ORIGINAL PAPER

THE HARMONIC FUNCTIONS OF POSITION VECTOR COMPONENTS IN THE THREE-DIMENSIONAL LORENTZIAN HEISENBERG GROUP $\mathbb{H}_3^1$

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Abstract. *In this paper we study the relationship between the Laplace-Beltrami operator Δ of the position vector field and the mean curvature vector field r of surfaces defined as a graph of functions in the three-dimensional Lorentzian Heisenberg group $\mathbb{H}_3^1$ which is endowed with left invariant Lorentzian metrics g_i, $(i = 1,2,3)$ and we prove that the surface as graph is minimal in $\mathbb{H}_3^1$, if and only if the components of the position vector field r are harmonic functions.*

Keywords: *Heisenberg group $\mathbb{H}_3^1$; left invariant Lorentzian metric; finite type surface; minimal surface; Laplacian operator; harmonic function.*

1. INTRODUCTION

During 1980-1990, B-Y. Chen introduced the notion of finite type immersion in the dimensional Euclidean and pseudo Euclidean spaces, this notion is a natural extension of minimal surfaces.

The classification of 1-type submanifolds of Euclidean space $\mathbb{E}^3$ [5-7], was done in 1966 by T. Takahashi. He proved that the submanifolds in $\mathbb{R}^m$ satisfy the differential equation $\Delta r = \mu r$ for some real number μ, if and only if, either the submanifold is a minimal submanifold of $\mathbb{R}^m$ $(\mu = 0)$, or a hypersphere of $\mathbb{R}^m$ centred at the origin $\mu \neq 0$.

Next, F. Dillen, J. Pas and L. Verstraelen, observed and proposed the study of submanifolds of $\mathbb{R}^m$ satisfying the following equation $\Delta r = Ar + B$ where A and B are two 3×3-real matrices.

The same and other authors studied several problems related to the subject of finite type particular surfaces like translation surfaces, the quadrics, surfaces of revolution, helicoidal surfaces.

The study of this notion of finite type was extended for surfaces in Euclidean 3-space $\mathbb{E}^3$ of specific form such that their Gauss map $\boldsymbol{N}$ satisfies an analogous or similar equation $\Delta \boldsymbol{N} = A\boldsymbol{N}$ where A is a 3×3-real matrice.

Similarly, many authors studied particular surfaces of finite type in the Euclidean, pseudo Euclidean and Lorentz-Minkowski 3-dimensional space satisfying the differential equation $\Delta^{II} r = Ar, \Delta^{III} r = Ar$, where A is a 3×3 - real matrice and Δ^{II}, Δ^{III} are respectively the Laplace operator with respect to the second and third fundamental form which are not degenerated.

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2. PRELIMINARIES

2.1. BASIC DEFINITIONS

Heisenberg group is a two-step nilpotent Lie group which is a subgroup of linear group $GL(3,\mathbb{R})$. It is known as a quantum physics model invariably studied by the theoretical physicists and mathematicians.

- The Heisenberg group is a 3-dimensional, connected. It is a 3-dimensional Riemannian manifold.
- The Heisenberg group $\mathbb{H}_3$ can be seen as the space $\mathbb{R}^3$ endowed with multiplication:

$$(\bar{x},\bar{y},\bar{z}) * (x,y,z) = (\bar{x}+x, \bar{y}+y, \bar{z}+z-\bar{x}y+x\bar{y}).$$

- The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$.

Definition1.1. *In Lorentzian Heisenberg space with the metric g*

- The vector X is called a space-like vector or time-like vector if and only if

$$g(X,X) = \epsilon = \begin{cases} 1, & X \ \ is\ spacelike\ vector \\ -1, & X \ \ is\ timelike\ vector \end{cases}$$

- If X is a vector in Lorentzian Heisenberg space we define the norm of X by

$$\|X\| = \sqrt{|g(X,X)|} \qquad .$$

- The vectors X and Y are said to be orthogonal if

$$g(X,Y) = 0$$

- *A vector X which satisfies $g(X,X) = \pm 1$ is called a unit vector.*

Let S be a regular surface in $\mathbb{H}_3$ define by

$$S:\ r(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),\ \ (u,v) \in D \subset \mathbb{R}^2.$$

Definition1.2. The mean curvature of the surface S in $\mathbb{H}_3$ is given by

$$H = \frac{EN + GL - 2FM}{2|EG - F^2|}, \tag{1}$$

where E, F and G are the coefficients of the first fundamental form and L, M and N are the coefficients of the second fundamental form of the surface S.

Definition1.3. *A surface S in the 3-dimensional Heisenberg Space is said to be minimal if it satisfies the condition $H = 0$ where H is the mean curvature of the surface S.*

Definition1.4. *The immersion* $(S,\ r)$ *is said to be of finite Chen-type (or finite-type) if the position vector* r *admits the following spectral decomposition*

$$r = r_0 + \sum_{1}^{k} r_i\ , \tag{2}$$

where r_0 *is a constant and* $r_i,\ (i = 1, 2, \dots, k)$ *are non-constant maps such that* $\Delta r_i = \mu_i r_i$, $\mu_i \in R, (i = 1, 2, \dots, k)$ *If all eigenvalues* μ_i *are different, then* S *is said to be of* k*-type.*

- Explicitly if the surface S in $\mathbb{E}^3$ is given by.

$$S\colon \mathrm{r}(\mathrm{u}, \mathrm{v}) = \big(\mathrm{r}_1(\mathrm{u}, \mathrm{v}), \mathrm{r}_2(\mathrm{u}, \mathrm{v}), \mathrm{r}_3(\mathrm{u}, \mathrm{v})\big),\ \ (\mathrm{u}, \mathrm{v}) \in \mathrm{D} \subset \mathbb{R}^2 \tag{3}$$

The well known equation, for all regular surfaces can be obtained

$$\Delta r = -2\boldsymbol{H}\ , \tag{4}$$

where Δ is the Laplace operator and $\boldsymbol{H}$ is the mean curvature vector field of S.
From (3), it is known that the minimal surfaces and spheres also verify the condition (3) shows that S is minimal surface, if and only if $r_i,\ (i = 1,2,3)$ are harmonic functions.

Definition1.5. *A surface* S *in the 3-dimensional Heisenberg Space is said to be Harmonic if it satisfies the condition* $\Delta r_i = 0$ *where* r_i *are the components of the position vector field* r.
If the matrix (g_{ij}) consists on the components of the induced any metric on S and (g^{ij}) its inverse and $\mathrm{D} = \det(g_{\mathrm{ij}})$, the Laplacian (Beltrami's operator) Δ on S is given by

$$\Delta = \frac{-1}{\sqrt{|\mathrm{D}|}} \sum_{\mathrm{i,j}} \frac{\partial}{\partial \mathrm{x}^{\mathrm{i}}} \left(\sqrt{|\mathrm{D}|} g^{\mathrm{ij}} \frac{\partial}{\partial \mathrm{x}^{\mathrm{j}}} \right). \tag{5}$$

If $\mathrm{S}\colon \mathrm{r}(u, v) = \big(\mathrm{r}_1(u, v), \mathrm{r}_2(u, v), \mathrm{r}_3(u, v)\big),\ (u, v) \in \mathrm{D} \subset \mathbb{R}^2$ is a function of class C^2 then, we set

$$\Delta \mathrm{r} = (\Delta \mathrm{r}_1, \Delta \mathrm{r}_2, \Delta \mathrm{r}_3). \tag{6}$$

In [6]. Rahmani, S suggested a classification of all left-invariant Lorentzian metric tensors on uni-modular Lie groups of dimension 3. Authors also deduced a classification of left invariant Lorentzian metrics on the Heisenberg group.

- Heisenberg group has three left invariant Lorentz metric which are

$$g_1 = \sum_{i=1}^{3} \omega_1^i = -\frac{1}{\mu^2} dx^2 + dy^2 + (xdy + dz)^2,$$
$$g_2 = \sum_{i=1}^{3} \omega_2^i = \frac{1}{\mu^2} dx^2 + dy^2 - (xdy + dz)^2,$$
$$g_3 = \sum_{i=1}^{3} \omega_3^i = dx^2 + (xdy + dz)^2 - \big((1 - x)dy + dz\big)^2.$$

These 1-forms ω_j^i , $(i, j = 1,2,3)$ in g_j, $(j = 1,2,3)$ are invariant by left translations in $\mathbb{H}_3^1$ and by rotations about (Oz) $-$axis. The left invariant orthonormal coframe is associated with the orthonormal left invariant frame.

3. GEOMETRIC PROPERTIES OF $(\mathbb{H}_3^1,\ \boldsymbol{g_i}),\ (\boldsymbol{i = 1,2,3})$

In this section we give some geometric properties of the three-dimensional Heisenberg group endowed with a left-invariant Lorentzian metric $\boldsymbol{g_i},\ (\boldsymbol{i = 1,2,3})$.

3.1. THE METRIC g_1

- The Lorentz metric $\boldsymbol{g_1}$ is

$$g_1 = -\frac{1}{\mu^2}dx^2 + dy^2 + (xdy + dz)^2,$$

where

$$\omega_1^1 = xdy + dz, \quad \omega_1^2 = dy, \quad \omega_1^3 = \frac{1}{\mu}dx,$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_1^1 = \partial_z, \quad e_1^2 = \partial_y - x\partial_z, \quad e_1^3 = \mu\partial_x,$$

where $\partial_u = \frac{\partial}{\partial u}$.

With the Lie brackets

$$[e_1^1,\ e_1^2] = [e_1^1,\ e_1^3] = 0, \qquad [e_1^2,\ e_1^3] = \mu e_1^1.$$

and

$$g_1(e_1^1, e_1^1) = g_1(e_1^2, e_1^2) = 1, \quad g_1(e_1^3, e_1^3) = -1.$$

The Levi-Civita connection of the left-invariant metric g_1 is given by

$$\begin{pmatrix} \nabla_{e_1^1} e_1^1 \\ \nabla_{e_1^1} e_1^2 \\ \nabla_{e_1^1} e_1^3 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} 0 \\ e_1^3 \\ e_1^2 \end{pmatrix}, \quad \begin{pmatrix} \nabla_{e_1^2} e_1^1 \\ \nabla_{e_1^2} e_1^2 \\ \nabla_{e_1^2} e_1^3 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} e_1^3 \\ 0 \\ e_1^1 \end{pmatrix}, \quad \begin{pmatrix} \nabla_{e_1^3} e_1^1 \\ \nabla_{e_1^3} e_1^2 \\ \nabla_{e_1^3} e_1^3 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} e_1^2 \\ -e_1^1 \\ 0 \end{pmatrix}.$$

3.2. THE METRIC g_2

- The Lorentz metric $\boldsymbol{g_2}$ is

$$g_2 = \frac{1}{\mu^2}dx^2 + dy^2 - (xdy + dz)^2,$$

where

$$\omega_2^1 = \frac{1}{\mu}dx, \quad \omega_2^2 = dy, \;\; \omega_2^3 = xdy + dz,$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_2^1 = \mu\partial_x, \;\; e_2^2 = x\partial_z - \partial_y, \quad e_2^3 = \partial_z \,.$$

With the Lie brackets

$$[e_2^1,\, e_2^3] = [e_2^2,\, e_2^3] = 0, \quad [e_2^1,\, e_2^2] = \mu e_2^3.$$

and

$$g_2(e_2^1, e_2^1) = g_2(e_2^2, e_2^2) = 1, \quad g_2(e_2^3, e_2^3) = -1.$$

The Levi-Civita connection of the left-invariant metric g_2 is given by

$$\begin{pmatrix} \nabla_{e_2^1} e_2^1 \\ \nabla_{e_2^1} e_2^2 \\ \nabla_{e_2^1} e_2^3 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} 0 \\ e_2^3 \\ e_2^2 \end{pmatrix}, \qquad \begin{pmatrix} \nabla_{e_2^2} e_2^1 \\ \nabla_{e_2^2} e_2^2 \\ \nabla_{e_2^2} e_2^3 \end{pmatrix} = -\frac{\mu}{2}\begin{pmatrix} e_2^3 \\ 0 \\ e_2^1 \end{pmatrix}, \qquad \begin{pmatrix} \nabla_{e_2^3} e_2^1 \\ \nabla_{e_2^3} e_2^2 \\ \nabla_{e_2^3} e_2^3 \end{pmatrix} = \frac{\mu}{2}\begin{pmatrix} e_2^2 \\ -e_2^1 \\ 0 \end{pmatrix}.$$

3.3. THE METRIC g_3

- The Lorentz metric $\boldsymbol{g_3}$ is

$$g_3 = dx^2 + (xdy + dz)^2 - \big((1-x)dy + dz\big)^2.$$

We have

$$\omega_3^1 = dx, \quad \omega_3^2 = xdz + dy, \quad \omega_3^3 = (1-x)dy - dz,$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$e_3^1 = \partial_x, \;\; e_3^2 = \partial_y + (1-x)\partial_z, \quad e_3^3 = \partial_y - x\partial_z.$$

With the Lie brackets

$$[e_3^1,\, e_3^2] = e_3^3 - e_3^2, \;\; [e_3^1,\, e_3^3] = e_3^3 - e_3^2, \;\; [e_3^2,\, e_3^3] = 0.$$

and

$$g_3(e_3^1, e_3^1) = g_3(e_3^2, e_3^2) = 1, \quad g_3(e_3^3, e_3^3) = -1.$$

The Levi-Civita connection of the left-invariant metric g_3 is given by

$$\begin{pmatrix}\nabla_{e_3^1}e_3^1\\ \nabla_{e_3^1}e_3^2\\ \nabla_{e_3^1}e_3^3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad \begin{pmatrix}\nabla_{e_3^2}e_3^1\\ \nabla_{e_3^2}e_3^2\\ \nabla_{e_3^2}e_3^3\end{pmatrix}=\begin{pmatrix}e_3^2-e_3^3\\ -e_3^1\\ -e_3^1\end{pmatrix},\quad \begin{pmatrix}\nabla_{e_3^3}e_3^1\\ \nabla_{e_3^3}e_3^2\\ \nabla_{e_3^3}e_3^3\end{pmatrix}=\begin{pmatrix}e_3^2-e_3^3\\ -e_3^1\\ -e_3^1\end{pmatrix}.$$

4. THE RELATIONSHIP BETWEEN THE LAPLACIAN AND THE MEAN CURVATURE IN $\mathbb{H}_3^1$

Let S be an immersed surface in $\mathbb{H}_3^1$ which is given as the graph of the function $z=f(x,y)$.

The position vector $r(x,y)$ of S is given by

$$r(x,y)=\big(x,y,f(x,y)\big),\ \ (x,y)\in D\subset\mathbb{R}^2,\quad f\in C^1(D).$$

Denote the Laplacian of the surface and the mean curvature vector field of S in $(\mathbb{H}_3^1,\ \boldsymbol{g_i}),\ \boldsymbol{i=1,2,3}$ by Δ_{g_i} and $\mathbf{H}_{g_i}$ respectively.

Theorem 4.1. A Beltrami formula in $(\mathbb{H}_3^1,\ \boldsymbol{g_i}),\ \boldsymbol{i=1,2,3}$ is given by $\Delta_{g_i}\boldsymbol{r}=\mathbf{2H}_{g_i}$ and the surface S is minimal if and only if $\boldsymbol{x,y,\ f}$ are harmonic functions.

Proof 4.1: Let S be a surface in $(\mathbb{H}_3^1,\ \boldsymbol{g_1})$, with

$$g_1=\ -\frac{1}{\mu^2}dx^2+dy^2+(xdy+dz)^2.$$

The first fundamental form of S is the induced metric of $\boldsymbol{g_1}$ on S,

$$\boldsymbol{g_1/S=\left(f_x^2-\frac{1}{\mu^2}\right)dx^2+\left(1+\left(x+f_y\right)^2\right)dy^2+2f_x\left(x+f_y\right)dxdy}.$$

Let $P=\boldsymbol{f_x},\ \boldsymbol{Q=f_y+x}\ \ \boldsymbol{where}\ \ \boldsymbol{f_u=\frac{\partial f}{\partial u}},$

we have

$$\begin{cases}\boldsymbol{P_x=f_{xx},\quad Q_y=f_{yy},}\\ \boldsymbol{P_y=f_{xy},\quad Q_x=f_{xy}+1,}\\ \boldsymbol{Q_x=P_y+1.}\end{cases}\tag{7}$$

- The tangent vectors, $\boldsymbol{r_x},\ \boldsymbol{r_y}$ are given by

$$\begin{cases}\boldsymbol{r_x=(1,0,P)=\partial_x+P\partial_z=Pe_1^1+\frac{1}{\mu}e_1^3}\\ \boldsymbol{r_y=(0,1,Q-x)=\partial_y+(Q-x)\partial_z=Qe_1^1+e_1^2.}\end{cases}\tag{8}$$

- The coefficients of the first fundamental form $E_1,F_1\ and\ G_1$ are given by

$$E_1 = g_1(r_x, r_x),\ \ F_1 = g_1(r_x, r_y) \quad and \quad G_1 = g_1(r_y, r_y),$$

we get

$$E_1 = P^2 - \frac{1}{\mu^2}, \qquad F_1 = PQ, \qquad G_1 = Q^2 + 1. \tag{9}$$

- The unit normal vector field $\boldsymbol{N_{g_1}}$ on S is given by

$$\boldsymbol{N_{g_1}} = \frac{1}{W_1}(1, -Q, \mu P),$$

where $W_1 = \sqrt{\epsilon(1 + Q^2 - \mu^2 P^2)}, \epsilon = \pm 1$.

- The coefficients of second fundamental form $L_1,\ M_1, and\ N_1$ are defined by

$$L_1 = -g_1(r_{xx}, N_{g_1}),\quad M_1 = -g_1(r_{xy}, N_{g_1}) \quad and \qquad N_1 = -g_1(r_{yy}, N_{g_1}).$$

In order to compute the coefficients of the second fundamental form of S we have to solve the following system of equations:

$$\begin{cases} r_{xx} = \nabla_{r_x} r_x = P_x \boldsymbol{e_1^1} + \boldsymbol{P e_1^2}, \\ r_{xy} = \nabla_{r_x} r_y = \nabla_{r_y} r_x = \left(P_y + \frac{1}{2}\right)\boldsymbol{e_1^1} + \frac{\boldsymbol{Q}}{\boldsymbol{2}}\boldsymbol{e_1^2} + \frac{\boldsymbol{\mu P}}{\boldsymbol{2}}\boldsymbol{e_1^3}, \\ r_{yy} = \nabla_{r_y} r_y = Q_y \boldsymbol{e_1^1} + \boldsymbol{\mu Q e_1^3}. \end{cases} \tag{10}$$

Which imply the coefficients of the second fundamental form of S are given by

$$L_1 = \frac{1}{W_1}(\boldsymbol{PQ} - \boldsymbol{P_x}), \quad M_1 = \frac{1}{W_1}\left(\frac{\boldsymbol{\mu^2}}{\boldsymbol{2}}\boldsymbol{P^2} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{Q} - \boldsymbol{P_y} - \frac{\boldsymbol{1}}{\boldsymbol{2}}\right), \qquad N_1 = \frac{1}{W_1}\left(\boldsymbol{\mu^2 PQ} - \boldsymbol{Q_y}\right). \tag{11}$$

Thus, the mean curvature $\mathbf{H}_{g_1}$ of S, with the help of (1), (9) and (11) is given by

$$\mathbf{H}_{g_1} = \frac{\boldsymbol{\mu^2}}{\boldsymbol{2W_2^3}}\left(\left(\frac{\boldsymbol{1}}{\boldsymbol{\mu^2}} - \boldsymbol{P^2}\right)\boldsymbol{Q_y} - (\boldsymbol{1} + \boldsymbol{Q^2})\boldsymbol{P_x} + \boldsymbol{PQ}(\boldsymbol{1} + \boldsymbol{2P_y})\right). \tag{12}$$

The Beltrami operator (4) associated to $\boldsymbol{g_1}$ is

$$\Delta_{g_1} = \frac{\boldsymbol{-1}}{\sqrt{|\boldsymbol{E_1 G_1} - \boldsymbol{F_1^2}|}}\left(\boldsymbol{\partial_x}\left(\frac{\boldsymbol{G_1 \partial_x} - \boldsymbol{F_1 \partial_y}}{\sqrt{|\boldsymbol{E_1 G_1} - \boldsymbol{F_1^2}|}}\right) - \boldsymbol{\partial_y}\left(\frac{\boldsymbol{F_1 \partial_x} - \boldsymbol{E_1 \partial_y}}{\sqrt{|\boldsymbol{E_1 G_1} - \boldsymbol{F_1^2}|}}\right)\right).$$

where $\left|\boldsymbol{E_1 G_1 - F_1^2}\right| = \frac{\epsilon}{\mu^2}(\boldsymbol{\mu^2 P^2 - Q^2 - 1}), \quad \boldsymbol{\epsilon = \pm 1}.$

4.1.1 Case 1: $\mathbf{N_{g_1}}$ is spacelike vector $\mathbf{g_1}(\mathbf{N_{g_1}}, \mathbf{N_{g_1}}) = \mathbf{1}$, (i.e. $\boldsymbol{\epsilon = 1}$).

In this case, the Beltrami operator Δ is denoted by $\Delta^{+}_{\boldsymbol{g_1}}$

$$\Delta^{+}_{\boldsymbol{g_1}} = \frac{-\mu^2}{\sqrt{\mu^2 P^2 - Q^2 - 1}}\left(\partial_x\left(\frac{(1+Q^2)\partial_x - PQ\partial_y}{\sqrt{\mu^2 P^2 - Q^2 - 1}}\right) - \partial_y\left(\frac{PQ\partial_x - \left(P^2 - \frac{1}{\mu^2}\right)\partial_y}{\sqrt{\mu^2 P^2 + Q^2 - 1}}\right)\right).$$

We get

$$\Delta^{+}_{\boldsymbol{g_1}} r = \frac{-\mu^2}{W_1^4}\left(\Delta^{+}_{11} r_x + \Delta^{+}_{12} r_y + (1+Q^2) W_2^2 r_{xx} + \left(P^2 - \frac{1}{\mu^2}\right) W_2^2 r_{yy} - 2PQ W_1^2 r_{xy}\right),$$

where

$$\begin{cases} \Delta^{+}_{11} = -2\mu^2 P^2 Q + 2\mu^2 P^2 Q P_y - Q^3 - Q - \mu^2 P P_x - \mu^2 Q^2 P P_x - \mu^2 P^3 Q_y + P Q_y, \\ \Delta^{+}_{12} = -2PQ^2 P_y - \frac{1}{\mu^2} Q Q_y + P^2 Q Q_y + Q^3 P_x + Q P_x - \mu^2 P^3 + P. \end{cases}$$

With the help of (7), (8) and (10), $\Delta^{+}_{\boldsymbol{g_1}} r$ simplifies to

$$\Delta^{+}_{\boldsymbol{g_1}} r = -\frac{\boldsymbol{\mu^2}}{\boldsymbol{W_2^4}}\begin{pmatrix} \left(\left(\frac{\mathbf{1}}{\boldsymbol{\mu^2}} - \boldsymbol{P^2}\right)\boldsymbol{Q_y} - (\mathbf{1} + \boldsymbol{Q^2})\boldsymbol{P_x} + \boldsymbol{PQ}(\mathbf{1} + \mathbf{2}\boldsymbol{P_y})\right)\boldsymbol{e_1^1} \\ -Q\left(\left(\frac{\mathbf{1}}{\boldsymbol{\mu^2}} - \boldsymbol{P^2}\right)\boldsymbol{Q_y} - (\mathbf{1} + \boldsymbol{Q^2})\boldsymbol{P_x} + \boldsymbol{PQ}(\mathbf{1} + \mathbf{2}\boldsymbol{P_y})\right)\boldsymbol{e_1^2} \\ \mu P\left(\left(\frac{\mathbf{1}}{\boldsymbol{\mu^2}} - \boldsymbol{P^2}\right)\boldsymbol{Q_y} - (\mathbf{1} + \boldsymbol{Q^2})\boldsymbol{P_x} + \boldsymbol{PQ}(\mathbf{1} + \mathbf{2}\boldsymbol{P_y})\right)\boldsymbol{e_1^3} \end{pmatrix},$$

and

$$\Delta^{+}_{\boldsymbol{g_1}} r = \underbrace{-\frac{\boldsymbol{\mu^2}}{\boldsymbol{W_2^3}}\left(\left(\frac{\mathbf{1}}{\boldsymbol{\mu^2}} - \boldsymbol{P^2}\right)\boldsymbol{Q_y} - (\mathbf{1} + \boldsymbol{Q^2})\boldsymbol{P_x} + \boldsymbol{PQ}(\mathbf{1} + \mathbf{2}\boldsymbol{P_y})\right)}_{-2H_{g_1}} \underbrace{\frac{\mathbf{1}}{\boldsymbol{W_2}}\begin{pmatrix} \boldsymbol{e_1^1} \\ -Q\ \boldsymbol{e_1^2} \\ \mu P \boldsymbol{e_1^3} \end{pmatrix}}_{\boldsymbol{N_{g_1}}}$$

So, we obtain

$$\Delta^{+}_{\boldsymbol{g_1}} r = -2H_{g_1}\boldsymbol{N_{g_1}}.$$

4.1.2 Case 2: $\mathbf{N_{g_1}}$ is timelike vector $\mathbf{g_1}(\mathbf{N_{g_1}}, \mathbf{N_{g_1}}) = -\mathbf{1}$, (i.e. $\boldsymbol{\epsilon = -1}$).

In this case, the Beltrami operator Δ is denoted by $\Delta^{-}_{\boldsymbol{g_1}}$

$$\Delta^{-}_{\boldsymbol{g_1}} = \frac{-\mu^2}{\sqrt{1+Q^2-\mu^2P^2}}\left(\partial_x\left(\frac{(1+Q^2)\partial_x - PQ\partial_y}{\sqrt{1+Q^2-\mu^2P^2}}\right) - \partial_y\left(\frac{PQ\partial_x - \left(P^2-\frac{1}{\mu^2}\right)\partial_y}{\sqrt{1+Q^2-\mu^2P^2}}\right)\right).$$

We obtain

$$\Delta^{-}_{\boldsymbol{g_1}} r = \frac{-\mu^2}{W_1^4}\left(\Delta^{-}_{11}r_x + \Delta^{-}_{12}r_y + (1+Q^2)W_2^2 r_{xx} + \left(\frac{1}{\mu^2}-P^2\right)W_1^2 r_{yy} - 2PQW_2^2 r_{xy}\right),$$

where

$$\begin{cases} \Delta^{-}_{11} = 2\mu^2P^2Q - 2\mu^2P^2QP_y + Q^3 + Q + \mu^2PP_x + \mu^2Q^2PP_x + \mu^2P^3Q_y - PQ_y, \\ \Delta^{-}_{12} = 2PQ^2P_y + \frac{1}{\mu^2}QQ_y - P^2QQ_y - Q^3P_x - QP_x + \mu^2P^3 - P. \end{cases}$$

We get also

$$\Delta^{-}_{\boldsymbol{g_1}} r = 2H_{g_1}\boldsymbol{N}_{\boldsymbol{g_1}}.$$

We have

$$\Delta^{+}_{\boldsymbol{g_1}} = -\Delta^{-}_{\boldsymbol{g_1}}$$

Therefore, we prove that

$$\Delta_{\boldsymbol{g_1}} r = 2\boldsymbol{H}_{\boldsymbol{g_1}}.$$

where $\boldsymbol{H}_{\boldsymbol{g_1}}$ is the mean curvature vector field of S in $(\mathbb{H}^{\mathbf{1}}_{\mathbf{3}}, \boldsymbol{g_1})$

4.2. Let now S be a surface in $(\mathbb{H}^1_3, g_2)$, with

$$g_2 = \frac{1}{\mu^2}dx^2 + dy^2 - (xdy+dz)^2.$$

- The tangent vectors, $\boldsymbol{r_x}, \boldsymbol{r_y}$ are given by

$$\begin{cases} \boldsymbol{r_x} = (\mathbf{1},\mathbf{0},\boldsymbol{P}) = \boldsymbol{\partial_x} + \boldsymbol{P\partial_z} = \frac{\mathbf{1}}{\boldsymbol{\mu}}\boldsymbol{e}^{\mathbf{1}}_{\mathbf{2}} + \boldsymbol{Pe}^{\mathbf{3}}_{\mathbf{3}}, \\ \boldsymbol{r_y} = (\mathbf{0},\mathbf{1},\boldsymbol{Q}-\boldsymbol{x}) = \boldsymbol{\partial_y} + (\boldsymbol{Q}-\boldsymbol{x})\boldsymbol{\partial_z} = -\boldsymbol{e}^{\mathbf{2}}_{\mathbf{2}} + \boldsymbol{Qe}^{\mathbf{3}}_{\mathbf{2}}. \end{cases} \quad (13)$$

- The coefficients of the first fundamental form E_2, F_2 *and* G_2 are given by

$$E_2 = \frac{1}{\mu^2} - P^2, \quad F_2 = -PQ, \quad G_1 = 1 - Q^2. \quad (14)$$

- The unit normal vector field $\boldsymbol{N}_{\boldsymbol{g_2}}$ on S is given by

$$\boldsymbol{N}_{g_2} = \frac{1}{W_2}(\mu P, -Q, 1),$$

where $W_2 = \sqrt{\epsilon(\mu^2 P^2 + Q^2 - 1)}, \quad \epsilon = \pm 1.$

We have the following system of equations:

$$\begin{cases} r_{xx} = \nabla_{r_x} r_x = \boldsymbol{P} \boldsymbol{e}_2^2 + P_x \boldsymbol{e}_2^3, \\ r_{xy} = \nabla_{r_x} r_y = \nabla_{r_y} r_x = \frac{\boldsymbol{\mu P}}{\mathbf{2}} \boldsymbol{e}_2^1 + \frac{\boldsymbol{Q}}{\mathbf{2}} \boldsymbol{e}_2^2 + \left(P_y + \frac{1}{2}\right) \boldsymbol{e}_2^3, \\ r_{yy} = \nabla_{r_y} r_y = \boldsymbol{\mu Q} \boldsymbol{e}_2^1 + Q_y \boldsymbol{e}_2^3. \end{cases} \tag{15}$$

- The coefficients of the second fundamental form of S are given by

$$L_2 = \frac{1}{W_2}(\boldsymbol{PQ} + \boldsymbol{P_x}), \quad M_2 = \frac{1}{W_2}\left(\boldsymbol{P_y} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{Q}^2 - \frac{\boldsymbol{\mu}^2}{\mathbf{2}}\boldsymbol{P}^2 + \frac{\mathbf{1}}{\mathbf{2}}\right), \quad N_2 = \frac{1}{W_2}\left(\boldsymbol{Q_y} - \boldsymbol{\mu}^2 \boldsymbol{PQ}\right). \tag{16}$$

Thus, the mean curvature $\mathbf{H}_{g_2}$ of S, with the help of (1), (14) and (16) is given by

$$\mathbf{H}_{g_2} = \frac{\boldsymbol{\mu}^2}{\mathbf{2}\boldsymbol{W}_2^3}\left(\left(\frac{\mathbf{1}}{\boldsymbol{\mu}^2} - \boldsymbol{P}^2\right)\boldsymbol{Q_y} + (\mathbf{1} - \boldsymbol{Q}^2)\boldsymbol{P_x} + \boldsymbol{PQ}(\mathbf{1} + \mathbf{2}\boldsymbol{P_y})\right). \tag{17}$$

The Beltrami operator (4) associated to $\boldsymbol{g_2}$ is

$$\Delta_{g_2} = \frac{-\mathbf{1}}{\sqrt{|\boldsymbol{E}_2\boldsymbol{G}_2 - \boldsymbol{F}_2^2|}}\left(\boldsymbol{\partial_x}\left(\frac{\boldsymbol{G}_2\boldsymbol{\partial_x} - \boldsymbol{F}_2\boldsymbol{\partial_y}}{\sqrt{|\boldsymbol{E}_2\boldsymbol{G}_2 - \boldsymbol{F}_2^2|}}\right) - \boldsymbol{\partial_y}\left(\frac{\boldsymbol{F}_2\boldsymbol{\partial_x} - \boldsymbol{E}_2\boldsymbol{\partial_y}}{\sqrt{|\boldsymbol{E}_2\boldsymbol{G}_2 - \boldsymbol{F}_2^2|}}\right)\right).$$

where $|\boldsymbol{E}_2\boldsymbol{G}_2 - \boldsymbol{F}_2^2| = \frac{\boldsymbol{\epsilon}}{\boldsymbol{\mu}^2}(\boldsymbol{\mu}^2\boldsymbol{P}^2 + \boldsymbol{Q}^2 - \mathbf{1}), \quad \boldsymbol{\epsilon} = \pm\mathbf{1}.$

4.2.1. Case 1: $\mathbf{N}_{\mathbf{g}_2}$ is spacelike vector $\mathbf{g}_2(\mathbf{N}_{\mathbf{g}_2}, \mathbf{N}_{\mathbf{g}_2}) = \mathbf{1}$, (i.e. $\boldsymbol{\epsilon} = \mathbf{1}$).

In this case, the Beltrami operator Δ is denoted by $\Delta_{g_2}^+$

$$\Delta_{g_2}^+ = \frac{-\mu^2}{\sqrt{\mu^2P^2 + Q^2 - 1}}\left(\partial_x\left(\frac{(1 - Q^2)\partial_x + PQ\partial_y}{\sqrt{\mu^2P^2 + Q^2 - 1}}\right) + \partial_y\left(\frac{PQ\partial_x + \left(\frac{1}{\mu^2} - P^2\right)\partial_y}{\sqrt{\mu^2P^2 + Q^2 - 1}}\right)\right).$$

We get

$$\Delta_{g_2}^+ r = \frac{-\mu^2}{W_2^4}\left(\Delta_{21}^+ r_x + \Delta_{22}^+ r_y + (1 - Q^2)W_2^2 r_{xx} + \left(\frac{1}{\mu^2} - P^2\right)W_2^2 r_{yy} + 2PQW_2^2 r_{xy}\right).$$

where

$$\begin{cases} \Delta_{21}^{+} = -2QQ_xW_2^2 - (1-Q^2)(\mu^2PP_x + QQ_x) + \left(P_yQ + PQ_y\right)W_2^2 - PQ\left(\mu^2PP_y + QQ_y\right), \\ \Delta_{22}^{+} = -2PP_yW_2^2 - \left(\frac{1}{\mu^2} - P^2\right)\left(\mu^2PP_y + QQ_y\right) + (P_xQ + PQ_x)W_2^2 - PQ(\mu^2PP_x + QQ_x). \end{cases}$$

With the help of (13), (15) and (17), we get

$$\Delta_{\boldsymbol{g_2}}^{+} r = 2H_{g_2}\boldsymbol{N}_{\boldsymbol{g_2}},$$

4.2.2. Case 2: $\mathbf{N_{g_2}}$ is timelike vector $\mathbf{g_2}\left(\mathbf{N_{g_2}},\ \mathbf{N_{g_2}}\right) = -\mathbf{1}$, $(\mathbf{i.e.}\ \boldsymbol{\epsilon} = -\mathbf{1})$.

In this case, the Beltrami operator Δ is denoted by $\Delta_{\boldsymbol{g_2}}^{-}$

$$\Delta_{\boldsymbol{g_2}}^{-} = \frac{-\mu^2}{\sqrt{1-\mu^2P^2-Q^2}}\left(\partial_x\left(\frac{(1-Q^2)\partial_x + PQ\partial_y}{\sqrt{1-\mu^2P^2-Q^2}}\right) + \partial_y\left(\frac{PQ\partial_x + \left(\frac{1}{\mu^2} - P^2\right)\partial_y}{\sqrt{1-\mu^2P^2-Q^2}}\right)\right).$$

We obtain

$$\Delta_{\boldsymbol{g_2}}^{-} r = \frac{-\mu^2}{W_2^4}\left(\Delta_{21}^{-}r_x + \Delta_{22}^{-}r_y + (1-Q^2)W_2^2r_{xx} + \left(\frac{1}{\mu^2} - P^2\right)W_2^2r_{yy} + 2PQW_2^2r_{xy}\right),$$

where

$$\begin{cases} \Delta_{21}^{-} = -2QQ_xW_2^2 + (1-Q^2)(\mu^2PP_x + QQ_x) + \left(P_yQ + PQ_y\right)W_2^2 + PQ\left(\mu^2PP_y + QQ_y\right), \\ \Delta_{22}^{-} = -2PP_yW_2^2 + \left(\frac{1}{\mu^2} - P^2\right)\left(\mu^2PP_y + QQ_y\right) + (P_xQ + PQ_x)W_2^2 + PQ(\mu^2PP_x + QQ_x). \end{cases}$$

We get also,

$$\Delta_{\boldsymbol{g_2}}^{-} r = -2H_{g_2}\boldsymbol{N}_{\boldsymbol{g_2}}.$$

We have

$$\Delta_{\boldsymbol{g_2}}^{+} = -\Delta_{\boldsymbol{g_2}}^{-}.$$

Therefore, we prove that

$$\Delta_{\boldsymbol{g_2}} r = 2\boldsymbol{H}_{\boldsymbol{g_2}}$$

where $\boldsymbol{H}_{\boldsymbol{g_2}}$ is the mean curvature vector field of S in $\left(\mathbb{H}_3^{\mathbf{1}},\ \boldsymbol{g_2}\right)$.

4.3. Let now S be a surface in $(\mathbb{H}_3^1,\ g_3)$*, with*

$$g_3 = dx^2 + (xdy + dz)^2 - \left((1-x)dy + dz\right)^2.$$

In this case the metric g_3 is flat, therefore, the space $\left(\mathbb{H}^{\mathbf{1}}_{\mathbf{3}},\ \boldsymbol{g_3}\right)$ is diffeomorphic to Lorentz Minkowski space $\mathbb{R}^3_1$. The result in $\left(\mathbb{H}^{\mathbf{1}}_{\mathbf{3}},\ \boldsymbol{g_3}\right)$ is a natural result of the Lorentz-Minkowski space $\mathbb{R}^3_1$. In [3] B.Y. Chen gives the proof of this result in $\mathbb{R}^3$. In the subsequent section, ananalytical proof of this result in the flat space $\left(\mathbb{H}^{\mathbf{1}}_{\mathbf{3}},\ \boldsymbol{g_3}\right)$ is given by

- The tangent vectors, $\boldsymbol{r_x}$, $\boldsymbol{r_y}$ are given by

$$\begin{cases} \boldsymbol{r_x = e^1_3 + Pe^2_3 - Pe^3_3}, \\ \boldsymbol{r_y = Qe^2_3} + (\mathbf{1} - \boldsymbol{Q})\boldsymbol{e^3_3}. \end{cases} \tag{18}$$

- The coefficients of the first fundamental form E_3, F_3 and G_3 are given by

$$E_3 = 1, \quad F_3 = P, \quad G_3 = 2Q - 1. \tag{19}$$

- The unit normal vector field $\boldsymbol{N_{g_3}}$ on S is given by

$$\boldsymbol{N_{g_3}} = \frac{1}{W_3}(-P, 1 - Q, Q\ ),$$

where $W_3 = \sqrt{\epsilon(P^2 + 1 - 2Q)}$, $\epsilon = \pm 1$.

- The coefficients of the second fundamental form of S with help of the following

$$\begin{cases} r_{xx} = \nabla_{r_x} r_x = P_x \boldsymbol{e^2_3} - P_x \boldsymbol{e^3_3}, \\ r_{xy} = \nabla_{r_x} r_y = \nabla_{r_y} r_x = \boldsymbol{Q_x e^2_3 - Q_x e^3_3}, \\ r_{yy} = \nabla_{r_y} r_y = \boldsymbol{-e^1_3 + Q_y e^2_3 - Q_y e^3_3}, \end{cases} \tag{20}$$

are given by

$$L_3 = \frac{-1}{W_3}\boldsymbol{P_x}, \quad M_3 = \frac{-1}{W_3} Q_x, \quad N_3 = \frac{-1}{W_3}\left(\boldsymbol{P} + \boldsymbol{Q_y}\right), \tag{21}$$

Thus, the mean curvature $\mathbf{H}_{g_3}$ of S is given by

$$\mathbf{H}_{g_3} = \frac{\mathbf{1}}{\boldsymbol{2W^3_3}}\left(\boldsymbol{2PP_y - 2QP_x + P_x - Q_y + P}\right). \tag{22}$$

The Beltrami operator (4) associated to $\boldsymbol{g_3}$ is

$$\Delta_{g_3} = \frac{\mathbf{-1}}{\sqrt{\left|\boldsymbol{E_3G_3 - F^2_3}\right|}}\left(\boldsymbol{\partial_x}\left(\frac{\boldsymbol{G_3\partial_x - F_3\partial_y}}{\sqrt{\left|\boldsymbol{E_3G_3 - F^2_3}\right|}}\right) - \boldsymbol{\partial_y}\left(\frac{\boldsymbol{F_3\partial_x - E_3\partial_y}}{\sqrt{\left|\boldsymbol{E_3G_3 - F^2_3}\right|}}\right)\right).$$

where $\left|\boldsymbol{E_3G_3 - F^2_3}\right| = \boldsymbol{\epsilon}(\boldsymbol{2Q - P^2} - \mathbf{1})$, $\boldsymbol{\epsilon} = \pm\mathbf{1}$.

4.3.1. Case 1: $\mathbf{N_{g_3}}$ is spacelike vector $\mathbf{g_3}\left(\mathbf{N_{g_3}},\ \mathbf{N_{g_3}}\right) = \mathbf{1}$, $(\mathbf{i.e.}\ \boldsymbol{\epsilon} = \mathbf{1})$.

In this case, the Beltrami operator Δ is denoted by $\Delta^{+}_{\boldsymbol{g_3}}$

$$\Delta^{+}_{\boldsymbol{g_3}} = \frac{-1}{\sqrt{\boldsymbol{2Q - P^2 - 1}}}\left(\partial_x\left(\frac{(2Q-1)\partial_x - P\partial_y}{\sqrt{\boldsymbol{2Q - P^2 - 1}}}\right) - \partial_y\left(\frac{P\partial_x - \partial_y}{\sqrt{\boldsymbol{2Q - P^2 - 1}}}\right)\right).$$

and

$$\Delta^{+}_{\boldsymbol{g_3}} r = \frac{-1}{W_3^4}\left(\Delta^{+}_{31} r_x + \Delta^{+}_{32} r_y + (2Q-1)W_3^2 r_{xx} + W_3^2 r_{yy} - 2PQW_3^2 r_{xy}\right),$$

where

$$\begin{cases} \Delta^{+}_{31} = 2PQP_x - 2P^2P_y - 2P^2 + 2Q - PP_x + PQ_y - 1, \\ \Delta^{+}_{32} = 2PP_y - 2QP_x + P_x - Q_y + P_y. \end{cases}$$

With the help of (18), (20) and (22), $\Delta^{+}_{\boldsymbol{g_3}} r$ simplifies to

$$\Delta^{+}_{\boldsymbol{g_3}} r = 2H_{g_3}\boldsymbol{N}_{\boldsymbol{g_3}},$$

4.3.2. Case 2: $\mathbf{N_{g_3}}$ is timelike vector $\mathbf{g_3}(\mathbf{N_{g_3}}, \mathbf{N_{g_3}}) = -\mathbf{1}$, (i.e. $\boldsymbol{\epsilon} = -\mathbf{1}$).

The Beltrami operator Δ is denoted by $\Delta^{-}_{\boldsymbol{g_3}}$

$$\Delta^{-}_{\boldsymbol{g_3}} = \frac{-1}{\sqrt{\boldsymbol{P^2 - 2Q + 1}}}\left(\partial_x\left(\frac{(2Q-1)\partial_x - P\partial_y}{\sqrt{\boldsymbol{P^2 - 2Q + 1}}}\right) - \partial_y\left(\frac{P\partial_x - \partial_y}{\sqrt{\boldsymbol{P^2 - 2Q + 1}}}\right)\right),$$

and

$$\Delta^{-}_{\boldsymbol{g_3}} r = \frac{-1}{W_3^4}\left(\Delta^{-}_{31} r_x + \Delta^{-}_{32} r_y + (2Q-1)W_3^2 r_{xx} + W_3^2 r_{yy} - 2PQW_3^2 r_{xy}\right),$$

where

$$\begin{cases} \Delta^{-}_{31} = 2P^2P_y + 2P^2 - 2PQP_x - 2Q - PP_x - PQ_y + 1, \\ \Delta^{-}_{32} = 2PP_y - 2QP_x + P_x - Q_y + P_y. \end{cases}$$

We get also,

$$\Delta^{-}_{\boldsymbol{g_3}} r = -2H_{g_3}\boldsymbol{N}_{\boldsymbol{g_3}}.$$

We have

$$\Delta^{+}_{\boldsymbol{g_3}} = -\Delta^{-}_{\boldsymbol{g_3}}.$$

Therefore, we prove that

$$\Delta_{\boldsymbol{g_3}} r = 2\boldsymbol{H}_{\boldsymbol{g_3}}$$

where $\boldsymbol{H}_{\boldsymbol{g_3}}$ is the mean curvature vector field of S in $(\mathbb{H}^{\mathbf{1}}_{\mathbf{3}}, \boldsymbol{g_3})$.

From the above, the relationship between the Laplace-Beltrami operator Δ of the position vector field and the mean curvature vector field $\boldsymbol{H}$ of surfaces defined as graph of functions in the three-dimensional Lorentzian Heisenberg group $\mathbb{H}_{\mathbf{3}}^{\mathbf{1}}$ is given by

$$\Delta r = 2\boldsymbol{H} \tag{23}$$

Using (6) and (23), we get the following corollary

Corollary 4.2. *Let S be an immersed surface in $(\mathbb{H}_3^1, \boldsymbol{g_i})$, $\boldsymbol{i = 1, 2, 3}$ which is given as the graph of the function $z = f(x, y)$ with the parametrization*

$$r(x, y) = \big(x, y, f(x, y)\big), \ \ (x, y) \in D \subset \mathbb{R}^2, \quad f \in C^1(D).$$

A surface S is minimal if and only if the components of position vector x, y and f are harmonic functions i.e.

$$\Delta x = \Delta y = \Delta f = 0.$$

5. CONCLUSION

Therefore, we have proved that in the three-dimensional Lorentzian Heisenberg group $\mathbb{H}_3^{\mathbf{1}}$ which is endowed with left invariant Lorentzian metric $\boldsymbol{g_i}$, $(\boldsymbol{i = 1, 2, 3})$ the surface as graph $\boldsymbol{S}$ is minimal in $\mathbb{H}_3^{\mathbf{1}}$ if and only if the components of position vector of **S** are harmonic functions.

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