Journal of Science and Arts Volume 22, Issue 1, pp. 55-62, 2022

ORIGINAL PAPER

THE CLP-COMPACT-OPEN TOPOLOGY ON $KC(X,Y)$

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Manuscript received: 09.11.2021; Accepted paper: 16.02.2022;
Published online: 30.03.2022.

Abstract. *In this paper, we introduce clp-compact-open topology on $KC(X,Y)$ and compare this topology with compact-open topology and the topology of uniform convergence. Then, we examine metrizability, completeness and countability properties of the clp-compact-open topology on $KC(X,Y)$.*

Keywords: *Function space; set-open topology; metrizable; separability; second countable.*

1. INTRODUCTION AND PRELIMINARIES

It is a well-known fact that there are many topologies on $C(X,Y)$ of all continuous functions from a Tychonoff space X to a metric space Y. A number of these natural topologies are the point-open topology, the compact-open topology, the open-cover topology, the uniform topology, the fine topology and the graph topology. The compact-open topology, which was introduced by Fox [1], is one of the commonly used topologies on function spaces, and has many applications in homotopy theory and functional analysis. Later on it was improved by Arens and Dugundji in [2, 3]. Since it is used to study uniformly convergent sequences of functions on compact subsets, it is also called the topology of uniform convergence on compact sets. Kundu and Garg [4] presented some results on the compact-open topology on $KC(X)$, the set of all real-valued functions on X, which are continuous on the compact subsets of X. Clearly $KC(X) = C(X)$ if and only if X is a $k_{\mathbb{R}}$-space. Therefore, more general and beneficial results can be presented if $KC(X)$ is used instead of $C(X)$.

In the present paper, we introduce clp-compact-open topology on $KC(X,Y)$ and compare this topology with some other known topologies such as the compact-open topology and the uniform topology. We investigate the properties of the clp-compact-open topology on $KC(X)$ such as submetrizability, metrizability, separability, and second countability.

Unless otherwise stated clearly, throughout this paper, all spaces are assumed to be Tychonoff (completely regular Hausdorff).

If X and Y are any two topological spaces with the same underlying set, then we use the notation $X = Y$, $X \leq Y$, and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X, and that the topology on Y is strictly finer than the topology on X. Topological space will be used as space. The topology of the space X will be represented by $\tau(X)$. If $A \subseteq X$ and $f \in C(X)$, then we use the notation $f|_A$ for the restriction of the function f to the set A. We denote by $\mathbb{R}$ the real line with the natural topology. Finally, the constant zero function in $C(X)$ is denoted by f_0.

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2. THE CLP-COMPACT-OPEN TOPOLOGY

In this section, we define the clp-compact-open topology on $KC(X,Y)$ and also give some equivalent definitions. Then we compare the clp-compact-open topology with the compact-open topology and the topology of uniform convergence.

A topological space X is clp-compact if every cover consisting of clopen (both open and closed) sets has a nite subcover [5]. The motivation for this fact that it is a generalization of both compact spaces and connected spaces. Clp-compactness is preserved by taking clopen subsets and continuous images. For details see [5] and [6].

A function $f: X \to Y$ is called compact-continuous [7] if restriction function $f|_A: X \to Y$ is continuous whenever A is a compact subspace of X. Let $KC(X,Y)$ denote the set of all compact-continuous functions from X to Y. Since the restriction function of every continuous function is also continuous, every continuous function is compact-continuous. Therefore, it is seen that $C(X,Y) \subseteq KC(X,Y)$.

A space X is a $k_{\mathbb{R}}$-space if it is a Tychonoff space and if every mapping $f: X \to \mathbb{R}$, whose restriction to every compact set $K \subset X$ is continuous, is continuous on X [8].

Recall that $C(X) = KC(X)$ if and only if the space X is $k_{\mathbb{R}}$-space. Also submetrizable space is a $k_{\mathbb{R}}$-space. It is clearly seen that for compact space X, $C(X) = KC(X)$.

Let α be a nonempty family of subsets of a space X. For $A \in \alpha$ and $V \in \tau(Y)$, set-open topology on $KC(X,Y)$ denoted by $KC_\alpha(X,Y)$ has a subbase consisting of the sets

$$S(A,V) = \{f \in KC(X,Y): f(A) \subseteq V\}.$$

If we take all compact subsets of X as the family, we get the compact-open topology and denote it by $KC_k(X,Y)$. If all clp-compact subsets of X are taken as the family, the obtained topology is called the clp-compact-open topology and denoted by $KC_{clp}(X,Y)$.

The topology of uniform convergence on members of α has as base at each point $f \in KC(X,Y)$ the family of all sets of the form

$$B_A(f,\varepsilon) = \{g \in KC(X,Y): d\big(f(x),g(x)\big) < \varepsilon, \forall x \in A\}$$

where $A \in \alpha$ and $\varepsilon > 0$. The space $KC(X,Y)$ having the topology of uniform convergence on is denoted by $KC_{\alpha,u}(X,Y)$. In the case that $\alpha = \{X\}$, the topology on $KC(X,Y)$ is called the topology of uniform convergence or uniform topology and denoted by $KC_u(X,Y)$.

Now, we can compare the topologies. Since every compact space is clp-compact, the following result can be obtained immediately.

Theorem 1. For any space X and any metric space Y, $KC_k(X,Y) \leq KC_{clp}(X,Y)$.

A space X is said to have nontrivial path if there are distinct points x_1 and x_2 in X, and a continuous function $p: [0;\ 1] \to X$ such that $p(0) = x_1$ and $p(1) = x_2$.

Theorem 2. For any space X and any metric space Y contains a nontrivial path, $KC_k(X,Y) = KC_{clp}(X,Y)$, if and only if every closed clp-compact subset of X is compact.

Proof: If every closed clp-compact subset of X is compact, then $KC_{clp}(X,Y) \leq KC_k(X,Y)$. Thus, we have $KC_k(X,Y) = KC_{clp}(X,Y)$.

Conversely, let $KC_k(X,Y) = KC_{clp}(X,Y)$ and let A be any closed clp-compact subset of X. Let $p: [0;\ 1] \to Y$, be a nontrivial path such that $p(0) \neq p(1)$. Let $h \in KC(X,Y)$ be defined as $h(x) = p(0)$ for all $x \in X$. Choose an $\varepsilon > 0$ such that $\varepsilon < d(p(0), p(1))$. Then, there should exist a compact subset F of X and $\delta > 0$ such that $B_F(h,\delta) \subseteq B_A(h,\varepsilon)$. Choose

$x_0 \in A\backslash F$. Since X is Tychonoff, there exists a continuous function $\phi: X \to [0;\ 1]$ such that $\phi(F) = \{0\}$ and $\phi(x_0) = 1$. Then, the function $g = \phi \circ p \in B_F(h, \delta)\backslash B_A(h, \varepsilon)$, a contradiction.

Corollary 1. If X is a compact space, then for any metric space Y, $KC_k(X, Y) = KC_{clp}(X, Y) = KC_u(X, Y)$.

We recall that in a zero-dimensional space, i.e. a space with a base of clopen sets, the notions of compactness and clp-compactness coincide (See [5, Proposition 1.12]). Then we obtain the following result.

Corollary 2. If X is a zero-dimensional space, then for any metric space Y, $KC_k(X, Y) = KC_{clp}(X, Y) \leq KC_u(X, Y)$.

Note that any countable regular space is zero-dimensional and consequently any countable compact Hausdorff space is zero-dimensional. Thus we can give the following result.

Corollary 3. If X is a countable space, then for any metric space Y,

$$KC_k(X, Y) = KC_{clp}(X, Y).$$

We recall that a non-empty subspace $Y \subseteq \mathbb{R}$ is zero-dimensional if and only if it does not contain any interval (see page 11 in [9]). Then, a subspace $Y \subseteq \mathbb{R}$ is zero-dimensional if and only if $\mathbb{R}\backslash Y$ is dense. It follows that the rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{P}$, and the Cantor set C are zero-dimensional.

Considering Theorem 4.1 and Theorem 4.5 in [10], it is clear that $KC_{clp}(X, Y) \neq KC_{clp,u}(X, Y)$. Therefore we will consider the clp-compact-open topology on $KC(X, Y)$ where Y is a zero-dimensional metric space contains a nontrivial path. The results to be obtained are clearly provided for $KC^*(X)$ the set of all bounded continuous real-valued functions on X, which are continuous on the compact subsets of X.

Theorem 3. For any space X and zero-dimensional metric space Y, the clp-compact-open topology on $KC(X, Y)$ is the same as the topology of uniform convergence on the clp-compact subsets of X, that is, $KC_{clp}(X, Y) = KC_{clp,u}(X, Y)$.

Proof: Assume that $S(A, V)$ is a subbasic open set in $KC_{clp}(X, Y)$ and $f \in S(A, V)$. Since $f(A)$ is compact and $f(A) \subseteq V$, then there exists $\varepsilon > 0$ such that $B(f(A), \varepsilon) \subseteq V$ [11, Corollary 4.1.14]. If $g \in B_A(f, \varepsilon)$ and $x \in A$, then we obtain $g(x) \in B(f(x), \varepsilon)$. Hence, we find $g(A) \subseteq V$, i.e. $g \in S(A, V)$. It follows that $B_A(f, \varepsilon) \subseteq S(A, V)$. Consequently, $KC_{clp}(X, Y) \leq KC_{clp,u}(X, Y)$.

Now, let $B_A(f, \varepsilon)$ be a basic neighborhood of f in $KC_{clp,u}(X, Y)$. Then, there exist $f(x_1), f(x_2), \ldots, f(x_n)$ such that $f(A) \subseteq \bigcup_{i=1}^{n} B(f(x_i), \varepsilon/3)$ since $f(A)$ is compact. If we take $V_i = B(f(x_i), \varepsilon/3)$ and $W_i = B(f(x_i), 2\varepsilon/3)$, we find $\overline{V_i} \subseteq W_i$. Also $f(A) \subseteq \bigcup_{i=1}^{n} V_i \subseteq \bigcup_{i=1}^{n} \overline{V_i}$. Let $A_i = A \cap f^{-1}(\overline{V_i})$, where each A_i is clp-compact. We have $f(A_i) \subseteq \overline{V_i} \subseteq W_i$ and so $f \in \bigcap_{i=1}^{n} S(A_i, W_i)$. Now we need to show that $\bigcap_{i=1}^{n} S(A_i, W_i) \subseteq B_A(f, \varepsilon)$. Suppose that $g \in \bigcap_{i=1}^{n} S(A_i, W_i)$ and $x \in \bigcup_{i=1}^{n} A_i$. Thus, there exists an i such that $x \in A_i$ and consequently, $f(x) \in \overline{V_i}$ and $g(x) \in W_i$. Since $d\big(f(x), g(x)\big) \leq d\big(f(x), f(x_i)\big) + d\big(f(x_i), g(x)\big) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$, then $g \in B_A(f, \varepsilon)$. Hence, $KC_{clp,u}(X, Y) \leq KC_{clp}(X, Y)$.

Corollary 4. For any space X and zero-dimensional metric space Y, $KC_k(X,Y) \leq KC_{clp}(X,Y) = KC_{clp,u}(X,Y) \leq KC_u(X,Y)$.

By Theorem 3, it is seen that the weak clp-compact-open topology on $KC(X,Y)$ can be achieved another way. For each clp-compact subset A of X and $\varepsilon > 0$, we define the seminorm p_A on $KC(X,Y)$ and $V_{A,\varepsilon}$, as follow: $p_A = \sup\{|f(x)|: x \in A\}$ and $V_{A,\varepsilon} = \{f \in KC(X,Y): p_A(f) < \varepsilon\}$. Let $\mathcal{V} = \{V_{A,\varepsilon}: A$ is clp-compact in $X, \varepsilon > 0\}$. Then for each $f \in KC(X,Y)$, $f + \mathcal{V} = \{f + V: V \in \mathcal{V}\}$ forms a neighborhood base at f. Since it is generated by a collections of seminorms this topology is locally convex and it is same as the clp-compact-open topology on $KC(X,Y)$. It is also easy to see that this topology is Hausdorff. Recall that if the topology of a locally convex Hausdorff space is generated by a countable family of seminorms, then it is metrizable (see page 119 in [12]).

Theorem 4. For any space X and zero-dimensional metric space Y contains a nontrivial path, $KC_{clp}(X,Y) = KC_u(X,Y)$ if and only if X is clp-compact.

Proof: Let $KC_{clp}(X,Y) = KC_u(X,Y)$. Let $p: [0;\ 1] \to Y$ be a continuous function such that $p(0) \neq p(1)$. Define $g: X \to Y$ by $g(x) = p(0)$ for all $x \in X$ and consider the basic open set $B_X(g,\varepsilon)$ in $KC_u(X,Y)$, where $\varepsilon = d(p(0),p(1))$. We show that $B_X(g,\varepsilon)$ is not open in $KC_{clp}(X,Y)$. Consider any clp-compact subset A of X. Since X is not clp-compact, there exists $x_0 \in X \setminus \overline{A}$. Since X is Tychonoff, there exist a continuous function $\phi: X \to [0;\ 1]$ such that $\phi(\overline{A}) = 0$ and $\phi(x_0) = 1$. Let $h = p \circ \phi \in KC(X,Y)$. Then $h \in B_A(g,\delta)$ for any $\delta > 0$, because for any $x \in A$, $d(h(x),g(x)) = d(p(\phi(x)),p(0)) = d(p(0),p(0)) = 0 < \delta$. But $h \notin B_X(g,\varepsilon)$ as $d(g(x_0),h(x_0)) = d(p(0),p(1)) = \varepsilon$. So $B_A(g,\delta) \nsubseteq B_X(g,\varepsilon)$ for any clp-compact subset A of X. This shows that $B_X(g,\varepsilon)$ is not open in $KC_{clp}(X,Y)$. So we arrive at a contradiction. Hence X is clp-compact.

Conversely, suppose that X is clp-compact. It follows that for each $f \in KC(X,Y)$ and each $\varepsilon > 0$, $B_X(f,\varepsilon)$ is an open set in $KC_{clp,u}(X,Y) = KC_{clp}(X,Y)$ (see Theorem 3). Consequently, $KC_{clp}(X,Y) = KC_u(X,Y)$.

Example 1. $\mathbb{R}$ is clp-compact because the only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$. Then for any metric space, we have $KC_k(\mathbb{R},Y) \leq KC_{clp}(\mathbb{R},Y) = KC_u(\mathbb{R},Y)$.

Example 2. Let X be connected space which is not compact. Then the space X is clp-compact. Hence for any metric space Y, we have $KC_k(X,Y) \leq KC_{clp}(X,Y) = KC_u(X,Y)$.

Example 3. The space of rational numbers $\mathbb{Q} \subseteq \mathbb{R}$ is zero-dimensional, but not compact [13, Example 30]. Also $\mathbb{Q}$ is not clp-compact. If $\mathbb{Q}$ is clp-compact, then $\mathbb{Q}$, being zero-dimensional, is compact. This leads to a contradiction Thus for any metric space Y, we obtain $KC_k(X,Y) = KC_{clp}(X,Y) \leq KC_u(X,Y)$.

Example 4. Let $X = [0,\omega_1)$ be the space of countable ordinals with the order topology [13, Example 42]. The space X is zero-dimensional, but not compact. Also X is not clp-compact. Hence for any metric space Y, we obtain $KC_k(X,Y) = KC_{clp}(X,Y) \leq KC_u(X,Y)$.

3. SOME TOPOLOGICAL PROPERTIES OF $KC^*_{clp}(X)$

In this section, we study the submetrizability, metrizability, first countability, complete metrizability, $\aleph_0$-space, cosmic, separability and second countability of $KC^*_{clp}(X)$.

A space X is said to be submetrizable if it has a weaker metrizable topology, equivalently if there exists a metrizable space Y and a continuous bijection $f: X \to Y$ from the space X onto Y.

Remark 1. (1) For any space X, if the set $\{(x, x): x \in X\}$ is G_δ-set (resp. Zero-set) in the product space $X \times X$, then X is said to have a G_δ-diagonal (resp. zero-set diagonal). Every submetrizable space X has a zero-set diagonal. Consequently, every submetrizable space X has a G_δ-diagonal since a zero-set is a G_δ-set [14].

(2) A space X is called an E_0-space if every point in the space is a G_δ-set. The submetrizable spaces are E_0-spaces [14].

Finally, recall that in submetrizable space, the notions of compactness, countably compactness and pseudocompactness coincide. By using this fact we obtain the following result.

Theorem 5. For any space X, the following are equivalent.

(1) $KC^*_{clp}(X)$ is submetrizable.
(2) Every pseudocompact subset of $KC^*_{clp}(X)$ is a G_δ-set in $KC^*_{clp}(X)$.
(3) Every countably compact subset of $KC^*_{clp}(X)$ is a G_δ-set in $KC^*_{clp}(X)$.
(4) Every compact subset of $KC^*_{clp}(X)$ is a G_δ-set in $KC^*_{clp}(X)$.
(5) $KC^*_{clp}(X)$ is an E_0-space.
(6) $KC^*_{clp}(X)$ has a zero-set-diagonal.
(7) $KC^*_{clp}(X)$ has a G_δ-diagonal.

Proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ and $(1) \Rightarrow (6) \Rightarrow (7) \Rightarrow (5)$ are all immediate.
$(5) \Rightarrow (1)$ Let $X = \bigcup_{n=1}^{\infty} A_n$ where each A_n is clp-compact. Let $S = \bigoplus\{A_n: n \in \mathbb{N}\}$ be the topological sum of the A_n and let $\phi: S \to X$ be the natural function. Then the induced function $\phi^*: KC^*_{clp}(X) \to KC^*_{clp}(S)$ defined by $\phi^*(f) = f \circ \phi$ is continuous. Now we shall show that ϕ^* is one-to-one. Let $\phi^*(g_1) = \phi^*(g_2)$. Then g_1 and g_2 are equal on $\bigcup_{n=1}^{\infty} A_n$. So $g_1 - g_2 \in \bigcap_{n=1}^{\infty} B_{A_n}(0, \varepsilon_n) = \{0\}$. Thus $g_1 = g_2$ and consequently ϕ^* is one-to-one. By Theorem 2.2 and Theorem 2.3 in [4], $KC^*_{clp}(\bigoplus\{A_n: n \in \mathbb{N}\})$ is homeomorphic to $\prod\{KC^*_{clp}(A_n): n \in \mathbb{N}\}$. But each $KC^*_{clp}(A_n) = C^*_{clp}(A_n)$ is metrizable by Theorem 2.8 in [15]. Since $KC^*_{clp}(S)$ is metrizable and ϕ^* is continuous injection, $KC^*_{clp}(X)$ is submetrizable.

A topological space is said to be hemiclp-compact if there exists a sequence of clp-compact sets $\{A_n\}$ in X such that for any clp-compact subset A of X, $A \subseteq A_n$ holds for some n.

Theorem 6. $KC^*_{clp}(X)$ is first countable if and only if X is hemiclp-compact.

Proof: Let $KC^*_{clp}(X)$ be rst countable. So f_0 has a countable base $\{W_n\}$, where each $W_n = S(A_{n1}, V_{n1}) \cap S(A_{n2}, V_{n2}) \cap \ldots \cap S(A_{nk_n}, V_{nk_n})$. Choose $A_n = \bigcup_{i=1}^{k_n} A_{ni}$. If A is any clp-compact subset of X, we need to show that $A \subseteq A_n$ for some $n \in \mathbb{N}$. Let $V = (0,1)$ and so $S(A, V)$ is a neighborhood of f_0. Then there exists W_n such that $W_n \subseteq S(A, V)$ for some $n \in \mathbb{N}$. Suppose that there exists $x \in A \backslash A_n$. Since X is Tychonoff, there exists $f \in KC^*(X)$ such that $f(x) = 1$ and $f(A_n) = \{0\}$. Then $f \notin S(A, V)$ but $f \in W_n$. This contradicts our supposition. Thus $A \subseteq A_n$ and hence X is hemiclp-compact.

Conversely, let $\{A_n\}$ be a sequence of clp-compact sets in X such that every clp-compact set in X is contained in A_n for some $n \in \mathbb{N}$, $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ and $\mathcal{V}$ be a countable local base at $x \in \mathbb{R}$. Let $\bigcap_{i=1}^{n} S(A_i, W_i)$ be any open set in $KC^*_{clp}(X)$ containing f and $f(x) \in W_i$. So there exist $K_{k_i} \in \mathcal{K}$ and $V_{k_i} \in \mathcal{V}$ such that $A_i \subseteq K_{k_i}$ and $V_{k_i} \subseteq W_i$, for $1 \leq i \leq n$. Clearly $f \in \bigcap_{i=1}^{n} S(K_{k_i}, V_{k_i}) \subseteq S(A_i, W_i)$. Hence, $KC^*_{clp}(X)$ is first countable since it has countable local base at f.

It is well known that a locally convex Hausdorff space is metrizable if and only if it is rst countable. Then the following result is obvious.

Corollary 5. $KC^*_{clp}(X)$ is metrizable if and only if X is hemiclp-compact.

Now, we show that even some properties of $KC^*_{clp}(X)$ weaker than first count-ability are equivalent to the metrizability of $KC^*_{clp}(X)$. So we begin with reminder of the definitions of these properties.

A subset S of a space X is said to have countable character if there exists a sequence $\{W_n : n \in \mathbb{N}\}$ of open subsets in X such that $S \subseteq W_n$ for each n and if W is any open set containing S, then $W_n \subseteq W$ for some n.

A space X is said to be of (pointwise) countable type if each (point) compact set is contained in a compact set having countable character. A space X is a q-space if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. Another property stronger than being a q-space is that of being an M-space, which can be characterized as a space that can be mapped onto a metric space by a quasi-perfect map (a continuous closed map in which inverse images of points are countably compact). Both a space of pointwise countable type and an M-space are q-spaces.

Theorem 7. For any space X, the following are equivalent.

(1) $KC^*_{clp}(X)$ is metrizable.
(2) $KC^*_{clp}(X)$ is first countable.
(3) $KC^*_{clp}(X)$ is of pointwise countable type.
(4) $KC^*_{clp}(X)$ is M-space.
(5) $KC^*_{clp}(X)$ is q-space.
(6) X is hemiclp-compact.

Proof: From the earlier discussions, we have $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (6)$, $(1) \Rightarrow (5) \Rightarrow (6)$ and $(2) \Rightarrow (6)$. Also by Theorem 6, $(1) \Leftrightarrow (2) \Leftrightarrow (7)$.
$(6) \Rightarrow (7)$ Suppose that $KC^*_{clp}(X)$ is a q-space. Hence, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of the zero function f_0 in $KC^*_{clp}(X)$ such that if for each n, $f_n \in U_n$ then $\{f_n : n \in \mathbb{N}\}$ has a cluster point in $KC^*_{clp}(X)$. Now for each n, there exist a clp-compact subset A_n of X and $\varepsilon_n > 0$ such that $f_0 \in B_{A_n}(f_0, \varepsilon_n) \subseteq U_n$.

Let A be a clp-compact subset of X. Suppose that A is not a subset of A_n for any $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A \backslash A_n$. So for each $n \in \mathbb{N}$ there exists a continuous function $f_n : X \to [0, n]$ such that $f_n(a_n) = n$ and $f_n(x) = 0$ for all $x \in A_n$. It is clear that $f_n \in B_{A_n}(f_0, \varepsilon_n)$. But the sequence $\{f_n\}_{n \in \mathbb{N}}$ does not have a cluster point in $KC^*_{clp}(X)$. Suppose that this sequence has a cluster point f in $KC^*_{clp}(X)$. Then for each $k \in \mathbb{N}$, there exists a positive integer $n_k > k$ such that $f_{n_k} \in B_A(f, 1)$. Thus, for all $k \in \mathbb{N}$, $f(a_{n_k}) >$

$f_{n_k}(a_{n_k}) - 1 = n_k - 1 \geq k$. But this means that f is unbounded on the clp-compact set A. Hence, the sequence $\{f_n\}_{n\in\mathbb{N}}$ cannot have a cluster point in $KC^*_{clp}(X)$ and consequently, $KC^*_{clp}(X)$ fails to be a q-space. Thus, X must be hemiclp-compact.

The topology of uniform convergence on the clp-compact subsets of X is actually generated by the uniformity of uniform convergence on these subsets. When this uniformity is complete, $KC^*_{clp}(X)$ is said to be uniformly complete. Recall that a uniform space X with an uniformity U is called uniformly complete if the uniformity U is complete. We say that the uniformity $\mathcal{U}$ on X is complete if every Cauchy net in X converges.

A space X is called Cech-complete if X is a G_δ-set in βX, the Stone-Cech compactication of X. A space X is called locally Cech-complete if every point $x \in X$ has a Cech-complete neighborhood.

Theorem 8. For any space X, $KC^*_{clp}(X)$ is uniformly complete.

Proof: Let (f_n) be a Cauchy net in $KC^*_{clp}(X)$. If A is a compact subset of X, then the net $(f_n|_A)$ is Cauchy in $KC^*_{clp}(A) = C^*_{clp}(A)$. But since $C^*_{clp}(A)$ is uniformly complete [15, Theorem 3.9], the net $(f_n|_A)$ converges to some f_A in $C^*_{clp}(A)$. Define $f: X \to \mathbb{R}$ by $f(x) = f_A(x)$ if $x \in A$. It can easily be seen that f is well defined and $f|_A = f_A$ for A for each compact subset A of X. Clearly $f \in KC(X)$. Also it is easy to see that (f_n) converges to f.

The following results gives a characterization of complete metrizable of the space $KC^*_{clp}(X)$.

Corollary 6. For any space X, the following are equivalent.

(1) $KC^*_{clp}(X)$ is complete metrizable.
(2) $KC^*_{clp}(X)$ is Cech-complete.
(3) $KC^*_{clp}(X)$ is locally Cech-complete.
(4) $KC^*_{clp}(X)$ is an open continuous image of Cech-complete space.
(5) $KC^*_{clp}(X)$ is metrizable.
(6) X is hemiclp-compact.

Since for a $k_{\mathbb{R}}$-space X, $C^*_{clp^*}(X)$ is complete by Theorem 3.9 in [15], then the following results can be given.

Corollary 7. For any space X, the following are equivalent.

(1) $C^*_{clp^*}(X)$ is complete metrizable.
(2) $C^*_{clp^*}(X)$ is Cech-complete.
(3) $C^*_{clp^*}(X)$ is locally Cech-complete.
(4) $C^*_{clp^*}(X)$ is an open continuous image of Cech-complete space.
(5) $C^*_{clp^*}(X)$ is metrizable.
(6) X is hemiclp-compact $k_{\mathbb{R}}$-space.

We now study countability properties of $KC^*_{clp}(X)$ such as $\aleph_0$-space, cosmic, separability and second countability.

A k-network for a space X is a family $\mathcal{K}$ of subsets of X such that whenever compact K is contained in open U, then there is a nite subset $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $K \subseteq \cup \mathcal{K}_0 \subseteq \mathcal{K}$. A space X is called a $\aleph_0$-space [16] if it has a countable k-network.

A space X is said to have a countable network if there exists a countable family $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that for each $x \in X$ and for each open set U containing x, there exists an A_n such that $x \in A_n \subseteq U$. A space X is called a cosmic space [16] if it has a countable network.

Recall that any $\aleph_0$-space is cosmic, any cosmic space is Lindelöf and separable. Also in metrizable space, the notions of second countability, $\aleph_0$-space and cosmic property coincide [16].

Theorem 9. For any space X, the following are equivalent.

(1) $KC^*_{clp}(X)$ is $\aleph_0$-space.
(2) $KC^*_{clp}(X)$ is cosmic space.
(3) $KC^*_{clp}(X)$ is second countable.
(4) $KC^*_{clp}(X)$ is separable.
(5) X is compact and metrizable.

Proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are all immediate.
$(4) \Rightarrow (5)$ It is easy to see that $C(X)$ is dense in $KC_{clp}(X)$. If $KC^*_{clp}(X)$ is separable, then $C^*_{clp}(X)$ is separable. By Theorem 3.11 in [15], X is compact and metrizable.
$(5) \Rightarrow (1)$ Let X be compact. Then $KC^*_k(X) = KC^*_{clp}(X) = KC^*_u(X)$ by Corollary 1 and also $KC^*_k(X) = C_k(X)$. Since compact metric space is second countable, X is second countable and so X is $\aleph_0$-space. Then $C_k(X)$ is $\aleph_0$-space by Corollary 4.1.3 in [17]. Hence $KC^*_{clp}(X)$ is $\aleph_0$-space.

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