}_n^4$ is 0."

3. SOME PARTICULAR GRAPHS IN THE SUBCLASSESRESULTS AND DISCUSSION

In this section, we take the first step towards solving Problem 1.1. To perform this, for all $n \geq 1$, we determine the subclasses to which a path graph, a complete graph and a cycle graph of order n belong. To calculate the energies and related graph invariants of these graphs, we need the following lemma for the sums of values of the sine and the cosine functions at a certain arithmetic progression. For the proof of Lemma 3.1, one can consult the book [16], see pages 77-78.

Lemma 3.1. [16]

$$\sum_{j=0}^{n} cos(\theta + \alpha j) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\cos\left(\theta + \frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

$$\sum_{j=0}^{n} sin(\theta + \alpha j) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\sin\left(\theta + \frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

Theorem 3.2. Let P_n be a path with $n \geq 2$ vertices. Then, $P_n \in \mathcal{G}_n^4$, that is $IE(P_n) < E(P_n) \leq \pi(P_n)$.

Proof. Let P_n be a path with n vertices. Then, its degree sequence is $(d) = (1,2,\dots,2,1)$ and hence $\pi(P_n) = 2 + (n-2)\sqrt{2}$. Moreover, the eigenvalues of $Q(P_n)$ are $q_j = 2 + 2\cos\left(\frac{\pi j}{n}\right)$ $(j = 1,2,\dots,n)$ (see [2]). Thus,

$$IE(P_n) = \sum_{j=1}^{n}\sqrt{2 + 2\cos\left(\frac{\pi j}{n}\right)}$$

$$= 2\sum_{j=1}^{n}\cos\left(\frac{\pi j}{2n}\right)$$

Then, by Lemma 3.1 and the sum formula for the sine function, we have

$$IE(P_n) = 2\left(-1 + \frac{\sin\left(\frac{(n+1)\pi}{4n}\right)\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4n}\right)} \right)$$

$$= 2\left(-1 + \frac{1}{2}\frac{\sin\left(\frac{\pi}{4n}\right) + \cos\left(\frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)} \right)$$

$$= -1 + \cot\left(\frac{\pi}{4n}\right).$$

On the other hand, the eigenvalues of the adjacency matrix of P_n are $\lambda_j = 2\ \cos\left(\frac{\pi j}{n+1}\right)$ $(j = 1,2, \ldots, n)$ (see [17]). Now, we can calculate the energy of P_n in two cases. For the former, suppose n is even.

$$\mathcal{E}(P_n) = 2\sum_{j=1}^{n}\left|\cos\left(\frac{\pi j}{n+1}\right)\right|$$

$$= 4\sum_{j=1}^{\frac{n}{2}}\cos\left(\frac{\pi j}{n+1}\right)$$

By Lemma 3.1 and a product-to-sum formula, we have

$$\mathcal{E}(P_n) = 4\frac{sin\left(\frac{(n+2)\pi}{4(n+1)}\right)cos\left(\frac{n\pi}{4(n+1)}\right)}{sin\left(\frac{\pi}{2(n+1)}\right)} - 4$$

$$= 2\frac{sin\frac{\pi}{2} + sin\left(\frac{\pi}{2(n+1)}\right)}{sin\left(\frac{\pi}{2(n+1)}\right)} - 4$$

$$= -2 + 2csc\left(\frac{\pi}{2(n+1)}\right).$$

For the latter, suppose *n* is odd. Similarly, by Lemma 3.1, we have

$$\mathcal{E}(P_n) = 4\sum_{j=1}^{n}\left|\cos\left(\frac{\pi j}{n+1}\right)\right|$$

$$= 4\frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{(n-1)\pi}{4(n+1)}\right)}{\sin\left(\frac{\pi}{2(n+1)}\right)} - 4.$$

By the difference formula for the cosine function,

$$\mathcal{E}(P_n) = 2\frac{\cos\left(\frac{\pi}{2(n+1)}\right) + \sin\left(\frac{\pi}{2(n+1)}\right)}{\sin\left(\frac{\pi}{2(n+1)}\right)} - 4$$

$$= -2 + 2\cot\left(\frac{\pi}{2(n+1)}\right).$$

Now, we show that $\mathcal{E}(P_n) > IE\,(P_n)$ Suppose n is odd. Since the cotangent function is decreasing on the interval $(0, \pi)$ we have

$$\cot\left(\frac{\pi}{2n}\right) < \cot\left(\frac{\pi}{2(n+1)}\right).$$

for all $n \geq 1$. Then,

$$2\cot\left(\frac{\pi}{2(n+1)}\right) - \cot\left(\frac{\pi}{4n}\right) > 2\cot\left(\frac{\pi}{2n}\right) - \cot\left(\frac{\pi}{4n}\right)$$

$$= 2cot\left(\frac{\pi}{2n}\right) - \frac{cot^2\left(\frac{\pi}{2n}\right) - 1}{2cot\left(\frac{\pi}{2n}\right)}$$

$$= \frac{4\cot^2\left(\frac{\pi}{2n}\right) - \cot^2\left(\frac{\pi}{2n}\right) - 1}{2\cot\left(\frac{\pi}{2n}\right)}$$

$$= \frac{3\cot^2\left(\frac{\pi}{2n}\right) - 1}{2\cot\left(\frac{\pi}{2n}\right)}.$$

On the other hand, one can show that

$$\frac{3\cot^2\left(\frac{\pi}{2n}\right) - 1}{2\cot\left(\frac{\pi}{2n}\right)} > 1 \Leftrightarrow \cot\left(\frac{\pi}{2n}\right) > 3$$

$$\Leftrightarrow n \geq 5,$$

Thus, we obtain $\mathcal{E}(P_n) \geq IE$ for all odd $n \geq 5$. Moreover, one can see that $\mathcal{E}(P_n) \geq IE(P_n)$ for $n = 1$ and $n = 3$, numerically. Thus, $\mathcal{E}(P_n) \geq IE(P_n)$ for all odd $n \geq 1$.

Now, suppose n is even. Since $\csc x - \cot x > 0$ for $0 < x < \frac{\pi}{2}$, we have $2\csc\left(\frac{\pi}{2(n+1)}\right) - \cot\left(\frac{\pi}{4n}\right) > 1$ and hence $\mathcal{E}(P_n) \geq IE(P_n)$ for all even $n \geq 5$. Thus, $\mathcal{E}(P_n) \geq IE(P_n)$ for all n ≥ 1.

Now, we show that $\mathcal{E}(P_n) \leq \pi(P_n)$. Suppose *n* is odd. Since $\cot x < \frac{1}{x}$ for $0 < x < \pi$,

$$\mathcal{E}(P_n) = -2 + 2\cot\left(\frac{\pi}{2(n+1)}\right) < -2 + \frac{4(n+1)}{\pi} = \frac{4}{\pi}n + \frac{4}{\pi} - 2.$$

On the other hand, one can show that

$$\frac{4}{\pi}n + \frac{4}{\pi} - 2 < \sqrt{2}n + 2 - 2\sqrt{2} \Leftrightarrow n > \frac{4\pi - 4 - 2\sqrt{2}\pi}{4 - \sqrt{2}\pi} \cong 0{,}72.$$

Thus, $\mathcal{E}(P_n) > \pi(P_n)$ if *n* is odd. Now, suppose *n* is even. The Laurent series of the cosecant functions is

$$\csc x = \sum_{k=0}^{n} \frac{(-1)^{k+1} 2(2^{2k-1} - 1)B_{2k}}{(2k)!}$$

$$= \frac{1}{x} + \frac{1}{6}x + \frac{1}{360}x^3 + \frac{31}{15120}x^5 + \ldots,$$

where B_{2k} is the $2k$ −th Bernoulli number. It is well-known that

$$B_{2k} = \frac{(-1)^{k-1} \cdot 2 \cdot (2k)!}{(2\pi)^{2k}} \zeta(2k),$$

where ζ is the Riemann zeta function, and $\zeta(2k) \leq \zeta(2) = \pi^2/6$ for every integer $k \geq 1$. Thus,

$$-2 + 2\csc\left(\frac{\pi}{2(n+1)}\right) < -2 + \frac{4(n+1)}{\pi} + \frac{2\pi}{3}\sum_{k=1}^{\infty}\left(\frac{1}{2(n+1)}\right)^{2k-1}$$

$$= -2 + \frac{4(n+1)}{\pi} + \frac{4\pi(n+1)}{3(2n+1)(2n+3)}$$

$$< \frac{4(n+1)}{\pi} - 1.4$$

In what follows one can show that

$$\frac{4(n+1)}{\pi} - 1.4 < \sqrt{2}n + 2 - 2\sqrt{2}$$

for all $n \geq 5$. Moreover, by a simple calculation, one can easily see that

$$-2 + 2\csc\left(\frac{\pi}{2(n+1)}\right) < \sqrt{2}n + 2 - 2\sqrt{2}$$

for each $n = 2,3,4$. Thus $\mathcal{E}(P_n) < \pi(P_n)$ for all $n \geq 2$. The proof is complete.

Theorem 3.3. Let C_n be a cycle graph with $n \geq 3$ vertices.

(i) If $n = 4k$ then $C_n \in \mathcal{G}_n^2$, that is $\pi^*(C_n) < E(C_n) < LEL(C_n)$,
(ii) If $n = 2k + 1$ then $C_n \in \mathcal{G}_n^3$, that is $LEL(C_n) < E(C_n) = IE(C_n)$,
(iii) If $n = 4k + 2$ then $C_n \in \mathcal{G}_n^4$, that is $IE(C_n) < E(C_n) \leq \pi(C_n)$.

Proof. (i) Let C_n be a cycle with $n = 4k \geq 4$ vertices. Its degree sequence is $(d) = (2,2,\ldots,2)$ and its conjugate degree sequence is $(d^*) = (n, n, 0, \ldots, 0)$. Thus $\pi^*(C_n) = 2\sqrt{n}$. On the other hand, the eigenvalues of its adjacency matrix are $2\cos\left(\frac{2\pi j}{n}\right)$ $(j = 0,1,2,\ldots,n-1)$ (see [17]) and hence its energy is

$$\mathcal{E}(C_n) = 2\sum_{j=1}^{4k-1}\left|\cos\left(\frac{\pi j}{2k}\right)\right| + 2$$

$$= 8\sum_{j=1}^{k-1}\cos\left(\frac{\pi j}{2k}\right) + 4.$$

By Lemma 3.1 and the sum formula for the sine function, we have

$$\mathcal{E}(C_n) = 4\sqrt{2}\cdot\frac{\sin\left(\frac{(k+1)\pi}{4k}\right)}{\sin\left(\frac{\pi}{4k}\right)} - 4$$

$$= 4\cdot\frac{\cos\left(\frac{\pi}{4k}\right) + \sin\left(\frac{\pi}{4k}\right)}{\sin\left(\frac{\pi}{4k}\right)} - 4$$

$$= 4\cot\left(\frac{\pi}{n}\right).$$

By the fact that $\cot x > \frac{1}{x} + \frac{1}{x-\pi}$ for $0 < x < \frac{\pi}{2}$, we have

$$4\cot\left(\frac{\pi}{n}\right) > \frac{4n(n-2)}{\pi(n-1)}.$$

Moreover, one can easily show that

$$\frac{4n(n-2)}{\pi(n-1)} > 2\sqrt{n}.$$

Thus, we obtain $\mathcal{E}(C_n) > \pi^*(C_n)$.

On the other hand, the Laplacian eigenvalues of C_n are $2 - 2\cos\left(\frac{2\pi j}{n}\right)$ $(j = 0,1,\ldots,n-1)$ (see [17]) and hence its Laplacian energy-like-invariant is

$$LEL(C_n) = 2\sum_{j=0}^{n-1} \sin\left(\frac{\pi j}{n}\right)$$

$$= 2\cot\left(\frac{\pi}{2n}\right).$$

Here the last step follows from Lemma 3.1. Now, let $\theta = \frac{\pi}{2n}$. It is clear that $\cot\theta$ and $\cot 2\theta$ are positive for all $n \geq 4$. Then, we have $\frac{\cot^2\theta - 1}{\cot\theta} < \cot\theta$ and hence $4\cot 2\theta < 2\cot\theta$. Thus, the proof of *(i)* is complete.

(ii) Let $n = 2k + 1$ for a positive integer k. In this case,

$$\mathcal{E}(C_n) = 2\sum_{j=0}^{n-1} \left|\cos\left(\frac{2\pi j}{n}\right)\right|$$

$$= 2 + 2\sum_{j=1}^{n-1} \left|\cos\left(\frac{2\pi j}{n}\right)\right|$$

$$= -2 + 4\sum_{j=0}^{\frac{n-1}{2}} \cos\left(\frac{\pi j}{n}\right)$$

$$= -2 + 4 \cdot \frac{\sin\left(\frac{(n+1)\pi}{4n}\right)\cos\left(\frac{(n-1)\pi}{4n}\right)}{\sin\left(\frac{\pi}{2n}\right)} \backslash$$

$$= 2\csc\left(\frac{\pi}{2n}\right)$$

and then it is clear that $\mathcal{E}(C_n) > LEL(C_n)$. On the other hand, the eigenvalues of the signless Laplacian matrix $Q(C_n)$ are $q_j = 2 + 2\cos\left(\frac{2\pi j}{n}\right)$ $(j = 0,1,2,\cdots,n-1)$ (see [18]) and hence the incidence energy of C_n is

$$IE(C_n) = -2 + 4\sum_{j=0}^{\frac{n-1}{2}} \cos\left(\frac{\pi j}{n}\right)$$

$$= -2 + 4 \cdot \frac{\sin\left(\frac{(n+1)\pi}{4n}\right)\cos\left(\frac{(n-1)\pi}{4n}\right)}{\sin\left(\frac{\pi}{2n}\right)}$$

$$= 2\csc\left(\frac{\pi}{2n}\right).$$

In this case, $\mathcal{E}(C_n) = IE(C_n)$. Thus the proof of *(ii)* is complete.

(iii) Let $n = 4k + 2$ for a positive integer k. The energy of C_n is

$$\mathcal{E}(C_n) = 4\sum_{j=0}^{\frac{n-2}{2}} \left|\cos\left(\frac{2\pi j}{n}\right)\right|$$

$$= 4 + 8\sum_{j=0}^{\frac{n-1}{8}} \cos\left(\frac{2\pi j}{n}\right)$$

$$= -4 + \frac{\sin\left(\frac{(n+2)\pi}{4n}\right)\cos\left(\frac{(n-2)\pi}{4n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

$$= -4 + 4 \cdot \frac{1 + \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

$$= 4\csc\left(\frac{\pi}{n}\right).$$

and the incidence energy of C_n is

$$IE(C_n) = -2 + 4\sum_{j=0}^{\frac{n-2}{2}} \left|\cos\left(\frac{\pi j}{n}\right)\right|$$

$$= -2 + 2\sqrt{2} \cdot \frac{\cos\left(\frac{(n-2)\pi}{4n}\right)}{\sin\left(\frac{\pi}{2n}\right)}$$

$$= 2\cot\left(\frac{\pi}{2n}\right).$$

Since $\frac{\mathcal{E}(C_n)}{IE(C_n)} = \frac{1}{\cos^2\left(\frac{\pi}{2n}\right)} > 1$, we have $\mathcal{E}(C_n) > IE(C_n)$. The proof of *(iii)* is complete.

Theorem 3.4. Let K_n be a complete graph with $n > 3$ vertices. Then $K_n \in \mathcal{G}_n^2$, that is $\mathcal{E}(K_n) \leq \pi^*(K_n)$.

Proof. The eigenvalues of a complete graph of order $n > 3$ are

$$n - 1, (-1)^{n-1}, (-1)^{n-1}, \dots, (-1)^{n-1},$$

(see [17]) and hence $\mathcal{E}(K_n) = 2n - 2$. On the other hand, $d_1 = d_2 = \cdots = d_n = n - 1$ and hence $d_1^* = d_2^* = \cdots = d_{n-1}^* = n$ and $d_n^* = 0$. Thus, $\pi^*(K_n) = (n-1)\sqrt{n}$. Since $n > 3$, we have $2(n-1) \leq \sqrt{n}(n-1)$, and hence $\mathcal{E}(K_n) \leq \pi^*(K_n)$.

CONCLUSIONS

In the paper, we present a conjecture on the inequalities among graph energies and some partial solutions to Problem 1.1 for a few particular graphs. We think the study on Conjecture 2.1 brings new techniques and problems to the literature of graph energies.

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