ORIGINAL PAPER

B-LIFT CURVES IN LORENTZIAN 3-SPACE

ANIL ALTINKAYA[1], MUSTAFA ÇALIŞKAN[1]

Manuscript received: 15.12.2021; Accepted paper: 23.02.2022;
Published online: 30.03.2022.

Abstract. *In this article, we define a new curve which is called B-Lift curve in 3-dimensional Lorentzian space and we obtain the Frenet vectors of the B-Lift curve. Also, we compare the Frenet vectors of the B-Lift curve and the natural lift curve.*
Keywords: *B-Lift; natural lift; Lorentzian space; Frenet vectors.*

1. INTRODUCTION

Riemannian geometry plays an important role in the development of the theoretical physics. However, it is insufficient to explain the special and general relativity. Hence, Lorentzian geometry has emerged which defined by a metric on a smooth manifold. For many years, Riemannian geometry and Lorentzian geometry developed separately. More recently, with the increasing interest of physicists, this difference has been reversed. Lorentzian space has differences and similarities with the Euclidean space. The curves in this space have Lorentzian characters such as spacelike, timelike or lightlike (null) [1].

The characterizations and the relationships among the curves are crucial problems in differential geometry. One of the most important to make this characterization is the Frenet-Serret formulas which were discovered independently by two French mathematicians J. F. Frenet (1847) and J. A. Serret (1851). These formulas are equations that use curvature and torsion to express the derivatives of the three vector fields that make up the Frenet frame in terms of vector fields $\{T, N, B\}$ for the study of space curves [2].

Many mathematicians have studied on the theory of curves. E. Ergün and M. Çalışkan [3] established the relationship between the Frenet vectors of the natural lift curve in the three-dimensional Lorentzian space. The definition of the natural lift curve was first encountered in an exercise in Thorpe's book [4]. According to the definition, it is a curve formed by combining the ends of the tangent vector of the curve.

In this study, based on Thorpe's definition, we defined the B-Lift curve and obtained the Frenet vectors of this curve in the three-dimensional Lorentzian space. Moreover, we introduce the equations of the Frenet vectors between the natural lift curve and the B-Lift curve. Finally, we give examples about these results and plotted these curves.

2. PRELIMINARIES

In this section, we give some basic definitions and theorems in differential geometry. Besides, we denote natural lift curve and examine the relationship between Frenet frames of natural lift curve in Lorentzian 3-space.

[1] Gazi University, Mathematics Department, 06560 Ankara, Turkey. E-mail: anilaltinkaya@gazi.edu.tr; mustafacaliskan@gazi.edu.tr.

The Lorentzian 3-space $\mathbb{R}_1^3$ is the real vector space $\mathbb{R}^3$ provided with Lorentzian inner product defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ [1].

A vector $x = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ is called spacelike if $\langle x, x \rangle > 0$ or $x = 0$, timelike if $\langle x, x \rangle < 0$ and lightlike (null) if $\langle x, x \rangle = 0$ and $x \neq 0$. An arbitrary curve $\gamma: I \subset \mathbb{R} \to \mathbb{R}_1^3$ are spacelike, timelike or lightlike (null), if $\gamma'(s)$ are spacelike, timelike or lightlike (null) at any $s \in I$, respectively. The norm of a vector $x = (x_1, x_2, x_3)$ defined as

$$\left||x|\right|_{IL} = \sqrt{|\langle x, x \rangle|} \quad [1].$$

If $\left||x|\right|_{IL} = 1$, then the vector x is called unit vector. Let x and y be vectors in $\mathbb{R}_1^3$, then the Lorentzian vector product is defined as

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2) \quad [5].$$

Assume that γ be a unit speed curve. $\{T(s), N(s), B(s)\}$ is called Frenet-Serret frame given by tangent, principal normal and binormal vectors, respectively. Now, we examine Frenet formulas depending on the Lorentzian character of the curve [6]:

Let γ be unit speed timelike curve. Then, T is a timelike vector, N and B are spacelike vectors. In this situation, we can write

$$N \times B = T\,,\ T \times N = -B,\ \ B \times T = -N.$$

Frenet formulas are given as

$$T' = \kappa N, N' = \kappa T + \tau B, B' = -\tau N.$$

Let γ be unit speed spacelike curve with spacelike binormal. Then, T and B are spacelike vectors, N is a timelike vector. In this situation, we can

$$N \times B = -T,\ T \times N = -B,\ B \times T = N.$$

Frenet formulas are given as follows

$$T' = \kappa N, N' = \kappa T + \tau B, B' = \tau N.$$

Let γ be unit speed spacelike curve with timelike binormal. Then, T and B are spacelike vectors, B is a timelike vector. In this situation, we can

$$N \times B = -T,\ T \times N = B,\ B \times T = -N.$$

Frenet formulas are given by

$$T' = \kappa N, N' = -\kappa T + \tau B, B' = \tau N.$$

Lemma 1. Let x and y be linearly independent spacelike vectors which span a spacelike vector subspace in $\mathbb{R}_1^3$. Then we have,

$$|<x,y>| \leq ||x||.||y||.$$

Hence we can write,

$$<x,y> = ||x||||y||cos\varphi \, ,$$

where φ is a Lorentzian spacelike angle between x and y [7].

Lemma 2. Let x and y be linearly independent spacelike vectors which span a timelike vector subspace in $\mathbb{R}_1^3$. Then we have,

$$|<x,y>| > ||x||.||y||.$$

Therefore we can write,

$$|<x,y>| = ||x||||y||cosh\varphi \, ,$$

where φ is Lorentzian timelike angle between x and y [7].

Lemma 3. Let x be spacelike vector and y be timelike vector in $\mathbb{R}_1^3$. Then we can write,

$$|<x,y>| = ||x||||y||sinh\varphi \, ,$$

where φ is Lorentzian timelike angle between x and y [7].

Lemma 4. Let x and y be two timelike vectors in $\mathbb{R}_1^3$. Then we can write,

$$<x,y> = ||x||||y||cosh\varphi \, ,$$

where φ is Lorentzian timelike angle between x and y [7].

Definition 1. Let $\gamma : I \to P$ be a parametrized curve, where $P \subset \mathbb{R}_1^3$ be a surface. We called an integral curve to the curve γ if

$$\gamma'(s) = X(\gamma(s))$$

where X is a differentiable vector field on P [1].

Definition 2. Let $\gamma : I \to P$ be a parametrized curve. The natural lift curve $\bar{\gamma} : I \to TP$ of the curve γ is defined by

$$\bar{\gamma}(s) = \big(\gamma(s), \gamma'(s)\big) = \gamma'(s)|_{\gamma(s)} \quad [1].$$

Let γ be a timelike curve. Then, $\bar{\gamma}$ is a spacelike curve with timelike or spacelike binormal [3].

i) Let $\bar{\gamma}$ be a spacelike curve with timelike binormal. We know the following equations between the Frenet frame $\{\bar{T}, \bar{N}, \bar{B}\}$ of the curve $\bar{\gamma}$ and the Frenet frame $\{T, N, B\}$ of the curve γ.

a) If W is spacelike vector, we have

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we have

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) Let $\bar{\gamma}$ be a spacelike curve with spacelike binormal. We know the following equations between the Frenet frame $\{\bar{T}, \bar{N}, \bar{B}\}$ of the curve $\bar{\gamma}$ and the Frenet frame $\{T, N, B\}$ of the curve γ.

a) If W is spacelike vector, we have

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ cosh\varphi & 0 & sinh\varphi \\ -sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we have

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ sinh\varphi & 0 & cosh\varphi \\ -cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let γ be a spacelike curve with spacelike binormal. Then, $\bar{\gamma}$ is a timelike curve. We know the following equations between the Frenet frame $\{\bar{T}, \bar{N}, \bar{B}\}$ of the curve $\bar{\gamma}$ and the Frenet frame $\{T, N, B\}$ of the curve γ [6].

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ cos\varphi & 0 & sin\varphi \\ sin\varphi & 0 & -cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let γ be a spacelike curve with timelike binormal. Then, $\bar{\gamma}$ is a spacelike curve with timelike or spacelike binormal [3].

i) Let $\bar{\gamma}$ is a spacelike curve with timelike binormal. We know the following equations between the Frenet frame $\{\bar{T}, \bar{N}, \bar{B}\}$ of the curve $\bar{\gamma}$ and the Frenet frame $\{T, N, B\}$ of the curve γ.

a) If W is spacelike vector, we get

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -sinh\varphi & 0 & cosh\varphi \\ -cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -cosh\varphi & 0 & sinh\varphi \\ -sinh\varphi & 0 & cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) Let $\bar{\gamma}$ is a spacelike curve with spacelike binormal. We know the following equations between the Frenet frame $\{\bar{T}, \bar{N}, \bar{B}\}$ of the curve $\bar{\gamma}$ and the Frenet frame $\{T, N, B\}$ of the curve γ.

a) If W is spacelike vector, we get

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

3. B-LIFT CURVES IN LORENTZIAN 3-SPACE

In this chapter, we denote the B-Lift curve in $\mathbb{R}_1^3$ which is the curve obtained by the end points of the binormal vectors of the main curve. We also introduce the Frenet vectors of the B-Lift curve and examine the relation between the Frenet vectors of the B-Lift curve and the natural lift curve. (In the calculations, specially the torsion will be taken as greater than zero.)

Definition 3. For any unit speed curve $\gamma : I \to P$, $\gamma_B : I \to TP$ is called the B-Lift curve in $\mathbb{R}_1^3$ of γ which provides the following equation:

$$\gamma_B(s) = \big(\gamma(s), B(s)\big) = B(s)|_{\gamma(s)}$$

where B is the binormal vector of the curve γ.

Let γ be a timelike curve. Then, γ_B is a spacelike curve with timelike or spacelike binormal.

i) Let γ_B be a spacelike curve with timelike binormal. We know the following equations between the Frenet frame $\{T_B, N_B, B_B\}$ of the curve γ_B and the Frenet frame $\{T, N, B\}$ of the curve γ:

a) If W is spacelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -cosh\varphi & 0 & -sinh\varphi \\ sinh\varphi & 0 & cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -sinh\varphi & 0 & -cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) Let γ_B be a spacelike curve with spacelike binormal. We know the following equations between the Frenet frame $\{T_B, N_B, B_B\}$ of the curve γ_B and the Frenet frame $\{T, N, B\}$ of the curve γ:

a) If W is spacelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let γ be a spacelike curve with spacelike binormal. Then, γ_B is a timelike curve. We know the following equations between the Frenet frame $\{T_B, N_B, B_B\}$ of the curve γ_B and the Frenet frame $\{T, N, B\}$ of the curve γ:

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ cos\varphi & 0 & sin\varphi \\ sin\varphi & 0 & -cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Let γ be a spacelike curve with timelike binormal. Then, $\gamma_B$is a spacelike curve with timelike or spacelike binormal.

i) Let γ_B is a spacelike curve with timelike binormal. We know the following equations between the Frenet frame $\{T_B, N_B, B_B\}$ of the curve γ_B and the Frenet frame $\{T, N, B\}$ of the curve γ:

a) If W is spacelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -cosh\varphi & 0 & sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) Let γ_B is a spacelike curve with spacelike binormal. We know the following equations between the Frenet frame $\{T_B, N_B, B_B\}$ of the curve γ_B and the Frenet frame $\{T, N, B\}$ of the curve γ:

a) If W is spacelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ sinh\varphi & 0 & -cosh\varphi \\ cosh\varphi & 0 & -sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) If W is timelike vector, we get

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ cosh\varphi & 0 & -sinh\varphi \\ sinh\varphi & 0 & -cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Corollary 1. Let γ be a timelike curve. Then, γ_B and $\bar{\gamma}$ are spacelike curve with timelike or spacelike binormal.

i) Let γ_B and $\bar{\gamma}$ be spacelike curves with timelike binormals. Then the following equations are provided:

$$T_B = -\bar{T},$$

$$N_B = -\bar{N},$$

$$B_B = \bar{B}.$$

ii) Let γ_B and $\bar{\gamma}$ be spacelike curves with spacelike binormals. Then the following equations are provided:

$$T_B = -\bar{T},$$

$$N_B = \bar{N},$$

$$B_B = -\bar{B}.$$

where $\{T_B, N_B, B_B\}$ and $\{\bar{T}, \bar{N}, \bar{B}\}$ are the Frenet vectors of the curves γ_B and $\bar{\gamma}$, respectively.
Corollary 2. Let γ be a spacelike curve with spacelike binormal. Then, γ_B and $\bar{\gamma}$ are timelike curve. We have the following equations:

$$T_B = \bar{T},$$

$$N_B = \bar{N},$$

$$B_B = \bar{B}.$$

Corollary 3. Let γ be a spacelike curve with timelike binormal. Then, γ_B and $\bar{\gamma}$ are spacelike curve with timelike or spacelike binormal. We have the following equations:

i) Let γ_B and $\bar{\gamma}$ be spacelike curves with timelike binormals. Then the following equations are provided:

$$T_B = \bar{T},$$

$$N_B = \bar{N},$$

$$B_B = -\bar{B}.$$

ii) Let γ_B and $\bar{\gamma}$ be spacelike curves with spacelike binormals. Then the following equations are provided:

$$T_B = \bar{T},$$

$$N_B = -\bar{N},$$

$$B_B = \bar{B}.$$

where $\{T_B, N_B, B_B\}$ and $\{\bar{T}, \bar{N}, \bar{B}\}$ are the Frenet vectors of the curves γ_B and $\bar{\gamma}$, respectively.

Example 1. Let γ be a unit speed timelike circular helix curve that is given by

$$\gamma(s) = (\sqrt{2}s, coss, sins).$$

After some calculations the Frenet vectors of the curve γ are given as follows:

$$T(s) = \left(\sqrt{2}, -sins, coss\right)$$
$$N(s) = (0, -coss, -sins)$$
$$B(s) = (-1, \sqrt{2}sins, \sqrt{2}coss).$$

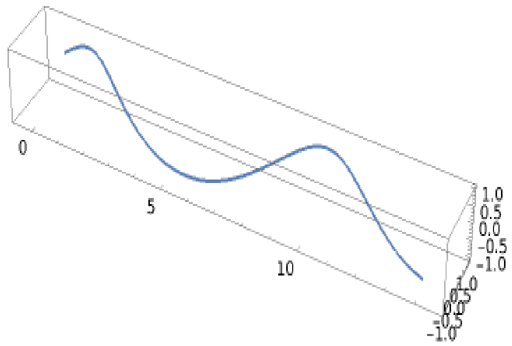

Figure 1. The circular helix

In Figure 1, we represented the circular helix curve $\gamma(s)$. Since $\gamma_B(s) = B(s)$, we have the following equation:

$$\gamma_B(s) = (-1, \sqrt{2}sins, \sqrt{2}coss).$$

B-Lift curve of the curve $\gamma(s)$ is represented in Figure 2 as follows:

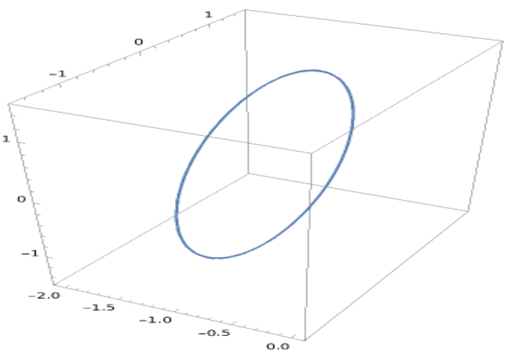

Figure 2. The circle

Example 2. Consider the unit speed spacelike hyperbolic helix with

$$\gamma(s) = (cosh\frac{s}{\sqrt{3}}, sinh\frac{s}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}s).$$

Then the Frenet vectors of the curve γ are given as following equations:

$$T(s) = \left(\frac{1}{\sqrt{3}}sinh\frac{s}{\sqrt{3}}, \frac{1}{\sqrt{3}}cosh\frac{s}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$$
$$N(s) = \left(cosh\frac{s}{\sqrt{3}}, sinh\frac{s}{\sqrt{3}}, 0\right)$$
$$B(s) = (\frac{\sqrt{2}}{\sqrt{3}}sinh\frac{s}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}cosh\frac{s}{\sqrt{3}}, -\frac{1}{\sqrt{3}}).$$

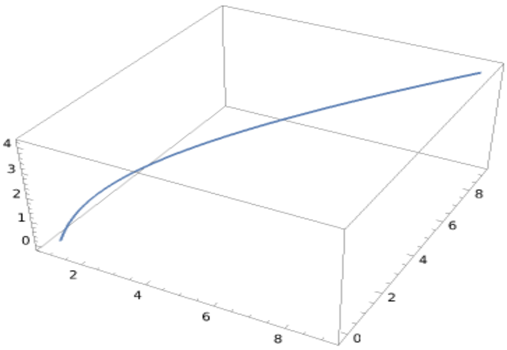

Figure 3. The unit speed spacelike hyperbolic helix

In Figure 3, we drew the unit speed spacelike hyperbolic helix curve $\gamma(s)$. Since $\gamma_B(s) = B(s)$, we have

$$\gamma_B(s) = (\frac{\sqrt{2}}{\sqrt{3}}sinh\frac{s}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}cosh\frac{s}{\sqrt{3}}, -\frac{1}{\sqrt{3}}).$$

B-Lift curve of the curve $\gamma(s)$ is showed in Figure 4 as follows:

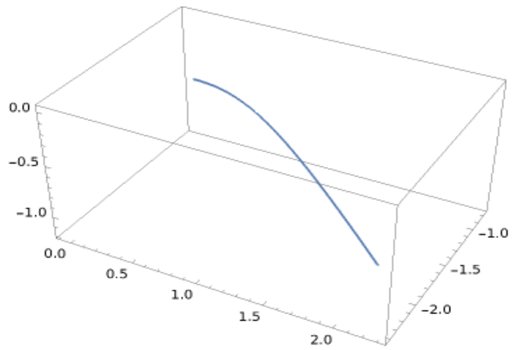

Figure 4. The unit speed timelike hyperbola

4. CONCLUSION

In this study, we introduced the B-Lift curve and examined the Frenet vectors of the B-Lift curve in Lorentzian 3-space. Furthermore, we obtained the relations between the Frenet frames of the natural lift curve and B-Lift curve.

REFERENCES

[1] O'Neill, B., *Semi Riemann Geometry*, Academic Press, New York and London, pp. 42-187, 1983.

[2] Do. Carmo, M., *Differential Geometry of Curves and Surfaces*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, pp. 1-114, 1976.

[3] Ergün, E., Çalışkan, M., *Journal of Science and Arts*, **45**(4), 869, 2018.

[4] Thorpe, J.A., *Elementary Topics in Differential Geometry*, Springer Verlag, New York, Heidelberg-Berlin, p. 61, 1979.

[5] Önder, M., Uğurlu, H.H., *Ain Shams Engineering Journal*, **4**, 507, 2013.

[6] Walrave, J., *Curves and Surfaces in Minkowski Space*, K.U. Leuven Faculteit, Der Wetenschappen, pp. 1-31, 1995.

[7] Ratcliffe, J.G., *Foundations of Hyperbolic Manifolds*, Springer-Verlag Inc., New York, pp. 54-98, 1994.