ORIGINAL PAPER

CONSTANT ANGLE RULED SURFACES DUE TO THE BISHOP FRAME IN MINKOWSKI 3-SPACE

GÜL UĞUR KAYMANLI[1], CUMALI EKICI[2], YASIN ÜNLÜTÜRK[3]

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Abstract. *In this present study, constant angle ruled surfaces are examined by using Bishop frame in three-dimensional Minkowski space. These surfaces are studied and classified based on the constant angle property of the surface in this research. Particularly, our analysis shows that although they are flat and so Weingarten surfaces, but they are not minimal ones.*

Keywords: *Bishop frame; constant angle surface; Minkowski space; ruled surface.*

1. INTRODUCTION

Constant angle surface is a surface whose normal vector makes a constant angle with a fixed direction. At the beginning, the application of constant angle surfaces in physics was studied namely for liquid crystals in [1] after giving a concept of constant angle surfaces for the product space $\boldsymbol{S}^2 \times \boldsymbol{R}$ in [2]. In addition, other ambient spaces, such as, namely for $\boldsymbol{S}^2 \times \boldsymbol{R}$, $\boldsymbol{H}^2 \times \boldsymbol{R}$ and $\boldsymbol{E}^3$, were also examined with the same property in [3]. That the normal constant angle surfaces are pieces of planes and the binormal constant angle surfaces are pieces of cylinders were shown by Nistor in [4]. After physical exploration, there have been studies of these constant angle surfaces in ambient spaces in details [5-15]. Inspiring by the studies above, in this paper, we present an examination of constant angle ruled surfaces by using Bishop frame in Minkowski 3-space. That is, we study ruled surface whose normal vector is parallel to tangent, normal, and binormal vectors of Bishop frame for the curve being both spacelike and timelike on the surface. We then show these surfaces are flat but not minimal.

2. MATERIALS AND METHODS

The three dimensional Minkowski space $\mathbb{E}_1^3$ is a real vector space $\mathbb{R}^3$ equipped with the metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is canonical coordinates in $\mathbb{R}^3$.

[1] Çankırı Karatekin University, Department of Mathematics, 18100 Çankırı, Turkey.
E-mail: gulugurk@karatekin.edu.tr.
[2] Eskişehir Osmangazi University, Department of Mathematics and Computer Sciences, 26040 Eskişehir, Turkey. E-mail: cekici@ogu.edu.tr.
[3] Kırklareli University, Department of Mathematics, 39100 Kırklareli, Turkey.
E-mail: yasinunluturk@klu.edu.tr.

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The norm of the vector ω is given by $\|\omega\| = \sqrt{|\langle \omega, \omega \rangle|}$. We say that a Lorentzian vector ω is spacelike, lightlike or timelike if $\langle \omega, \omega \rangle > 0$ or $\omega = 0$, $\langle \omega, \omega \rangle = 0$ and $\omega \neq 0$, $\langle \omega, \omega \rangle < 0$, respectively [8, 16].

A ruled surface is a surface generated by a moving a line along a curve in space [17]. Therefore, it has a parametrization of the form

$$\varphi(\upsilon, \nu) = \omega(\upsilon) + \nu\delta(\upsilon),$$

where ω is called the directrix and δ is the director curve.

Let φ be a smooth surface with a diagonalizable shape operator in $\mathbb{E}_1^3$, the first fundamental form of the surface φ is given as $I = Ed\upsilon^2 + 2Fd\upsilon d\nu + Gd\nu^2$, where

$$E = <\varphi_\upsilon, \varphi_\upsilon>, F = <\varphi_\upsilon, \varphi_\nu>, G = <\varphi_\nu, \varphi_\nu>.$$

The second fundamental form of φ is defined as $II = ed\upsilon^2 + 2fd\upsilon d\nu + gd\nu^2$, where

$$e = <\varphi_{\upsilon\upsilon}, U>,\ f = <\varphi_{\upsilon\nu}, U>,\ g = <\varphi_{\nu\nu}, U>$$

and U is the unit normal vector field of φ. We have $<U, U> = \pm\xi = \pm 1$ depending on the surface $\varphi(\upsilon, \nu)$ being timelike and spacelike. The Gaussian and mean curvatures are written as

$$K = \frac{eg - f^2}{\xi(EG - F^2)}, \text{ and } H = \frac{eG - 2fF + gE}{2\xi\left(EG - F^2\right)}$$

respectively [8, 16]. A necessary and sufficient condition for a curve to be a flat and minimal is its Gaussian and mean curvatures vanish identically, respectively [3]. A surface satisfying the relation $\frac{\partial K}{\partial \upsilon}\frac{\partial H}{\partial \nu} - \frac{\partial K}{\partial \nu}\frac{\partial H}{\partial \upsilon} = 0$ is said to be Weingarten surface [18].

CASE 1: Let $\omega(\upsilon)$ be a timelike space curve. The derivative equations of the Bishop frame for the timelike space curve when tangent vector (timelike), normal vector (spacelike) and binormal vector (spacelike) are written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where $T \wedge N_1 = -N_2, T \wedge N_2 = N_1, N_1 \wedge N_2 = T, <N_1, N_1> = <N_2, N_2> = 1$ and $<T, T> = -1$.

CASE 2: Let $\omega(\upsilon)$ be a spacelike space curve. The derivative equations of the Bishop frame for the spacelike space curve when tangent vector (spacelike) are written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ \epsilon k_1 & 0 & 0 \\ \epsilon k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where

$$T \wedge N_1 = -\epsilon N_2, T \wedge N_2 = -\epsilon N_1, N_1 \wedge N_2 = -T, \ <T,T>=1,$$

and

$$<N_1,N_1> = -<N_2,N_2> = -\epsilon.$$

3. RESULTS AND DISCUSSION

3.1 CASE 1

Ruled surface is given by

$$\varphi(\upsilon,\nu) = \omega(\upsilon) + \nu\delta(\upsilon),$$

where δ be any vector filed $\delta(\upsilon) = x_1 T + x_2 N_1 + x_3 N_2$ where x_1, x_2, x_3 are smooth functions. Partial derivatives of the surface $\varphi(\upsilon,\nu)$ are given

$$\varphi_\upsilon = (1+\nu(x_1' + k_1 x_2 + k_2 x_3))T + \nu(x_2' + k_1 x_1)N_1 + \nu(x_3' + k_2 x_1)N_2,$$

and

$$\varphi_\nu = x_1 T + x_2 N_1 + x_3 N_2.$$

The cross product of these partial derivatives is given

$$\varphi_\upsilon \wedge \varphi_\nu = \nu(x_1(k_1 x_3 - k_2 x_2) + x_2' x_3 - x_2 x_3')T + (x_3 + \nu(k_2(x_3^2 - x_1^2) + k_1 x_2 x_3 + x_3 x_1' - x_1 x_3'))N_1$$
$$+ (-x_2 + \nu(k_1(x_1^2 - x_2^2) - k_2 x_2 x_3 + x_1 x_2' - x_2 x_1'))N_2.$$

One can easily say that the normal vector of the surface as

$$U = U_1 T + U_2 N_1 + U_3 N_2,$$

where $U_1 = U_{11} + \nu U_{12}, U_2 = U_{21} + \nu U_{22}$ and $U_3 = U_{31} + \nu U_{32}$ such that

$$\begin{cases} U_{11} = 0 \\ U_{12} = x_1(k_1 x_3 - k_2 x_2) + x_2' x_3 - x_2 x_3' \\ U_{21} = x_3 \\ U_{22} = k_2(x_3^2 - x_1^2) + k_1 x_2 x_3 + x_3 x_1' - x_1 x_3' \\ U_{31} = -x_2 \\ U_{32} = k_1(x_1^2 - x_2^2) - k_2 x_2 x_3 + x_1 x_2' - x_1' x_2. \end{cases}$$

3.1.1. A constant angle ruled surface parallel to tangent vector

In this subsection, we take the normal vector of the ruled surface $\varphi(\upsilon,\nu),$ which is linearly dependent the tangent vector of the timelike curve $\omega(\upsilon)$ into account. Therefore we have the following conditions:

$$U_1 \neq (0,0), U_2 = U_3 = (0,0).$$

In order to examine these conditions, we need take $x_2 = x_3 = 0.$ However, it is contradiction being $U_1 \neq (0,0).$

3.1.2. A constant angle ruled surface parallel to normal vector

In this subsection, we take the normal vector of the ruled surface $\varphi(\upsilon,\nu),$ which is linearly dependent the normal vector of the timelike curve $\omega(\upsilon).$ Therefore we have the following conditions:

$$U_2 \neq (0,0), U_1 = U_3 = (0,0).$$

Since $U_{31} = 0,$ x_2 must be vanishing. Therefore, we have following equalities

$$\begin{cases} U_{11} = 0 \\ U_{12} = k_1 x_1 x_3 \\ U_{21} = x_3 \\ U_{22} = k_2(x_3^2 - x_1^2) + x_3 x_1' - x_1 x_3' \\ U_{31} = 0 \\ U_{32} = k_1 x_1^2. \end{cases}$$

Using this equations and the condition for surface being parallel to normal vector, we get $x_1 = 0$ since U_{32} and U_{12} must be vanishing. So $x_3 \neq 0.$ In this case, we get $\delta(\upsilon) = x_3 N_2.$ Therefore the equation of constant angle ruled surface is obtained as

$$\varphi(\upsilon,\nu) = \omega(\upsilon) + \nu x_3 N_2. \qquad (1)$$

Using the equation (1), basic calculations show that the coefficients of the first fundamental form are

$$E = \nu^2 x_3'^2 - (1 + \nu k_2 x_3)^2,$$
$$F = \nu x_3 x_3',$$
$$G = x_3^2.$$

The coefficients of the second fundamental form are given

$$e = k_1(1+\nu k_2 x_3),$$
$$f = 0,$$
$$g = 0.$$

Gauss and mean curvatures are calculated by

$$K = 0,$$

and

$$H = -\frac{k_1}{2\xi(1+\nu k_2 x_3)}, \tag{2}$$

respectively.

Corollary 3.1. The constant angle ruled surface (1) is flat.

Corollary 3.2. The constant angle ruled surface (1) is minimal if and only if the Bishop curvature k_1 vanishes.
This result is plainly seen from the mean curvature equation (2)

Corollary 3.3. The constant angle ruled surface (1) is a Weingarten surface.

3.1.3. A constant angle ruled surface parallel to binormal vector

In this subsection, we consider the normal vector of ruled surface $\varphi(\upsilon,\nu)$, is linearly dependent the binormal vector of the timelike curve $\omega(\upsilon)$. Therefore we have the following conditions:

$$U_3 \neq (0,0), U_1 = U_2 = (0,0).$$

In order to examine these conditions, we need to analyse $x_3 = 0$ since $U_2 = (0,0)$.

$$\begin{cases} U_{11} = 0, \\ U_{12} = -k_2 x_1 x_2, \\ U_{21} = 0, \\ U_{22} = -k_2 x_1^2, \\ U_{31} = -x_2, \\ U_{32} = k_1(x_1^2 + x_2^2) + x_1 x_2' - x_1' x_2. \end{cases}$$

Then one can get $x_1 = 0$ and $x_2 \neq 0$.

In this case, we can get $\delta(\upsilon) = x_2 N_1$. Therefore the equation of constant angle ruled surface is obtained as

$$\varphi(\upsilon,\nu) = \omega(\upsilon) + \nu x_2 N_1. \tag{3}$$

Using the equation (3), basic calculations show that the coefficients of the first fundamental form are

$$E = -(1+\nu k_1 x_2)^2 + \nu^2 x_2'^2,$$
$$F = \nu x_2 x_2',$$
$$G = x_2^2.$$

The coefficients of the second fundamental form are given

$$e = -k_2(1+\nu k_1 x_2),$$
$$f = 0,$$
$$g = 0.$$

Gauss and mean curvatures are calculated by

$$K = 0,$$

and

$$H = \frac{k_2}{2\xi(1+\nu k_1 x_2)}, \tag{4}$$

respectively. We can give the results: the Gaussian curvature vanishes, the ruled surfaces is a flat one as follows:

Corollary 3.4. The constant angle ruled surface (3) is flat.

Also, the mean curvature equation (4) vanishes when the Bishop curvature k_2 becomes zero.

Corollary 3.5. The constant angle ruled surface (3) is minimal if and only if the Bishop curvature k_2 vanishes.

Corollary 3.6. The constant angle ruled surface (3) is a Weingarten surface.

3.2. CASE 2

Ruled surface is given by

$$\varphi(\upsilon,\nu) = \omega(\upsilon) + \nu\delta(\upsilon),$$

where δ be any vector filed $\delta(\upsilon) = x_1 T + x_2 N_1 + x_3 N_2$, where x_1, x_2, x_3 are smooth functions. Partial derivatives of the surface $\varphi(\upsilon,\nu)$ are given

$$\varphi_\upsilon = (1+\nu(x_1' + \epsilon(k_1 x_2 + k_2 x_3)))T + \nu(x_2' + k_1 x_1)N_1 + \nu(x_3' - k_2 x_1)N_2,$$

and

$$\varphi_\nu = x_1 T + x_2 N_1 + x_3 N_2.$$

The cross product of these partial derivatives is given

$$\varphi_\upsilon \wedge \varphi_\nu = \nu(-x_1(k_1x_3+k_2x_2)+x_2x_3'-x_2'x_3)T+(-\epsilon x_3+\epsilon\nu(x_1x_3'-x_3x_1'-k_2x_1^2)-\nu(k_1x_2x_3+k_2x_3^2))N_1$$
$$+(-\epsilon x_2+\epsilon\nu(k_1x_1^2+x_1x_2'-x_2x_1')-\nu(k_1x_2^2+k_2x_2x_3))N_2.$$

One can easily say that the normal vector of the surface as

$$U = U_1T+U_2N_1+U_3N_2,$$

where $U_1 = U_{11}+\nu U_{12}, U_2 = U_{21}+\nu U_{22}$ and $U_3 = U_{31}+\nu U_{32}$ such that

$$\begin{cases} U_{11} = 0, \\ U_{12} = -x_1(k_1x_3+k_2x_2)+x_2x_3'-x_2'x_3, \\ U_{21} = -\epsilon x_3, \\ U_{22} = \epsilon(x_1x_3'-x_3x_1'-k_2x_1^2)-x_3(k_1x_2+k_2x_3), \\ U_{31} = -\epsilon x_2, \\ U_{32} = \epsilon(k_1x_1^2+x_1x_2'-x_2x_1')-x_2(k_1x_2+k_2x_3). \end{cases}$$

3.2.1. A constant angle ruled surface parallel to tangent vector

In this subsection, we have the normal vector of the ruled surface $\varphi(\upsilon,\nu)$, which is linearly dependent the tangent vector of the spacelike curve $\omega(\upsilon)$. Therefore we have the following conditions:

$$U_1 \neq (0,0), U_2 = U_3 = (0,0).$$

In order to examine these conditions, we need take $x_2 = x_3 = 0.$ However, it is contradiction being $U_1 \neq (0,0).$

3.2.2. A constant angle ruled surface parallel to normal vector

In this subsection, we consider the normal vector of the ruled surface $\varphi(\upsilon,\nu)$, which is linearly dependent the normal vector of the spacelike curve $\omega(\upsilon)$. Therefore we have the following conditions:

$$U_2 \neq (0,0), U_1 = U_3 = (0,0).$$

Since $U_{31} = 0,$ x_2 must be vanishing. Therefore, we have following equalities:

$$\begin{cases} U_{11} = 0, \\ U_{12} = -k_1 x_1 x_3, \\ U_{21} = -\epsilon x_3, \\ U_{22} = \epsilon(x_1 x_3{}' - x_3 x_1{}' - k_2 x_1^2) - k_2 x_3^2, \\ U_{31} = 0, \\ U_{32} = \epsilon k_1 x_1^2. \end{cases}$$

Using this equations and the condition for surface being parallel to normal vector, we get $x_1 = 0$ since U_{32} and U_{12} must be vanishing. So $x_3 \neq 0$. In this case, we get $\delta(\upsilon) = x_3 N_2$. Therefore the equation of constant angle ruled surface is obtained as

$$\varphi(\upsilon, \nu) = \omega(\upsilon) + \nu x_3 N_2. \tag{5}$$

Using the equation (5), basic calculations show that the coefficients of the first fundamental form are

$$E = \epsilon \nu^2 x_3{}'^2 + (1 + \epsilon \nu k_2 x_3)^2,$$
$$F = \epsilon \nu x_3 x_3{}',$$
$$G = \epsilon x_3^2.$$

The coefficients of the second fundamental form are given

$$e = -\epsilon k_1 - \nu k_1 k_2 x_3,$$
$$f = 0,$$
$$g = 0.$$

Gauss and mean curvatures are calculated by

$$K = 0,$$

and

$$H = -\frac{k_1}{2\xi\epsilon(1 + \epsilon \nu k_2 x_3)}, \tag{6}$$

respectively.

Obtaining the values of Gaussian and mean curvatures, we present the following results:

Corollary 3.7. The constant angle ruled surface (5) is flat.

Corollary 3.8. The constant angle ruled surface (5) is minimal if and only if the Bishop curvature k_1 vanishes.
This corollary is straigthforwardly understood from the mean curvature equation (6)

Corollary 3.9. The constant angle ruled surface (5) is a Weingarten surface.

3.2.3. A constant angle ruled surface parallel to binormal vector

In this subsection, we take the normal vector of the ruled surface $\varphi(\upsilon,\nu)$, is linearly dependent the binormal vector of the spacelike curve $\omega(\upsilon)$. Therefore we have the following conditions:

$$U_3 \neq (0,0), U_1 = U_2 = (0,0).$$

In order to examine these conditions, we need to analyse $x_3 = 0$ since $U_2 = (0,0)$.

$$\begin{cases} U_{11} = 0, \\ U_{12} = -k_2 x_1 x_2, \\ U_{21} = 0, \\ U_{22} = -\epsilon k_2 x_1^2, \\ U_{31} = -\epsilon x_2, \\ U_{32} = \epsilon(k_1 x_1^2 + x_1 x_2' - x_2 x_1') - k_1 x_2^2. \end{cases}$$

Then one can get $x_1 = 0$ and $x_2 \neq 0$.

In this case, we can get $\delta(\upsilon) = x_2 N_1$. Therefore the equation of constant angle ruled surface is obtained as

$$\varphi(\upsilon,\nu) = \omega(\upsilon) + \nu x_2 N_1. \tag{7}$$

Using the equation (7), basic calculations show that the coefficients of the first fundamental form are

$$E = (1 + \epsilon \nu k_1 x_2)^2 - \epsilon \nu^2 x_2'^2,$$
$$F = -\epsilon \nu x_2 x_2',$$
$$G = -\epsilon x_2^2.$$

The coefficients of the second fundamental form are given

$$e = -\epsilon k_2 (1 + \epsilon \nu k_1 x_2),$$
$$f = 0,$$
$$g = 0.$$

Gauss and mean curvatures are calculated by

$$K = 0,$$

and

$$H = \frac{k_2}{-2\xi\epsilon(1 + \epsilon \nu k_1 x_2)}, \tag{8}$$

respectively.

By means of these curvatures, we obtain the following conclusions:

Corollary 3.10. The constant angle ruled surface (7) is flat.

Corollary 3.11. The constant angle ruled surface (7) is minimal if and only if the Bishop curvature k_2 vanishes.

This result is directly shown from the equation (8).

Corollary 3.12. The constant angle ruled surface (7) is a Weingarten surface.

4. CONCLUSION

In this work, we characterized constant angle ruled surfaces via the Bishop frame in Minkowski 3-space. Based on the constant angle property of the surfaces, we studied the ruled surfaces according to the mentioned frame. Finally, we presented some results about constant angle ruled surface.

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