ORIGINAL PAPER

THE SHAPE OPERATOR OF THE BEZIER SURFACES IN MINKOWSKI-3 SPACE

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Abstract. *Bezier surfaces are commonly used in Computer-Aided Geometric Design since it enables in geometric modeling of the objects. In this study, the shape operator of the timelike and spacelike surfaces has been analyzed in Minkowski-3 space. Then, the obtained results were applied to a numeric example.*

Keywords: *Minkowski space; Bezier surface; shape operator; timelike and spacelike surface.*

1. INTRODUCTION

Bezier curves and surfaces which are the elements of computer-aided geometric design were discovered by Pierre Bezier and De Casteljau on the purpose of using them to design car bodies. Bezier surfaces have a set of some algorithmic properties that analyze and interpret the shapes. Thus, CAGD and geometric modeling are used commonly. As a parametric curve, the Bezier curve is described by Bernstein's basic polynomials and has a control polygon. Bezier surfaces seen as a product of two Bezier curves have the same feature as well. Until today, various studies related to Bezier curves and surfaces have been conducted in several studies. Some basic concepts of Bezier curves and surfaces are given in detail in [1-7]. Ye and his colleagues (2010), by examining a new sort of the fundamental functions of the Bezier curves having one and two shape parameters, studied on some models for these structures [8]. As for Sun and his colleagues (2019) investigated the particular coordinate networks on the Bezier surfaces [9]. Incesu (2008, 2003) and Yılmaz (2009) calculated the geometric properties of the Bezier surfaces, shape operator, Gauss, and the mean curvature [10-12]. Lang ve Röschel (1992) studied on the metric properties of the developable rational Bezier surfaces [13]. On the other hand, Minkowski geometry firstly discovered by German mathematician and physicist Herman Minkowski (1864-1909). Minkowski tried to solve the problems in relativity theory handled in mathematical physics by using geometric methods. Minkowski and Einstein, his teacher of Russian origin from Zurich Polytechnic school, gave a dimension quality to time by showing that indeed time concept was an inseparable part of the space. In addition to three dimensions described with width-length-height (coordinates x-y-z**)** of an object, the time is taken as a fourth dimension in the Minkowski geometry. There is a distance concept bearing an important resemblance to the distance concept in the Euclid geometry. Definitions and theorems related to curves and surfaces in the Minkowski 3-space are given in the source [14] in detail. Georgiev (2009)

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examined some basic metric properties to form the substructure of the spacelike Bezier surface in the Minkowski 3-space [15]. Also, Ugail and his colleagues (2011) worked on the calculation of the basic forms of the Bezier surfaces and the solution of the Plateau-Bezier problem in Minkowski-3 space [16]. Kuşak Samancı and Çelik (2017) analyzed the surface normal of the timelike and the spacelike Bezier surfaces, coefficients of the basic forms, Gauss and mean curvatures, and also shape operator in the Minkowski 3-space for the first time [17, 18]. İncesu (2019) researched the equivalence relations LS (3) and the ratio of similarity of the Bezier surfaces [19].

In this study, the first and second fundamental basis form of the timelike and spacelike Bezier surface, Gauss and mean curvature were calculated in Minkowski space. Furthermore, the coefficients of the matrix corresponding to the shape operator were acquired. By utilizing these coefficients, Gauss and mean curvatures were calculated. At the end of the chapter, an explanatory numeric example was given.

2. MATERIALS AND METHODS

Let R^3 be three dimensional Euclid space. The space $R_1^3 = \left(R^3, g(,)\right)$ defined by the Lorentzian inner product $g(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ is called Minkowski space for the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. When $g(\vec{w},\vec{w}) > 0$ or $\vec{w} = 0$, the vector $\vec{w}$ taken in Minkowski-3 space is called a spacelike vector, when $g(\vec{w},\vec{w}) < 0$, it's called a timelike vector, and then when $g(\vec{w},\vec{w}) = 0$ and $\vec{w} \neq 0$, it's called a null (spacelike) vector. The spacelike and timelike vectors among these vectors are termed as non-degenerate vectors as well. Suppose that $\vec{k} = (k_1, k_2, k_3)$ and $\vec{l} = (l_1, l_2, l_3)$ taken in $\mathbb{R}_1^3$ space are any two vectors. The cross product of two vectors $\vec{k}$ and $\vec{l}$ in Minkowski 3-space is calculated by the equation $\vec{k} \wedge_L \vec{l} = -\begin{vmatrix} e_1 & e_2 & -e_3 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{vmatrix}$. Assume that M surface having parameterization $b = b(u,v)$ defined by $b: U \subset R^2 \to R_1^3$ in R_1^3 Minkowski space is a surface. The tangent plane T_PM which is spanned by vectors $b_u(P)$ and $b_v(P)$ in the Minkowski space, $k \geq 1$, is called a tangent plane passed through a point P of regular plane M of the classes C^k.

The unit normal vector field in P point of the M surface is $N = \dfrac{b_u \wedge_L b_v}{\left\| b_u \wedge_L b_v \right\|} = \dfrac{b_u \wedge_L b_v}{\sqrt{-\varepsilon\left(E.G - F^2\right)}}$.

Matrix representation of the first fundamental form on any T_pM plane of the M surface is acquired by $W = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where $E = g(b_u, b_u)$, $F = g(b_u, b_v)$, $G = g(b_v, b_v)$ are the coefficients of the first fundamental form of the surface, $\det W = EG - F^2$. The non-degenerate surfaces (timelike, spacelike) in Minkowski space are defined by this means. If the surface is timelike (spacelike), then $\det W$ is negative (positive). For non-degenerate surfaces to be represented simultaneously in Minkowski space, the demonstration $g(N,N) = \varepsilon$ is used. If the surface is timelike, the normal N is a spacelike vector since the tangent plane is timelike, and it yields $g(N,N)=\varepsilon=1$ equality. If the surface is spacelike, N normal is a timelike vector since the tangent plane is spacelike, and it yields equality $g(N,N)=\varepsilon=-1$. Here, it is calculated by the equation $\|b_u \wedge_L b_v\| = \sqrt{-\varepsilon\left(E.G-F^2\right)} = \sqrt{-\varepsilon.\det W}$. The coefficients of the second fundamental form of the surface M on any plane T_PM

$$e = -g\left(Ab_u, b_u\right) = -g\left(N_u, b_u\right) = g\left(N, b_{uu}\right)$$

$$f = -g\left(Ab_u, b_v\right) = -g\left(N_v, b_u\right) = -g\left(b_v, N_u\right) = g\left(N, b_{uv}\right)$$

$$g = -g\left(Ab_v, b_v\right) = -g\left(N_v, b_v\right) = g\left(N, b_{vv}\right)$$

are calculated with the Lorentzian inner product. Also, with the help of coefficients (E,F,G) of the first fundamental form and coefficients (e,f,g) of the second fundamental form, the shape operator of the surface is defined with the matrix $A=\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.\begin{pmatrix} e & f \\ f & g \end{pmatrix}$. The mean curvature H and the Gauss curvature K for an undegenerate surface are obtained by $H=\frac{\varepsilon}{2} traceA$, $K=\varepsilon \det A$ with the help of the matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ corresponding to shape operator [12-20].

Definition 2.1. A Bezier curve of degree n is defined by the equation $b^n(t)=\sum_{i=0}^{n} b_i B_i^n(t)$ for $n+1$ control points $b_0, b_1, ..., b_n$ in Euclid space. Here, $B_i^n(t)$ is the n-th degree Bernstein polynomials, it is represented by the equation $B_i^n(t)=\binom{n}{i} t^i (1-t)^{n-i}$. The binomial coefficient provides the condition $\binom{n}{i}=\frac{n!}{i!(n-i)!}$ for $0 \le i \le n$.

Theorem 2.2. Let $b_i \in R^3$ be the control points, the first-order derivative of the Bezier curve $b^n(t)$ is obtained by the formula

$$\frac{db^n}{dt}(t)=\sum_{i=0}^{n-1} b_i^{(1)} B_i^{n-1}(t) \tag{2.1}$$

where $b_i^{(1)}=n\left(b_{i+1}-b_i\right)$.

Definition 2.3. Assume that $B_i^n(u)$ and $B_j^m(v)$ are the nth and mth Bernstein base functions with the parameters u and v, respectively. $0 \le i \le n$, $0 \le j \le m$, for the Bezier surface

$(u,v)\in[0,1]\times[0,1]$ with control points b_{ij}, it's expressed with the equation $b(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}B_i^n(u)B_j^m(v)$. Parameter curves of the Bezier surface are specific Bezier curves. Particularly, parameter curves $b(u,0)$, $b(u,1)$, $b(0,v)$, $b(1,v)$ are a Bezier curve which is in the form of four edges of the Bezier curve. Bezier surfaces yield endpoint interpolation with the corner points $b(0,0)=b_{00}$, $b(1,0)=b_{n0}$, $b(0,1)=b_{0m}$, $b(1,1)=b_{nm}$. The set $CH\{b_{00},...,b_{nm}\}$, as in the convex polyhedron involving the Bezier surface, yields feature $b(u,v)\in CH\{b_{00},...,b_{n,m}\}$ for $\forall(u,v)\in[0,1]\times[0,1]$. Besides, T is a three-dimensional affine transformation, the Bezier surfaces are invariant under the affine transformation due to the equality $T\left(\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}B_i^n(u).B_j^m(v)\right)=\sum_{i=0}^{n}\sum_{j=0}^{m} T(b_{ij})B_i^n(u).B_j^m(v)$, [1-12].

Theorem 2.4. The first-order partial derivative of the Bezier surface $b(u,v)$ according to the parameters u and v is obtained by

$$b_u(u,v)=\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}B_j^m(v)\ ,\qquad b_v(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}B_j^{m-1}(v), \tag{2.2}$$

where $b_{ij}^{(1,0)}=n\left(b_{(i+1)j}-b_{ij}\right)$ and $b_{ij}^{(0,1)}=m\left(b_{i(j+1)}-b_{ij}\right)$, [1-7].

Corollary 2.5. The values of the first-order partial derivatives of the Bezier surface $b(u,v)$ according to the parameters u and v at the minimum point $(u,v)=(0,0)$ are calculated by

$$b_u(0,0)=n.(b_{10}-b_{00})=b_{00}^{10},\qquad b_v(0,0)=m.(b_{01}-b_{00})=b_{00}^{(0,1)},\ \ [10\text{-}12]. \tag{2.3}$$

Theorem 2.6. The second-order partial derivatives of the Bezier surface $b(u,v)$ according to the parameters u and v are calculated with the following equations.

$$b_{uu}(u,v)=\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^m(v) \tag{2.4}$$

$$b_{uv}(u,v)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)}B_i^{n-1}B_j^{m-1}(v) \tag{2.5}$$

$$b_{vv}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m-2} b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v) \tag{2.6}$$

where $b_{ij}^{(2,0)}=n.(n-1)\left(b_{(i+2)j}-2b_{(i+1)j}+b_{ij}\right)$, $b_{ij}^{(1,1)}=nm\left(b_{(i+1)(j+1)}-b_{(i+1)j}-b_{i(j+1)}+b_{ij}\right)$ and $b_{ij}^{(0,2)}=m.(m-1)\left(b_{i(j+1)}-2b_{i(j+1)}+b_{ij}\right)$.

Theorem 2.7. The second-order partial derivatives of the Bezier surface $b(u,v)$ for parameters u and v at the minimum point $(u,v)=(0,0)$ is represented by the following equations [10-12],

i) $b_{uu}(0,0)=n(n-1)(b_{20}-2b_{10}+b_{00})=b_{00}^{(2,0)}$ (2.7)

ii) $b_{uv}(0,0)=nm(b_{11}-b_{10}-b_{01}+b_{00})=b_{00}^{(1,1)}$ (2.8)

iii) $b_{vv}(0,0)=m(m-1)(b_{02}-2b_{01}+b_{00})=b_{00}^{(0,2)}$ (2.9)

3. RESULTS AND DISCUSSION

In this study, the coefficients of the first fundamental form of the timelike and spacelike Bezier surfaces were calculated by following a method different from in [16] and by generalizing in the Minkowski space described in [15]. Then, by calculating the second fundamental form, Gauss, and mean curvatures of the Bezier surface for the first time, the matrix form of the shape operator of the surface was calculated in the Minkowski-3 surface. At the very and, a quantitative example was given.

Definition 3.1. The Bezier surface which is defined in the form of $b(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}b_{ij}B_i^n(u)B_j^m(v)$ for the control points b_{ij} in Minkowski-3 space defined by the Minkowski inner product is called the non-degenerate Bezier surface in Minkowski-3 space. If the normal of the surface is $g(N,N)=1$, it is called a timelike Bezier surface, but if $g(N,N)=-1$, it is called a spacelike Bezier surface.

Theorem 3.2. The coefficients (E,F,G) of the first fundamental form of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space are obtained by

$$E=g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right) \tag{3.1}$$

$$F=g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$
(3.2)

$$G=g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right). \tag{3.3}$$

Proof: The coefficients of the first fundamental form of the Bezier surface are obtained by using the first-order partial derivative given in the equation (2.2) with the Lorentzian inner product metric. The coefficient E is obtained by

$$E=g\left(b_u(u,v),b_u(u,v)\right)$$

$$=\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2+\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2-\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2$$
$$=g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right).$$

Similarly, the coefficients F and G are obtained.

Corollary 3.3. The coefficients (E,F,G) of the first fundamental form of the timelike (spacelike) Bezier surface $b(u,v)$ at the min point $(u,v)=(0,0)$ in the Minkowski-3 space are obtained by

$$E=g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right),\ F=g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right),\ G=g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right). \tag{3.4}$$

Corollary 3.4. The first fundamental form of the timelike (spacelike) Bezier surface at the point $(u,v)=(0,0)$ in Minkowski-3 space is calculated by

$$ds^2=g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)d_u{}^2$$
$$+2g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)d_u.d_v$$
$$+g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)d_v{}^2.$$

Corollary 3.5. The first fundamental form of the timelike (spacelike) Bezier surface at the point $(u,v)=(0,0)$ in Minkowski-3 space is obtained by $ds^2=g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right)d_u{}^2+2g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)d_u.d_v+g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right)d_v{}^2$. Let the components of the vector $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ to provide the abbreviation in the equations to be

$$\lambda_1=\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)-\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$\lambda_2=\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)$$

$$\lambda_3=\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right).$$

Now, the metric properties for the timelike (spacelike) Bezier surfaces in Minkowski space were proven. In the following results, if the Bezier surface is timelike, then $\varepsilon=1$ will be taken in the equalities; if the surface is spacelike, then $\varepsilon=1$ in the equalities.

Theorem 3.6. The normal vector field N on the non-degenerate Bezier surface $b(u,v)$ in the Minkowski-3 space is obtained by

$$N = \frac{\lambda}{\sqrt{-\varepsilon g(\lambda,\lambda)}} \tag{3.5}$$

where if the Bezier surface is timelike, then $\varepsilon = 1$, if it is spacelike, then $\varepsilon = 1$.

Proof: The unit normal vector of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski 3-space is obtained by using the Lorentzian cross product of the tangent vector of the curve for the parameters given in the equation (2.2)

$$N(u,v) = \frac{b_u \wedge_L b_v}{\|b_u \wedge_L b_v\|_L} = \frac{\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)} B_i^{n-1}(u) B_j^m(v) \wedge_L \sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)} B_i^n(u) B_j^{m-1}(v)}{\left\| \sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)} B_i^{n-1}(u) B_j^m(v) \wedge_L \sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)} B_i^n(u) B_j^{m-1}(v) \right\|_L} \tag{3.6}$$

where the demonstration $g(N,N) = \varepsilon$ is used for the timelike and spacelike surfaces to be represented simultaneously. If the Bezier surface is timelike, $g(N,N) = 1$, if it's spacelike, it's seen as $g(N,N) = -1$. Also, the norm $\|b_u \wedge_L b_v\| = \sqrt{|E.G - F^2|} = \sqrt{-\varepsilon(E.G - F^2)} = \sqrt{-\varepsilon.\det W}$ is used [14]. When the demonstration involving ε is written while taking a norm in the equation (3.6), the surface normal is computed by

$$N = \frac{(\lambda_1,\lambda_2,\lambda_3)}{\sqrt{|\lambda_1^2 + \lambda_2^2 - \lambda_3^2|}} = \frac{(\lambda_1,\lambda_2,\lambda_3)}{\sqrt{-\varepsilon(\lambda_1^2 + \lambda_2^2 - \lambda_3^2)}} = \frac{\lambda}{\sqrt{-\varepsilon g(\lambda,\lambda)_L}}$$

Corollary 3.7. The equation of the normal vector field N on the Bezier surface which is timelike (spacelike) in the Minkowski-3 space at the min point $(u,v) = (0,0)$ is

$$N(0,0) = \frac{b_{00}^{(1,0)} \wedge_L b_{00}^{(0,1)}}{\left\| b_{00}^{(1,0)} \wedge_L b_{00}^{(0,1)} \right\|_L}.$$

Theorem 3.8. The determinant of the first fundamental form of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space is calculated by $\det W = -g(\lambda,\lambda)$.

Proof: When the determinant of the first fundamental form of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space is written in its place in the equations $\det W = EG - F^2$ and the product and square functions are organized by utilizing equations (3.1), (3.2) ve (3.3), the following result is obtained.

$$\det W=\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2$$
$$-\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2$$
$$-\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2$$

$$=-\left(\begin{array}{l}+\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2\\+\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}z_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}z_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2\\-\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}x_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\sum_{i=0}^{n}\sum_{j=0}^{m-1}y_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)-\sum_{i=0}^{n}\sum_{j=0}^{m-1}x_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\sum_{i=0}^{n-1}\sum_{j=0}^{m}y_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)^2\end{array}\right)$$
$$=-g(\lambda,\lambda)$$

Corollary 3.9. The coefficients $\lambda_1=\left(y_{00}^{(0,1)}z_{00}^{(1,0)}-y_{00}^{(1,0)}z_{00}^{(0,1)}\right)$, $\lambda_2=\left(x_{00}^{(1,0)}z_{00}^{(0,1)}-x_{00}^{(0,1)}z_{00}^{(1,0)}\right)$, $\lambda_3=\left(x_{00}^{(1,0)}y_{00}^{(0,1)}-x_{00}^{(0,1)}y_{00}^{(1,0)}\right)$ are obtained for the point $(u,v)=(0,0)$ of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski space. Using the equation (3.4), the determinant of the first fundamental form is calculated by

$$\begin{aligned}\det W&=EG-F^2\\&=g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right).g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right)-g^2\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)\\&=-\left(\lambda_1{}^2+\lambda_2{}^2-\lambda_3{}^2\right)=-g(\lambda,\lambda).\end{aligned}$$

Theorem 3. 10. The coefficients of the second form of the timelike (spacelike) Bezier surface are computed by

$$e=g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^m(v),\lambda\right)\Big/\sqrt{-\varepsilon g(\lambda,\lambda)} \tag{3.7}$$

$$f=g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)\Big/\sqrt{-\varepsilon g(\lambda,\lambda)} \tag{3.8}$$

$$g=g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)\Big/\sqrt{-\varepsilon g(\lambda,\lambda)}. \tag{3.9}$$

Proof: The coefficients e, f, g of the second fundamental form of the timelike (spacelike) Bezier surface $b(u,v)$ are calculated to be the coefficient e of the second fundamental form by using the second-order partial derivative given in the equations (2.4) (2.5) and (2.6), and the formulas of the normal vector field N given in the equation (3.6).

$$e = g\left(b_{uu}\left(u,u\right), N\left(u,v\right)\right)$$

$$= \frac{-\det\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right), \sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)} B_i^{n-1}\left(u\right) B_j^{m}\left(v\right), \sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)} B_i^{n}\left(u\right) B_j^{m-1}\left(v\right)\right)}{\left\| \sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)} B_i^{n-1}\left(u\right) B_j^{m}\left(v\right) \wedge_L \sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)} B_i^{n}\left(u\right) B_j^{m-1}\left(v\right) \right\|_L}$$

$$= \frac{-\sum_{i=0}^{n-2}\sum_{j=0}^{m} x_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\left(-\lambda_1\right) - \sum_{i=0}^{n-2}\sum_{j=0}^{m} y_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\left(-\lambda_2\right) - \sum_{i=0}^{n-2}\sum_{j=0}^{m} z_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\left(\lambda_3\right)}{\sqrt{\left|\lambda_1^2 + \lambda_2^2 - \lambda_3^2\right|}}$$

$$= \frac{\sum_{i=0}^{n-2}\sum_{j=0}^{m} x_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\lambda_1 + \sum_{i=0}^{n-2}\sum_{j=0}^{m} y_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\lambda_2 - \sum_{i=0}^{n-2}\sum_{j=0}^{m} z_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right)\lambda_3}{\sqrt{-\varepsilon\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2\right)}}$$

$$= \frac{g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right), \lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}$$

Similarly, the coefficient f and g of the second fundamental form are obtained by using the Theorem (2.6) and the Theorem (3.6).

Corollary 3.11. The coefficients (e, f, g) of the second fundamental form of the timelike (spacelike) Bezier surface at the min point $\left(u,v\right)=\left(0,0\right)$ in the Minkowski-3 space are obtained by the equations

$$e = \frac{g\left(b_{00}^{(2,0)}, \lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}},\ f = \frac{g\left(b_{00}^{(1,1)}, \lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}},\ g = \frac{g\left(b_{00}^{(0,2)}, \lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}} \tag{3.10}$$

where $\lambda = \left(\lambda_1, \lambda_2, \lambda_3\right)$ is defined by the components $\lambda_1 = \left(y_{00}^{(0,1)} z_{00}^{(1,0)} - y_{00}^{(1,0)} z_{00}^{(0,1)}\right)$, $\lambda_2 = \left(x_{00}^{(1,0)} z_{00}^{(0,1)} - x_{00}^{(0,1)} z_{00}^{(1,0)}\right)$ and $\lambda_3 = \left(x_{00}^{(1,0)} y_{00}^{(0,1)} - x_{00}^{(0,1)} y_{00}^{(1,0)}\right)$.

Theorem 3.12. The Gauss and the mean curvature of the timelike (spacelike) Bezier surface $b\left(u,v\right)$ in the Minkowski-3 space are obtained with the following equations.

$$K = -\frac{\varepsilon}{g^2\left(\lambda,\lambda\right)} \cdot \left(\begin{array}{l} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)} B_i^{n-2}\left(u\right) B_j^{m}\left(v\right), \lambda\right) . g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2} b_{ij}^{(0,2)} B_i^{n}\left(u\right) B_j^{m-2}\left(v\right), \lambda\right) \\ -g^2\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)} B_i^{n-1}\left(u\right) B_j^{m-1}\left(v\right), \lambda\right) \end{array} \right) \tag{3.11}$$

$$H=\frac{-\varepsilon}{2\sqrt{-\varepsilon.g^{3}\left(\lambda,\lambda\right)_{L}}}\left(\begin{array}{l} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}\left(u\right)B_j^{m}\left(v\right),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right)\right)\\ -2g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}\left(u\right)B_j^{m-1}\left(v\right),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right)\right)\\ +g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}\left(u\right)B_j^{m-2}\left(v\right),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right)\right) \end{array}\right) \tag{3.12}$$

Proof: The Gauss and the mean curvature of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski space are obtained by utilizing the Theorem (3.2) and the Theorem (3.10). The Gauss curvature of the Bezier surface is calculated by

$$K=\varepsilon\left(\frac{eg-f^{2}}{EG-F^{2}}\right) \tag{3.13}$$

$$=\varepsilon.\frac{\dfrac{g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}\left(u\right)B_j^{m}\left(v\right),\lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}.\dfrac{g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}\left(u\right)B_j^{m-2}\left(v\right),\lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}-\dfrac{g^{2}\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}\left(u\right)B_j^{m-1}\left(v\right),\lambda\right)}{\left(\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}\right)^{2}}}{-g\left(\lambda,\lambda\right)}$$

$$=-\frac{\varepsilon}{g^{2}\left(\lambda,\lambda\right)}.\left(g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}\left(u\right)B_j^{m}\left(v\right),\lambda\right).g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}\left(u\right)B_j^{m-2}\left(v\right),\lambda\right)-g^{2}\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}\left(u\right)B_j^{m-1}\left(v\right),\lambda\right)\right).$$

As for the mean curvature H, it is obtained by

$$H=\varepsilon\frac{1}{2}.\left(\frac{eG-2fF+gE}{EG-F^{2}}\right) \tag{3.14}$$

$$=\frac{-\varepsilon}{2g\left(\lambda,\lambda\right)}\left(\begin{array}{l} \dfrac{g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}\left(u\right)B_j^{m}\left(v\right),\lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right)\right)\\ -2\dfrac{g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}\left(u\right)B_j^{m-1}\left(v\right),\lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}\left(u\right)B_j^{m-1}\left(v\right)\right)\\ +\dfrac{g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}\left(u\right)B_j^{m-2}\left(v\right),\lambda\right)}{\sqrt{-\varepsilon g\left(\lambda,\lambda\right)}}g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}\left(u\right)B_j^{m}\left(v\right)\right) \end{array}\right)$$

$$=\frac{-\varepsilon}{2\sqrt{-\varepsilon.g^3(\lambda,\lambda)_L}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -2g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ +g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2} b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v)\right) \end{pmatrix}.$$

Corollary 3.13. The Gauss and the mean curvature of the timelike (spacelike) Bezier surface $b(u,v)$ at the minimum point $(u,v)=(0,0)$ in Minkowski space is computed by

$$K=\frac{-\varepsilon}{g^2(\lambda,\lambda)}\left(g\left(b_{00}^{(2,0)},\lambda\right)g\left(b_{00}^{(0,2)},\lambda\right)-g^2\left(b_{00}^{(1,1)},\lambda\right)\right)$$

$$H=\frac{-\varepsilon}{2\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(b_{00}^{(2,0)},\lambda\right).g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right) \\ -2.g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right) \\ +g\left(b_{00}^{(0,2)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right) \end{pmatrix}.$$

Theorem 3.14. The coefficients of the matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ corresponding to the shape operator of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space are calculated by

$$a_{11}=-\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \end{pmatrix}$$

$$a_{12}=-\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2} b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \end{pmatrix}$$

$$a_{21}=+\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m} b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1} b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m} b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v)\right) \end{pmatrix}$$

$$a_{22}=+\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v)\right)\end{pmatrix}.$$

Proof: The coefficients of the matrix corresponding to the shape operator of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space are calculated with the equations are obtained by writing the statements of the Theorem (3.2) and the Theorem (3.10) in their place in the matrix multiplication

$$A=\frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E\end{pmatrix}.\begin{pmatrix} e & f \\ f & g\end{pmatrix}.$$

When we calculate the components obtained from this matrix multiplication separately, the first component of the matrix is obtained by

$$a_{11}=\frac{eG-fF}{EG-F^2}$$

$$=-\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$+\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$=-\frac{1}{\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right) \\ -g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)\end{pmatrix}.$$

Similarly, other matrix components can be found.

Teorem 3.15. The Gauss and the mean curvature of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space are calculated with the following equations by utilizing the shape operator of the surface.

$$K=-\frac{\varepsilon}{g^2(\lambda,\lambda)}.\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right).g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right) \\ -g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)^2\end{pmatrix}$$

$$H=\frac{-\varepsilon}{2\sqrt{-\varepsilon.g^{3}(\lambda,\lambda)_{L}}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)\\ -2g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)\\ +g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v)\right)\end{pmatrix}.$$

Proof: Alternatively, the Gauss and the mean curvature of the timelike (spacelike) Bezier surface $b(u,v)$ in the Minkowski-3 space can be obtained by utilizing from the coefficients of the matrix corresponding to shape operator as well. Thus, The Gauss curvature of the timelike (spacelike) Bezier surface is calculated with the equation

$$K=\varepsilon\det(A)=\varepsilon\left(a_{11}.a_{22}-a_{12}.a_{21}\right) \tag{3.15}$$

$$=\frac{\varepsilon\left(\lambda_3^2-\lambda_1^2-\lambda_2^2\right)}{g^{3}(\lambda,\lambda)}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)\\ -\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)^2\end{pmatrix}$$

$$=\frac{-\varepsilon g(\lambda,\lambda)}{g^{3}(\lambda,\lambda)}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)\\ -\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)^2\end{pmatrix}$$

$$=-\frac{\varepsilon}{g^{2}(\lambda,\lambda)}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^{n}(u)B_j^{m-2}(v),\lambda\right)\\ -\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)^2\end{pmatrix}.$$

The mean curvature H is obtained by the equation

$$H=\varepsilon.\frac{1}{2}IzA=\frac{\varepsilon}{2}\left(a_{11}+a_{22}\right) \tag{3.16}$$

$$=-\frac{\varepsilon}{2\sqrt{-\varepsilon.g^{3}(\lambda,\lambda)}}g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^{m}(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$+\frac{\varepsilon}{2\sqrt{-\varepsilon.g^{3}(\lambda,\lambda)}}g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$+\frac{\varepsilon}{2\sqrt{-\varepsilon.g^{3}(\lambda,\lambda)}}g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^{m}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^{n}(u)B_j^{m-1}(v)\right)$$

$$-\frac{\varepsilon}{2\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^n(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right)$$

$$=-\frac{\varepsilon}{2\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(\sum_{i=0}^{n-2}\sum_{j=0}^{m}b_{ij}^{(2,0)}B_i^{n-2}(u)B_j^m(v),\lambda\right)g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\right) \\ -2g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}b_{ij}^{(1,1)}B_i^{n-1}(u)B_j^{m-1}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n}\sum_{j=0}^{m-1}b_{ij}^{(0,1)}B_i^n(u)B_j^{m-1}(v)\right) \\ +g\left(\sum_{i=0}^{n}\sum_{j=0}^{m-2}b_{ij}^{(0,2)}B_i^n(u)B_j^{m-2}(v),\lambda\right)g\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v),\sum_{i=0}^{n-1}\sum_{j=0}^{m}b_{ij}^{(1,0)}B_i^{n-1}(u)B_j^m(v)\right) \end{pmatrix}.$$

Corollary 3.16. The coefficients of the matrix A corresponding to the shape operator of the timelike (spacelike) Bezier surface $b(u,v)$ at the point $(u,v)=(0,0)$ are calculated by the equality

$$a_{11}=\frac{-1}{\sqrt{-\varepsilon g^3(\lambda,\lambda)}}\left(g\left(b_{00}^{(2,0)},\lambda\right)g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right)-.g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)\right)$$

$$a_{12}=\frac{-1}{\sqrt{-\varepsilon g^3(\lambda,\lambda)}}.\left(g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right)-g\left(b_{00}^{(0,2)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)\right)$$

$$a_{21}=\frac{1}{\sqrt{-\varepsilon g^3(\lambda,\lambda)}}.\left(g\left(b_{00}^{(2,0)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)-g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right)\right)$$

$$a_{22}=\frac{1}{\sqrt{-\varepsilon g^3(\lambda,\lambda)}}\left(g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)-g\left(b_{00}^{(0,2)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right)\right).$$

Theorem 3.17. Through the coefficients of the shape operator of the timelike (spacelike) Bezier surface $b(u,v)$ at the min point $(u,v)=(0,0)$, the Gauss and the mean curvatures are calculated by the formulas

$$K=\frac{-\varepsilon}{g^2(\lambda,\lambda)}\left(g\left(b_{00}^{(2,0)},\lambda\right)g\left(b_{00}^{(0,2)},\lambda\right)-g^2\left(b_{00}^{(1,1)},\lambda\right)\right)$$

$$H=\frac{-\varepsilon}{2\sqrt{-\varepsilon.g^3(\lambda,\lambda)}}\begin{pmatrix} g\left(b_{00}^{(2,0)},\lambda\right).g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right) \\ -2.g\left(b_{00}^{(1,1)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right) \\ +g\left(b_{00}^{(0,2)},\lambda\right)g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right) \end{pmatrix}$$

in the Minkowski-3 space.

Proof: The Gauss and the mean curvature at the minimum point of the surface are calculated by writing the coefficient values in the Corollary (3.16) into the equations (3.15) ve (3.16).

3. A NUMERIC EXAMPLE

The equation of the quadratic timelike Bezier surface $b(u,v)$ of which control points are b_{00} = (2,1,3), b_{01} = (4,2,6), b_{02} = (2,6,7), b_{10} = (3,3,8), b_{11} = (3,6,8), b_{12} = (3,7,8), b_{20} = (4,3,6), b_{21} = (4,6,9), b_{22} = (4,8,9) in the Minkowski space is calculated by

$$b(u,v)=\sum_{i=0}^{2}\sum_{j=0}^{2}B_i^2(u)B_j^2(v).b_{ij}=\sum_{i=0}^{2}\left(\sum_{j=0}^{2}B_j^2(v).b_{ij}\right)B_i^2(u)$$

$$=\begin{pmatrix}\left((1-u)^2.(1-v)^2.(2,1,3)+2(1-u)^2.v(1-v).(2,4,6)+(1-u)^2v^2.(2,6,7)\right)\\+\left(2.u.(1-u).(1-v)^2.(3,3,8)+4u.v(1-u).(1-v).(3,6,8)+2.u(1-u).v^2.(3,7,8)\right)\\+\left(u^2.(1-v)^2.(4,3,6)+2u^2.v(1-v).(4,6,9)+u^2v^2.(4,8,9)\right)\end{pmatrix}$$

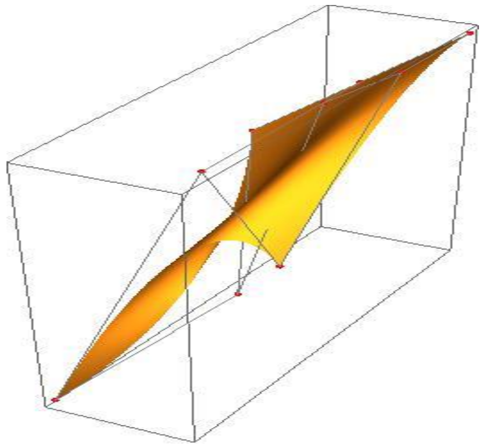

Figure 1. Quadratic timelike Bezier surface.

Let analyze the surface metric of this described quadratic timelike Bezier surface at the point $b(0,0)$. The first-order partial derivatives of the quadratic timelike Bezier surface is obtained by the representation $b_{ij}^{(1,0)}=n\left(b_{(i+1)j}-b_{ij}\right)$ and $b_{ij}^{(0,1)}=m\left(b_{i(j+1)}-b_{ij}\right)$. In that case, the first-order partial derivatives of the quadratic Bezier surface are calculated as $b_{10}^{(1,0)}=(2,0,-4)$, $b_{01}^{(0,1)}=(0,4,2)$ $b_{00}^{(1,0)}=(2,4,10)$, $b_{00}^{(0,1)}=(0,6,6)$ $b_{01}^{(1,0)}=(2,4,4)$. The coefficients E,F,G of the first fundamental form of the quadratic timelike Bezier surface are calculated respectively as $E=g\left(b_{00}^{(1,0)},b_{00}^{(1,0)}\right)=-80$, $F=g\left(b_{00}^{(1,0)},b_{00}^{(0,1)}\right)=-36$ $G=g\left(b_{00}^{(0,1)},b_{00}^{(0,1)}\right)=0$ which are from the Corollary (3.3). The second-order derivatives of the quadratic timelike Bezier surface are obtained to be $ds^2=-80du^2-72.du.dv$ from the Corollary (3.5). The second-order derivatives of the quadratic timelike Bezier surface are got as $b_{uu}(0,0)=(0,-4,-14)$, $b_{uv}(0,0)=(0,0,-12)$, $b_{vv}(0,0)=(0,-2,-4)$. The normal vector field N of the quadratic timelike Bezier surface is the unit normal vector $N(u,v)\big|_{(u,v)=(0,0)}=(1,1/3,1/3)$ of the surface from the Corollary (3.7). The equation $g\left(N(0,0),N(0,0)\right)=1^2+(1/3)^2-(1/3)^2=1$ is obtained with the help of the Minkowski metric. Therefore, the normal vector N is spacelike. The coefficients *e*, *f*, *g* of the second fundamental form of the quadratic timelike Bezier surface are calculated respectively to be

$e = g\left(b_{uu}\left(0,0\right), N\left(0,0\right)\right) = 10/3$, $f = g\left(b_{uv}\left(0,0\right), N\left(0,0\right)\right) = 4$, $g = g\left(b_{vv}\left(0,0\right), N\left(0,0\right)\right) = 2/3$. Since $\det W = EG - F^2 = -(36)^2 < 0$ the surface is timelike. Since the surface is timelike, it's taken to be $\varepsilon = 1$. The coefficients of the matrix corresponding to the shape operator of the quadratic timelike Bezier surface is obtained to be $a_{11} = -\frac{1}{9}, a_{12} = \frac{25}{162}, a_{21} = -\frac{1}{54}, a_{22} = -\frac{17}{243}$. By utilizing the shape operator of the quadratic timelike Bezier surface, the Gauss and the mean curvature are found respectively as $K = 31/2916$, $H = -22/243$ from the equations (3.15) and (3.16).

4. CONCLUSION

In this study, it was discussed that the analysis of the Bezier surfaces commonly used in the Computer-Aided Geometric Desing (CAGD) in the Minkowski space in terms of geometric perspective. After obtaining the coefficients of the first and the second fundamental form of the timelike and the spacelike Bezier surfaces, the Gauss and the mean curvatures, the shape operator is calculated in the Minkowski space for the first time.

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