ORIGINAL PAPER

A NOTE ON LAMBERT'S LAW INVOLVING INCOMPLETE I-FUNCTIONS

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Abstract. *Many researchers are interested in Lambert's law because of its relevance in light attenuation owing to the characteristics of a material through which the light passes. In this paper, we developed the Lambert's law that involves incomplete I-functions. Next, while taking a course in the constraints of incomplete I-functions, we give a few special cases of our model, and also mention some known results.*

Keywords: *incomplete Gamma function; incomplete I-functions; incomplete $\bar{I}$ −function, Mellin-Barnes type integrals.*

Mathematics Subject Classification: *03C65, 33B15, 33B20*

1. INTRODUCTION

The Lambert 's law takes into account the attenuation of the light to the properties of a material through which the light travels. The law is commonly applied to measurements of chemical analysis and is used to understand attenuation in physical optics, photons, neutrons, or rarified gases.

This study introduces the Lambert's law that contains incomplete I-functions. We think back to the commonly applied incomplete Gamma functions $\Gamma(s,x)$ and $\gamma(s,x)$ described as:

$$\gamma(s,x) := \int_0^x t^{s-1} e^{-t} dt \qquad (\Re(s) > 0; x \geqq 0) \tag{1}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1} e^{-t} dt \qquad (x \geqq 0; \Re(s) > 0 \quad \text{when} \quad x = 0), \tag{2}$$

respectively, follow the formula of decomposition presented by:

$$\Gamma(s,x) + \gamma(s,x) = \Gamma(s) \qquad (\Re(s) > 0). \tag{3}$$

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ were presented and analyzed in the following manner (see, [1]):

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$$ {}^{\gamma}I_{p,q}^{m,n}(z) = {}^{\gamma}I_{p,q}^{m,n}\left[z \left| \begin{matrix} (\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p) \\ (\eta_1,\Im_1;B_1),(\eta_2,\Im_2;B_2),\cdots,(\eta_q,\Im_q;B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_{£} \phi(s,x)\ z^s\ ds \tag{4} $$

and

$$ {}^{\Gamma}I_{p,q}^{m,n}(z) = {}^{\Gamma}I_{p,q}^{m,n}\left[z \left| \begin{matrix} (\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p) \\ (\eta_1,\Im_1;B_1),(\eta_2,\Im_2;B_2),\cdots,(\eta_q,\Im_q;B_q) \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_{£} \Phi(s,x)\ z^s\ ds \tag{5} $$

for all $z \neq 0$, where

$$ \phi(s,x) = \frac{\{\gamma(1-\zeta_1+\ell_1 s,x)\}^{A_1}\prod_{j=1}^{m}\{\Gamma(\eta_j-\Im_j s)\}^{B_j}\prod_{j=2}^{n}\{\Gamma(1-\zeta_j+\ell_j s)\}^{A_j}}{\prod_{j=n+1}^{p}\{\Gamma(\zeta_j-\ell_j s)\}^{A_j}\prod_{j=m+1}^{q}\{\Gamma(1-\eta_j+\Im_j s)\}^{B_j}} \tag{6} $$

and

$$ \Phi(s,x) = \frac{\{\Gamma(1-\zeta_1+\ell_1 s,x)\}^{A_1}\prod_{j=1}^{m}\{\Gamma(\eta_j-\Im_j s)\}^{B_j}\prod_{j=2}^{n}\{\Gamma(1-\zeta_j+\ell_j s)\}^{A_j}}{\prod_{j=n+1}^{p}\{\Gamma(\zeta_j-\ell_j s)\}^{A_j}\prod_{j=m+1}^{q}\{\Gamma(1-\eta_j+\Im_j s)\}^{B_j}}. \tag{7} $$

The incomplete I-functions $I_{p,q}^{m,n}(z)$ and $I_{p,q}^{m,n}(z)$ defined in (4) and (5) exist for all $x \geq 0$ under the same contour and conditions as described in [2]. For $A_1 = 1$, these functions fulfill the following relation:

$$ {}^{\gamma}I_{p,q}^{m,n}(z) + {}^{\Gamma}I_{p,q}^{m,n}(z) = I_{p,q}^{m,n}(z) \tag{8} $$

for the familiar I-function.

In science and engineering, several studies and uses of the incomplete I-function have been documented (see, for example, recent works [3–7]). The incomplete I-functions are the generalizations of incomplete $\bar{I}$-functions, I-function, incomplete H-functions, and H-functions etc., which are given under:

(i) Considering that $x = 0$ for the (5), incomplete I-function ${}^{\Gamma}I_{p,q}^{m,n}(z)$ reduce to the I-function [2] as follows:

$$ {}^{\Gamma}I_{p,q}^{m,n}\left[z \left| \begin{matrix} (\zeta_1,\ell_1;A_1:0),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p) \\ (\eta_1,\Im_1;B_1),\cdots,(\eta_q,\Im_q;B_q) \end{matrix} \right. \right] = I_{p,q}^{m,n}\left[z \left| \begin{matrix} (\zeta_1,\ell_1;A_1),\cdots,(\zeta_p,\ell_p;A_p) \\ (\eta_1,\Im_1;B_1),\cdots,(\eta_q,\Im_q;B_q) \end{matrix} \right. \right]. \tag{9} $$

(ii) Setting B_j $(j = 1,\cdots,m) = 1$, the functions (4) and (5) reduce to the incomplete $\bar{I}$-functions [1] as follows:

$$\begin{aligned}&{}^{\gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p)\\(\eta_j,\Im_j;1)_{1,m},(\eta_j,\Im_j;B_j)_{m+1,q}\end{matrix}\right.\right]\\&={}^{\gamma}\bar{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p)\\(\eta_j,\Im_j;1)_{1,m},(\eta_j,\Im_j;B_j)_{m+1,q}\end{matrix}\right.\right]\end{aligned} \tag{10}$$

and

$$\begin{aligned}&{}^{\Gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p)\\(\eta_j,\Im_j;1)_{1,m},(\eta_j,\Im_j;B_j)_{m+1,q}\end{matrix}\right.\right]\\&={}^{\Gamma}\bar{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_2,\ell_2;A_2),\cdots,(\zeta_p,\ell_p;A_p)\\(\eta_j,\Im_j;1)_{1,m},(\eta_j,\Im_j;B_j)_{m+1,q}\end{matrix}\right.\right].\end{aligned} \tag{11}$$

(iii) Setting B_j $(j=1,\cdots,q)=1$ and A_j $(j=1,\cdots,p)=1$, the functions (4) and (5) reduce to the incomplete H-functions [8] as follows:

$${}^{\gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;1:x),(\zeta_j,\ell_j;1)_{1,p}\\(\eta_j,\Im_j;1)_{1,q}\end{matrix}\right.\right]=\gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1,x),(\zeta_j,\ell_j)_{2,p}\\(\eta_j,\Im_j)_{1,q}\end{matrix}\right.\right] \tag{12}$$

and

$${}^{\Gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;1:x),(\zeta_j,\ell_j;1)_{1,p}\\(\eta_j,\Im_j;1)_{1,q}\end{matrix}\right.\right]=\Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1,x),(\zeta_j,\ell_j)_{2,p}\\(\eta_j,\Im_j)_{1,q}\end{matrix}\right.\right]. \tag{13}$$

(iv) Setting B_j $(j=1,\cdots,q)=1$, A_j $(j=1,\cdots,p)=1$ and $x=0$ in (13), the function (13) reduces to the H-function [9] as follows:

$${}^{\Gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;1:0),(\zeta_j,\ell_j;1)_{2,p}\\(\eta_j,\Im_j;1)_{1,q}\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_j,\ell_j)_{1,p}\\(\eta_j,\Im_j)_{1,q}\end{matrix}\right.\right]. \tag{14}$$

2 MAIN RESULTS

Throughout in this paper, letting $\mathfrak{v}$ be the incident light intensity of λ wavelength, and t the medium thickness. Also, we suppose on any time $\mathfrak{v}_1$ and t_1 represent the partial changes in intensity of incident light and thickness of the medium, respectively.

Theorem 1. If $\mathfrak{v}>\mathfrak{v}_1, t>t_1$ and $x\geq 0$ then the subsequent rule shall be observed:

$$\int {}^{\Gamma}I_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p},(1+\mathfrak{v},\mathfrak{v}_1;1),\\(\mathfrak{v},\mathfrak{v}_1;1),(\eta_j,\Im_j;B_j)_{1,q}\end{matrix}\right.\right]d\mathfrak{v} \tag{15}$$

$$= -\mu\, {}^{\Gamma}I_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(-t,t_1;1),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\Im_j;B_j)_{1,q},(1-t,t_1;1)\end{matrix}\right.\right]$$
$$+\,\mathfrak{C}\, {}^{\Gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\Im_j;B_j)_{1,q}\end{matrix}\right.\right],$$

where μ is the attenuation coefficient and $\mathfrak{C}$ is a constant of integration.

Proof: The Lambert Law is mathematically described this way

$$\frac{d\mathfrak{v}}{dt} = -\mu\mathfrak{v}$$

on integration, we obtain

$$\int \frac{d\mathfrak{v}}{\mathfrak{v}} = -\mu t + \mathfrak{C}$$

where μ is attenuation coefficient and $\mathfrak{C}$ is a constant of integration. The above equation may be described to

$$\int \frac{\Gamma(\mathfrak{v})}{\Gamma(\mathfrak{v}+1)}\,d\mathfrak{v} = -\mu\frac{\Gamma(t+1)}{\Gamma(t)} + \mathfrak{C}.$$

If, t is replaced by $t + t_1 s$ (medium thickness will increase) and $\mathfrak{v}$ is replaced by $\mathfrak{v} - \mathfrak{v}_1 s$ (light intensity will diminish) and multiplying both sides by $\frac{1}{2\pi i}\Phi(s,x)z^s$, then integrate in the direction of the contour with respect to s, we get the needed result by taking account of (5).

Theorem 2. If $\mathfrak{v} > \mathfrak{v}_1, t > t_1$ and $x \geq 0$ then the subsequent rule shall be observed:

$$\int {}^{\gamma}I_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p},(1+\mathfrak{v},\mathfrak{v}_1;1),\\(\mathfrak{v},\mathfrak{v}_1;1),(\eta_j,\Im_j;B_j)_{1,q}\end{matrix}\right.\right]d\mathfrak{v}$$
$$= -\mu\, {}^{\gamma}I_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(-t,t_1;1),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\Im_j;B_j)_{1,q},(1-t,t_1;1)\end{matrix}\right.\right] \tag{16}$$
$$+\,\mathfrak{C}\, {}^{\gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\Im_j;B_j)_{1,q}\end{matrix}\right.\right],$$

where μ is the attenuation coefficient and $\mathfrak{C}$ is a constant of integration.

Proof: The claim (16) of the Theorem 2 can be determined using the same lines as the Theorem 1.

3. SPECIAL CASES

We shall report a few important limiting cases of key outcomes in this section.

If $B_j\ (j=1,\cdots,m)=1$ is set, the incomplete I-functions (4) and (5) will reduce the incomplete $\overline{I}$-functions defined by (11), then Theorems 1 and 2 will give the following corollaries:

Corollary 1. If $\mathfrak{v}>\mathfrak{v}_1, t>t_1$ and $x\geq 0$ then the following law is hold:

$$\begin{aligned}\int {}^{\Gamma}\overline{I}_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p},(1+\mathfrak{v},\mathfrak{v}_1;1)\\(\mathfrak{v},\mathfrak{v}_1;1),(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q}\end{matrix}\right.\right]d\mathfrak{v}\\ =-\mu\,{}^{\Gamma}\overline{I}_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(-t,t_1;1),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q},(1-t,t_1;1)\end{matrix}\right.\right]\\ +\mathfrak{C}\,{}^{\Gamma}\overline{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q}\end{matrix}\right.\right].\end{aligned}\tag{17}$$

Corollary 2. If $\mathfrak{v}>\mathfrak{v}_1, t>t_1$ and $x\geq 0$ then the following law is hold:

$$\begin{aligned}\int {}^{\gamma}\overline{I}_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p},(1+\mathfrak{v},\mathfrak{v}_1;1)\\(\mathfrak{v},\mathfrak{v}_1;1),(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q}\end{matrix}\right.\right]d\mathfrak{v}\\ =-\mu\,{}^{\gamma}\overline{I}_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(-t,t_1;1),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q},(1-t,t_1;1)\end{matrix}\right.\right]\\ +\mathfrak{C}\,{}^{\gamma}\overline{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1:x),(\zeta_j,\ell_j;A_j)_{2,p}\\(\eta_j,\mathfrak{J}_j;1)_{1,m},(\eta_j,\mathfrak{J}_j;B_j)_{m+1,q}\end{matrix}\right.\right].\end{aligned}\tag{18}$$

If $x=0$, the incomplete I-functions (5) will reduce the incomplete I-functions defined by (9), then Theorems 1 will give the following corollary:

Corollary 3. If $x=0$, then the following law is hold:

$$\begin{aligned}\int I_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1),\cdots,(\zeta_p,\ell_p;A_p),(1+\mathfrak{v},\mathfrak{v}_1;1)\\(\mathfrak{v},\mathfrak{v}_1;1),(\eta_1,\mathfrak{J}_1;B_1),\cdots,(\eta_q,\mathfrak{J}_q;B_q)\end{matrix}\right.\right]d\mathfrak{v}\\ =-\mu\,I_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1),\cdots,(\zeta_p,\ell_p;A_p),(-t,t_1;1)\\(\eta_1,\mathfrak{J}_1;B_1),\cdots,(\eta_q,\mathfrak{J}_q;B_q),(1-t,t_1;1)\end{matrix}\right.\right]\\ +\mathfrak{C}\,I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\zeta_1,\ell_1;A_1),\cdots,(\zeta_p,\ell_p;A_p),\\(\eta_1,\mathfrak{J}_1;B_1),\cdots,(\eta_q,\mathfrak{J}_q;B_q)\end{matrix}\right.\right],\end{aligned}\tag{19}$$

In addition, specializing the parameters in (4) and (5), we may obtain internal blood pressure equations as special cases for functions expressed in section 1 of our main results as follows:

1. Using the relationship (9), we may obtain the Lambert's Law involving I-function.

2. Using the relationships (12) and (13), we obtain the known results presented by Bansal *et al.* [10].

3. Using the relationship (14), we obtain the known results presented by Srivastava [11].

4. Furthermore, utilising the connection relations provided in [8] (see equations (6.3) and (6.4)), between the incomplete Fox-Wright functions and the incomplete H-functions, similar results may be obtained using Fox-Wright functions.

4. CONCLUSION

Lambert 's Law was presented for incomplete I-functions. In addition, we draw attention to some new special cases by specializing the incomplete I-function parameters and also indicating some known results. The results of this study are very useful in the measurement study of chemical analysis.

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