ORIGINAL PAPER

POST QUANTUM-HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS INVOLVING THE NOTION OF $(p,q)^b$ −INTEGRAL

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Abstract. *In this investigation, we established some new post quantum-Hermite-Hadamard inequalities for differentiable convex function with critical point by using the notion of $(p,q)^b$-integeral .The perseverance of this article is to establish different results on the left hand side of $(p,q)^b$- Hermite-Hadamard inequality for differentiable convex function along with critical a point. Some Special cases are obtain for different $(p,q)^b$- Hermite Hadamard inequalies at critical point c for some specific values of q.*

Keywords: $(p,q)^b$ −*derivative,* $(p,q)^b$ − *integeral; Hermite–Hadamard's inequality; critical point; convex functions.*

1. INTRODUCTION

1.1. CRITICAL POINT AND CONVEXITY

Critical point is an extensively used term in various branches of mathematics, always connected to the derivative of a function or mapping. Various aspects of critical point with functions are listed below:

For functions of a real variabale, a critical point is defined as a point in the domain of function where the function is either not differentiable or the derivative is equal to zero.

When dealing with complex variables, a critical point is a point in the domain of a function where it is either not holomorphic or the derivative is equal to zero.

Likewise, for a function of several real variables, a critical point is a value in its domain where the gradient is undefined or is equal to zero.

Convex functions are basic tools for constructing literature on mathematical inequalities. A function $f: I \dot{\subseteq} \to \mathbb{R}$ is said to be convex function on I, if

$$f\big((1-t)x+ty\big) \leq (1-t)f(x)+tf(y) \;\; \forall\, x,y \in I, t \in [0,1].$$

In recent years considerable attention has been given in studying numerous aspects of convex functions. This concept has been extended and generalized in different directions, see [1-8].

A large number of inequalities were obtained by means of convex functions see [9-12]. A classical inequality for convex functions is the Hermite–Hadamard inequality, and is given as:

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$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

where $f: I \dot{\subseteq} \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with (see [13]).

1.2. INTRODUCTION

Quantum calculus or $q-$calculus is an approach pertinent to the classic study of calculus, but it is mainly arranged for derivation of q -analogous results without use of limits. Post-quantum or $(p,q)-$ calculus is a generalization of $q-$calculus and it is the next stage ahead of the q-calculus. The idea of q calculus was first introduced by Euler who started his study in the earlier years of the 18th century. In $q-$calculus, the classical derivative is replaced by the q -difference operator in order to deal with non-differentiable functions; for more details see [14-15]. In recent years, the topic of $q-$calculus has attracted the attention of several scholars. Therefore $q-$calculus bridges a connection between mathematics and physics Applications of $q-$calculus can be found in various fields of mathematics and physics, interested readers are referred to [16–19].

In 2014, Tariboon and Ntouyas [20] investigated the $q-$analogue of Hermite-Hadamard's inequality,

$$f(\frac{qa+b}{1+q}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{qf(a)+f(b)}{1+q}.$$

In 2018, Alp et al [21] proved the $(p,q)-$Hermite –Hadamard inequality.

$$f(\frac{qa+pb}{p+q}) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)dx \leq \frac{qf(a)+pf(b)}{p+q}.$$

In 2020, Alp et al [22] studied the generalized $(p,q)^b-$Hermite–Hadamard inequality for differentiable convex functions,

$$max\{I_1, I_2\} \leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, b_{d_{p,q}x} \leq \frac{qf(b)+pf(a)}{p+q}.$$

where

$$I_1 = f\left(\frac{qb+pa}{p+q}\right)$$

$$I_2 = f\left(\frac{pb+qa}{p+q}\right) + \frac{(p-q)(b-a)}{p+q} f'\left(\frac{pb+qa)}{p+q}\right)$$

The organization of current paper is as follows: In Section 2, a short explanation of the concepts of $(p,q)-$calculus and some associated works in this direction is given. In Section 3 we present some new post quantum Hermite –Hadamard integeral inequalities for differentiable convex functions along with critical point and also present some examples satisfying main outcomes. Section 4 contains some conclusions and more directions for future research. We hope that the study initiated in this paper may motivate new research in this area.

2. PRELIMINARIES OF (p, q) − CALCULUS

Throughout this paper, we consider [a, b] ⊂ ℝ be an interval and $0 < q < p \leq 1$ constants. The definations for $(p, q)^b$–derivative and $(p, q)^b$– integeral were given in [22].

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The $(p, q)^b$ −derivative of f at t ∈[a,b] is characterized by the expression

$$^{b}D_{\mathrm{p,q}}\, f(t) = \frac{f(qt+(1-q)b)-f(pt+(1-p)b)}{(p-q)(b-t)}, \ t \neq b \tag{1}$$

For $x = b$, we state $^{b}D_{\mathrm{p,q}}\, f(a) = \lim_{x\to b}\ {}^{b}D_{\mathrm{p,q}}\, f(x)$ if it exist and is finite.

Definition 2.2 Let $f : [a; b] \to \mathbb{R}$ be a continuous function. The definite $(p, q)^b$-integral of f on $[a, b]$ is defined as:

$$\int_{t}^{b} f(\mathrm{x})\, b_{d_{(p,q)}\mathrm{x}} = (p-q)(b-t)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left\{\frac{q^n}{p^{n+1}}t + \left(1-\frac{q^n}{p^{n+1}}\right)b\right\} \tag{2}$$

with $0 < q < p \leq 1$.

The proofs of the following theorems were given in [23].

Theorem 2.1. Suppose that f:[a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a, b)$ and let q be a constant with $0 < q < 1$.Then we have

$$f\left(\frac{q(a+c)+(1-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right)$$
$$\leq \frac{1}{(b-a)}\int_a^b f(x)\quad {}_a d_q x\ \leq \frac{qf(a)+f(b)}{1+q}.$$

Theorem 2.2. let f:[a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a, b)$ and let q be a constant with $0 < q < 1$.Then we have

$$f\left(\frac{(1-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(1-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+(b-a)}{1+q}\right)$$

$$\leq \frac{1}{(b-a)}\int_a^b f(x)\quad {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}$$

Theorem 2.3.

$$\begin{bmatrix} Generalized\ q - Hermite\ Hadamard\ Inequality\ for\ convex \\ differentiable\ functions \end{bmatrix}$$

Let $f: [a, b] \to \mathbb{R}$ be a differentiable convex function on (a, b) such that $f'(c) = 0$ for $c \in (a, b)$ and $0 < q < 1$, Then we have

$$max\{I_1, I_2\} \leq \frac{1}{(b-a)} \int_a^b f(x) \quad {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

where

$$I_1 = f\left(\frac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right),$$

$$I_2 = f\left(\frac{(1-q)a+q(c+b)}{1+q}\right) + f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+(b-a)}{1+q}\right)$$

3. MAIN OUTCOMES

In this section, we present some new post quantum Hermite –Hadamard integeral inequalities for differentiable convex functions along with critical point and also present some examples satisfying main outcomes.

Theorem 3.1. Suppose that f: [a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < p \leq 1$.Then we have

$$\begin{aligned} f\left(\frac{q(b+c)+(p-q)a}{p+q}\right) + f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right) \\ \leq \frac{1}{p(b-a)} \int_{pa+(1-p)b}^{b} f(x)\, b \quad d_{p,q}x \leq \frac{qf(b)+pf(a)}{p+q} \end{aligned} \tag{3}$$

Proof: Since the function f is differentiable on (a,b) , there exist a tangent line at the point $\frac{q(b+c)+(p-q)a}{p+q} \in (a,b)$ **,** given by

$$\begin{aligned} h(x) = f\left(\frac{q(b+c)+(p-q)a}{p+q}\right) \\ + f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(x - \frac{q(b+c)+(p-q)a}{p+q}\right) \end{aligned} \tag{4}$$

Since f is convex function on [a, b], it follows that $h(x) \leq f(x)$ for all $x \in [a,b]$. Applying $(p,q)^b$ −integeration of Eq (3.2) on $[a,b]$, we have

$$\begin{aligned} \int_{\boldsymbol{pa}+(\boldsymbol{1}-\boldsymbol{p})\boldsymbol{b}}^{\boldsymbol{b}} h(x) \boldsymbol{b}_{d_{p,q}x} \\ = \int_{\boldsymbol{pa}+(\boldsymbol{1}-\boldsymbol{p})\boldsymbol{b}}^{\boldsymbol{b}} \Bigg[f\left(\frac{q(b+c)+(p-q)a}{p+q}\right) \\ + f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(x - \frac{q(b+c)+(p-q)a}{p+q}\right)\Bigg] \boldsymbol{b}_{d_{p,q}x} \end{aligned}$$

$$=p(b-a)f\left(\frac{q(b+c)+(p-q)a}{p+q}\right)+$$
$$f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\int_{\boldsymbol{pa}+(\boldsymbol{1}-\boldsymbol{p})\boldsymbol{b}}^{\boldsymbol{b}}\boldsymbol{x}\,{}_{\boldsymbol{b}}d_{p,q}x-p(b-a)\frac{q(b+c)+(p-q)a}{p+q}\right)$$

$$=p(b-a)f\left(\frac{q(b+c)+(p-q)a}{p+q}\right)+$$
$$f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(p(b-a)\frac{(qb+pa)}{p+q}-p(b-a)\frac{q(b+c)+(p-q)a}{p+q}\right)$$

$$=p(b-a)\left[f\left(\frac{q(b+c)+(p-q)a}{p+q}\right)+f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right)\right]$$
$$\leq\int_{\boldsymbol{pa}+(\boldsymbol{1}-\boldsymbol{p})\boldsymbol{b}}^{\boldsymbol{b}}f(x)\,{}_{\boldsymbol{b}}d_{p,q}x$$

On the other hand, f is convex function, we obtain

$$\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x))\,{}_{\boldsymbol{b}}d_{p,q}x$$
$$=\frac{1}{p(b-a)}\left[(p-q)p(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}(pa+(1-p)b)+\left(1-\frac{q^n}{p^{n+1}}\right)b\right)\right]$$

$$\leq(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left[\frac{q^n}{p^{n+1}}(pf(a)+(1-p)f(b))+\left(1-\frac{q^n}{p^{n+1}}\right)f(b)\right]$$
$$=(p-q)\left[\frac{pf(a)}{p^2-q^2}-\frac{pf(b)}{p^2-q^2}+\frac{f(b)}{(p-q)}\right]$$
$$=\frac{qf(b)+pf(a)}{p+q}.$$

The proof is complete.

Remark 3.2: Consider Theorem 3.1, if $q\in\left(0,\frac{c-a}{b-a}\right), then\ \frac{q(b+c)+(p-q)a}{p+q}\in[c,a)$. We can reduce the left-hand side of Theorem 3.1 as:

$$f\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x)\,{}_{\boldsymbol{b}}d_{p,q}x\leq\frac{qf(b)+pf(a)}{p+q}$$

Since $f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right)\geq 0.$

Remark 3.3: In Remark 3.2, if $c\to b^+$, then $\frac{c-a}{b-a}\to 1^-$. Since $q\in(0,1)$, we have $\frac{q(b+c)+(p-q)a}{p+q}\in(a,b)$.

We can reduce the left hand side of Theorem 3.1 as:

$$f\left(\frac{qb+(p-q)a}{p+q}\right) \leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\,\boldsymbol{b}_{d_{p,q}x} \leq \frac{qf(b)+pf(a)}{p+q}$$

Since $f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right) \geq 0.$

Corollary 3.4. Assume that $f:[a,b] \to \mathbb{R}$ is a differentiable convex function on (a,b) such that $f'\left(\frac{a+b}{2}\right) = 0$ **for** $0 < q < p \leq 1$.Then we have

$$f\left(\frac{q\left(b+\frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right)+(p-q)a}{p+q}\right) + f'\left(\frac{q\left(b+\frac{\boldsymbol{a+b}}{\boldsymbol{2}}\right)+(p-q)a}{p+q}\right)\left(\frac{q(a-b)}{2(p+q)}\right)$$
$$\leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\,\boldsymbol{b}_{d_{p,q}x} \leq \frac{qf(b)+pf(a)}{p+q} \;.$$

Corollary 3.5. Assume that $f:[a,b] \to \mathbb{R}$ is a differentiable convex function on (a,b) such that $f'(0) = 0$ for $0 \in (a,b)$ and $0 < q < p \leq 1$.Then we have

$$f\left(\frac{qb+(p-q)a}{p+q}\right) + f'\left(\frac{q(b)+(p-q)a}{p+q}\right)\left(\frac{qa}{(p+q)}\right)$$
$$\leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\,\boldsymbol{b}_{d_{p,q}x} \leq \frac{qf(b)+pf(a)}{p+q}.$$

Theorem 3.6. Let f:[a,b]→ $\mathbb{R}$ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < p \leq 1$.Then we have

$$\begin{aligned} &f\left(\frac{(p-q)b+q(c+a)}{p+q}\right) \\ &\quad + f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right) \\ &\leq \frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\,\boldsymbol{b}_{d_{p,q}x} \leq \frac{qf(b)+pf(a)}{p+q} \end{aligned} \tag{5}$$

Proof: Since function f is differentiable on (a,b), there exist a tangent line at the point

$\frac{(p-q)b+q(c+a)}{p+q} \in (a,b)$, which is given by

$$\begin{aligned} k(x) = &f\left(\frac{(p-q)b+q(c+a)}{p+q}\right) \\ &+ f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(x-\frac{(p-q)b+q(c+a)}{p+q}\right). \end{aligned} \tag{6}$$

Sine f is convex on$[a,b]$, it follows that $k(x) \leq f(x)$ for all $x \in [a,b]$.
Applying (p,q)-integration, we get

$$\int_{pa+(1-p)b}^{b} k(x)\,\boldsymbol{b}_{d_{p,q}x}$$

$$=\int_{pa+(1-p)b}^{b}\left[f\left(\frac{(p-q)b+q(c+a)}{p+q}\right)+f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(x-\frac{(p-q)b+q(c+a)}{p+q}\right)\right]\boldsymbol{b}_{d_{p,q}x}$$

$$=p(b-a)f\left(\frac{(p-q)b+q(c+a)}{p+q}\right)+f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)$$

$$\left(\int_{pa+(1-p)b}^{b}x\boldsymbol{b}_{d_{p,q}x}-p(b-a)\frac{(p-q)b+q(c+a)}{p+q}\right)$$

$$=p(b-a)f\left(\frac{(p-q)b+q(c+a)}{p+q}\right)+f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left[p(b-a)\left(\left(\frac{qb+pa}{p+q}\right)-\frac{(p-q)b+q(c+a)}{p+q}\right)\right]$$

$$=p(b-a)\left[f\left(\frac{(p-q)b+q(c+a)}{p+q}\right)+f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right)\right]$$

$$\leq\int_{pa+(1-p)b}^{b}f(x)\,\boldsymbol{b}_{d_{p,q}x}$$

The proof is complete.

Remark 3.7. Consider Theorem 3.6, if $q\in\left(\frac{1}{2},\frac{c-b}{a-b}\right],then\ f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\leq 0\ \ and\ \left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right)<0$ we can reduce the left hand side of Theorem 3.6 as:

$$f\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x)\,\boldsymbol{b}_{d_{p,q}x}\leq\frac{qf(b)+pf(a)}{p+q},$$

Meanwhile $f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right)\geq 0$.

Remark 3.8. In Remark 3.7, if $c\to a^-$, then $q\to 1^-$,Since $q\in\left(\frac{1}{2},1\right)$, we have $\frac{(p-q)b+q(c+a)}{p+q}\in\left(\frac{2a+b}{3},a\right)$.

We can reduce the left hand side of Theorem 3.6 as:

$$f\left((p-q)(\frac{2b+a}{3}+qa)\right)\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x)\,\boldsymbol{b}_{d_{p,q}x}\leq\frac{qf(b)+pf(a)}{p+q},$$

Since $f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right)\geq 0$.

Theorem 3.9. Generalized $(p,\ q)^b$ – Hermite-Hadamard inequality for convex differentiable functions: Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b) such that $f'(c)=0$ for $c\in(a,b)$ and $0<q<p\leq 1$. Then we have:

$$max\{I_1,I_2\}\leq\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b}f(x)\,\boldsymbol{b}_{d_{p,q}x}\leq\frac{qf(b)+pf(a)}{p+q}. \tag{7}$$

where

$$I_1 = f\left(\frac{q(b+c)+(p-q)a}{p+q}\right) + f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right),$$

$$I_2 = f\left(\frac{(p-q)b+q(c+a)}{p+q}\right) + f'\left(\frac{(p-q)b+q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c)+p(a-b)}{p+q}\right)$$

Proof: Combining of (3.1) and (3.3) yields (3.5). This completes the proof.

Example 3.10. Define the function $f(x) = x^2$ on $[-1,3]$, and let $q \in (0,1), p = 1$.Applying Theorem 3.1 with $a = -1, b = 3$ and $c = 0$,the left-hand side becomes:

$$f\left(\frac{q(b+c)+(p-q)a}{p+q}\right) + f'\left(\frac{q(b+c)+(p-q)a}{p+q}\right)\left(\frac{q(a-c)}{p+q}\right)$$
$$-\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, b \quad d_{p,q}x$$
$$= f\left(\frac{4q-p}{p+q}\right) + f'\left(\frac{4q-p}{p+q}\right)\left(\frac{-q}{p+q}\right) - \frac{1}{4p}\left[4p(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n(-4p+3)}{p^{n+1}} + \left(1-\frac{q^n}{p^{n+1}}\right)3\right)\right]$$
$$= \frac{q^4-q^3-q^2-16q}{(p^2+pq+q^2)(p+q)^2} \le 0$$

For the right hand side, we have:

$$\frac{1}{3-(-1)}\int_{-4p+3}^{3} x^2\, b \quad d_{p,q}x - \frac{qf(3)+pf(-1)}{P+q} = \frac{16p^2}{(p^2+pq+q^2)} - \frac{24p}{P+q} + 9 - \frac{9q+p}{p+q} \le 0\ .$$

Example 3.11. Define the function $f(x) = x^2$ on$[-1,1]$, and let $q \in (0,1), p = 1$. Applying Corollary 3.5 with $a = -1, b = 1$and $c = 0$, the left side becomes:

$$f\left(\frac{q(b)+(p-q)a}{p+q}\right) + f'\left(\frac{q(b)+(p-q)a}{p+q}\right)\left(\frac{qa}{(p+q)}\right)$$
$$-\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, \boldsymbol{b}_{d_{p,q}x}$$
$$= f\left(\frac{2q-p}{p+q}\right) + f'\left(\frac{2q-p}{p+q}\right)\left(\frac{q}{p+q}\right) - (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{-2q^n}{p^n}+1\right)$$
$$= \frac{p^2+4q^2-4pq}{(p+q)^2} + \frac{2q(p-2q)}{(p+q)^2} - \frac{p^3+p^2q-pq^2+p^2q-pq^2+q^3}{(p+q)(p^2+pq+q^2)} \le 0$$

For the right hand side,we have:

$$\frac{1}{p(1-(-1))}\int_{-2p+1}^{1} x^2\, b \quad d_{p,q}x - \frac{qf(1)+pf(-1)}{P+q} = \frac{4p}{(p^2+pq+q^2)} - \frac{4p}{P+q} + 1 - \frac{q+p}{p+q} \le 0.$$

Example 3.12. Define the function $f(x) = x^2$ on $[-3,1]$, and let $q \in (0,1), p = 1$. Applying Theorem 3.6 with $a = -3, b = 1$ and $c = 0$, the left side becomes:

$$f\left(\frac{(p-q)b + q(c+a)}{p+q}\right) + f'\left(\frac{(p-q)b + q(c+a)}{p+q}\right)\left(\frac{q(2b-a-c) + p(a-b)}{p+q}\right)$$
$$-\frac{1}{p(b-a)}\int_{pa+(1-p)b}^{b} f(x)\, b \quad d_{p,q}x$$
$$= f\left(\frac{-4q+p}{p+q}\right) + f'\left(\frac{-4q+p}{p+q}\right)\left(\frac{-4p+5q}{p+q}\right)$$
$$-\frac{1}{4p}\left[4(p-q)p\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{(-4p+1)q^n}{p^{n+1}} + \left(1 - \frac{q^n}{p^{n+1}}\right)\right)\right]$$
$$= \frac{16q^2 - 8pq + p^2}{(p+q)^2} + \frac{-40q^2 - 22pq + 8p^2}{(p+q)^2} - \frac{16p^2}{p^2 + pq + q^2} + \frac{8p}{p+q} - 1 \leq 0$$

For the right hand side, we have

$$\frac{1}{p(1-(-3))}\int_{-4p+1}^{1} x^2\, b \quad d_{p,q}x - \frac{qf(1) + pf(-3)}{P+q}$$
$$= \frac{16p^2}{(p^2 + pq + q^2)} - \frac{8p}{P+q} + 1 - \frac{q+9p}{p+q} \leq 0$$

4. CONCLUSION

In this study, we have obtained some new results for the $(p,q)^b$- calculus of Hermite–Hadamard inequalities for differentiable convex function with the critical point along with some examples. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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