ORIGINAL PAPER

SOME PROPERTIES OF (*n*, *m*)-POWER-HYPONORMAL IN SEMI HILBERTIAN SPACE

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Abstract. *In [1] (Kikete Wabuya, Luketero Wanyonyi, and Justus Mile) show that if an operator (n,m) hyponormal is isometrically equivalent to an operator S, then S is also (n ,m) hyponormal operator. In this paper, we prove results in the same spirit but in a semi Hilbertian space, i.e., spaces generated by positive semi-definite sesquilinear forms. This kind of spaces appears in many problems concerning linear and bounded operators on Hilbert spaces and is intensively studied in the present; some of the basic properties of this class are studied. Moreover, the product, tensor product and the sum of finite numbers of this type are discussed.*

Keywords: *Semi-Hilbertian space; (n, m)-power-hyponormal operators; n-power-hyponormal operators; n-normal operators.*

1. INTRODUCTION

First, we assume that H is a semi Hilbertian space. Let us, however, recall some notations that will be met below. We said that T is unitary if $T\ T^{\#}= T^{\#}T = I$, isometry if $T^{\#}T = I$ co-isometry if $T\ T^{\#} = I$, normal if $T\,T^{\#}= T^{\#}T$, n-normal if $T^{\mathrm{n}}\ T^{\#}= T^{\#}T^{\mathrm{n}}$ hyponormal if $T\,T^{\#} \leq T^{\#}T$, n-hyponormal if $T^{\mathrm{n}}\,T^{\#} \leq T^{\#}T^{\mathrm{n}}$. An operator T is said to be: (n, m) power hyponormal if

$$T^{\mathrm{n}}\,(T^{m})^{\#} \leq (T^{m})^{\#}\,T^{\mathrm{n}}$$

for some positive integers n and m. This class of operators will be doned by $[(n,m)HN]$. Clearly, if $n = m = 1$, then (n,m)-power hyponormal becomes hyponormal, and if $m = 1$, then (n, 1)-power hyponormal becomes n-hyponormal.

2. BASIC DEFINITIONS

Definition 2.1. [2] Two operators S, T are said to be:
- Unitarily equivalent if there exists a unitary operator U such that $T = U^{\#}SU$,
- D-unitarily equivalent if there exists a unitary operator U such that $T^{D} = U^{\#}S^{D}U$.

Theorem 2.2. [3] If T is (n,m) power hyponormal operator then:

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(1) T is (m, n) power hyponormal operator.
(2) T^k is (n, m) power hyponormal operator for $k \in N$.
(3) αT is (n, m) power hyponormal operator for $\alpha \in R$.
(4) T^{mn} is n, m-hyponormal operator.

Proof 2.3:

- We have $T^n (T^m)^\# \leq (T^m)^\# T^n$,
  In the same time we have: $T^m (T^n)^\# = (T^n (T^m)^\#)^\#$
  So $T^m (T^n)^\# \leq ((T^m)^\# T^n)^\# = (T^n)^\# T^m$
  Hence T is (m, n) power hyponormal

$$(T^k)^n((T^k)^m)^\# = \underbrace{(T^nT^nT^n\ldots\ldots)}_{k\ times}\underbrace{((T^mT^mT^m\ldots))^\#}_{k\ times} \quad (1)$$

$$= \underbrace{(T^nT^nT^n\ldots\ldots)}_{(k-1)\ times}\ T^n (T^m)^\#\ \underbrace{((T^mT^mT^m\ldots))^\#}_{(k-1)\ times} \quad (2)$$

$$\leq \underbrace{(\boldsymbol{T^nT^nT^n}\ldots\ldots)}_{(k-1)\ times}\ (\boldsymbol{T^m})^\#\ \boldsymbol{T^n}\ \underbrace{((\boldsymbol{T^mT^mT^m}\ldots))^\#}_{(k-1)\ times} \quad (3)$$

$$= ((T^m)^\# T^n)^k = ((T^k)^m)^\#(T^k)^n \quad (4)$$

Hence T^k is (n, m) power hyponormal operator for $k \in N$.

- We have

$$(\alpha T)^n((\alpha T)^m)^\# = \alpha^n T^n (\alpha^m (T^m))^\# \quad (1)$$

$$= \alpha^n T^n (\alpha^m)^\# (T^m)^\# = \alpha^n (\alpha^m)^\# (T^m)^\# T^n \quad (2)$$

$$= ((\alpha T)^m)^\# (\alpha T)^n \quad (3)$$

Hence αT is (m, n) power hyponormal

$$(T^n)^m((T^m)^n)^\# = \underbrace{(T^nT^nT^n\ldots\ldots)}_{m\ times}\ \underbrace{((T^mT^mT^m\ldots))^\#}_{n\ times}$$

Using the same method of the first demonstration we find

$$T^{nm}(T^{nm})^\# = (T^n)^m((T^m)^n)^\# \leq ((T^m)^n)^\#(T^n)^m$$

Hence T^{nm} is nm-hyponormal.

3. MAIN RESULTS

Theorem 3.1. [3] Let S *and* T two operators in a semiHilbertian space such that $S = U\,T\,U^\#$ where U is an isometry.
So, if T is (n, m) hyponormal then S is also (n, m)hyponormal.

Proof 3.2: Let S and T two isometrically equivalent operators so there exists anisometry U such that$S = U\,T\,U^\#$then$S^\# = U\,T\,U^\#$.

Now we need to find S^n and $S^{\#^m}$

$$S^n = (U\,T\,U^\#)^n \Rightarrow S^n = (UTU^\#)\,(UTU^\#)\,(UTU^\#)\,\dots\dots..(UTU^\#) \quad (1)$$

$$S^n = UT\,(U^\# U)\,T\,(\,U^\# U)\,T(\,U^\# U)\ \dots\dots..(\,U^\# U)\,T\,U^\# \Rightarrow S^n = U\,T^n U^\# \quad (2)$$

Similarly, $S^m = UT^m U^\#$.
So $(S^m)^\# = (UT^m U^\#)^\# \Rightarrow (S^\#)^m = U(T^m)^\# U^\#$.
Therefore,

$$S^n(S^m)^\# = U\,T^n U^\# U\,(T^m)^\# U^\# = U\,T^n (T^m)^\# U^\# \leq U(T^m)^\# T^n U^\# (1)$$

$$U(T^m)^\# T^n U^\# = U(T^m)^\# U^\# U\,T^n U^\# = (S^m)^\# S^n (2)$$

Hence, S is $(n, m) -$ power hyponormal.

Corollary 3.3. Let $T\ and\ S$ two operators such that: $T\ is\ (n, m)$ −power hyponormal operator. If $S = U^\#\,T\,U$ with U being a co-isometry, then S is also (n, m) −power hyponormal.

Proof 3.4: We have

$$S = U^\#\,T\,U$$

so

$$S^\# = (U^\#\,T\,U)^\# \Rightarrow S^\# = U^\#\,T\,U$$

and

$$S^n = (U^\# TU)^n \Rightarrow S^n = (U^\# TU)(U^\# TU)(U^\# TU)\,\dots\dots..(U^\# TU) \quad (1)$$

$$\Rightarrow S^n = U^\# T(UU^\#)T(UU^\#)T(UU^\#)\ \dots\dots..(UU^\#)\,T\,U \quad (2)$$

$$\Rightarrow S^n = U^\# T^n\,U \quad (3)$$

Similarly, we find $S^m = U^\# T^m U \Rightarrow (S^m)^\# = (U^\# T^m U)^\#$
Therefore,

$$S^n(S^m)^\# = U^\# T^n\,U\,U^\# (T^m)^\# U$$

$$= U^\# T^n (T^m)^\# U \leq U^\# (T^m)^\# T^n$$

$$= (S^m)^\# S^n$$

Proposition 3.5. Let Tis (n, m) power-hyponormal, if S is unitary equivalent of T, then S is (n, m) power hyponormal operator.

Proof 3.6: Let T be an (n, m) power hyponormal operator , since S is unitary equivalent of T then there exists a unitary operator U such that $S = UTU^\#$, it is easly to chek that

$$S^n = UT^nU^\# \text{ and } S^\# = UT^\#U^\#$$

We have

$$S^n(S^\#)^m = (UT^nU^\#)(U(T^\#)^mU^\#) = UT^nU^\#U(T^\#)^mU^\# = UT^n(T^\#)^mU^\# \quad (1)$$

$$UT^n(T^\#)^mU^\# \leq UT^{\#m}T^nU^\# = (UT^{\#m}U^\#)(UT^nU^\#) = (S^\#)^mS^n \quad (2)$$

Hence, $S^n(S^\#)^m \leq (S^\#)^mS^n$ then S is (n,m) power hyponormal operator.

The following discusses the conditions for product and sum of two (n,m) power hyponormal operators to be (n,m) power hyponormal.

Proposition 3.7. If T, S are commuting (n,m) power hyponormal operators such that $ST^\# = T^\#S$ and $TS^\# = S^\#T$ then TS is (n,m) power hyponormal operator.

Proof 3.8: Since $ST = TS$ and $T^\# = T^\#S$, then $(T^\#)^mS^n = S^n(T^\#)^m$

$$(ST)^n((ST)^\#)^m = (S^nT^n)(S^mT^m)^\# = S^nT^n(T^\#)^m(S^\#)^m \leq S^n(T^\#)^mT^n(S^\#)^m \quad (1)$$

$$S^n(T^\#)^mT^n(S^\#)^m = (T^\#)^m S^nT^n(S^\#)^m$$

$$= (T^\#)^m S^n(S^\#)^mT^n \leq (T^\#)^m(S^\#)^mS^nT^n \quad (2)$$

$$= ((ST)^\#)^m(ST)^n \quad (3)$$

Example 3.9. Let $T = \begin{pmatrix} -3 & 2 \\ 0 & 3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ two operators. A simple calculation shows that

$$T^* = \begin{pmatrix} -3 & 0 \\ 2 & 3 \end{pmatrix}, T^\# = \begin{pmatrix} -3 & 0 \\ 2/3 & 3 \end{pmatrix}$$

A direct calculation show that S is of class [(2, 3) H] but T is not [H].

Example 3.10. Let

$$T = \begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

A simple calculation shows that

$$T^* = \begin{pmatrix} -2 & -1 \\ 0 & 2 \end{pmatrix}, \qquad T^\# = \begin{pmatrix} 1/2 & -5/2 \\ -3/2 & -1/2 \end{pmatrix}$$

Therefore T is a (3, 2)-Hyponormal and (2, 3)-Hyponormal.

Proposition 3.11. Let T, S are (n,m) power hyponormal operators for some positive integers n and m such that $TS^{\#} = S^{\#}T$, $ST^{\#} = T^{\#}S$ and $TS = ST = 0$. Then $(S+T)$ are (n,m) power hyponormal operators for some positive integers n and m.

Proof 3.12: Under assumption we have

$$(S+T)^{\#m}(S+T)^n = (S^{\#m}+T^{\#m})(S^n+T^n) \quad (1)$$

$$= S^{\#m}S^n + S^{\#m}T^n + T^{\#m}S^n + T^{\#m}T^n \quad (2)$$

$$= S^{\#m}S^n + T^nS^{\#m} + S^nT^{\#m} + T^{\#m}T^n \quad (3)$$

$$\geq S^nS^{\#m} + T^nS^{\#m} + S^nT^{\#m} + T^nT^{\#m} \quad (4)$$

$$= (S^n+T^n)(S^{\#m}+T^{\#m}) = (S+T)^n(S+T)^{\#m} \quad (5)$$

Exemple 3.13. Let

$$T = \begin{pmatrix} -3 & 2 \\ 0 & 3 \end{pmatrix}, S = \begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

A direct calculation show that S is $[(2,3)H]$, T is $[(2,3)\ H]$ and $T+S$ is $[(2,3)\ H]$ but $TS \neq ST$ and $TS^{\#} \neq S^{\#}T$.

4. DIRECT SUM AND TENSOR PRODUCT

In the following theorem we will prove the stability of the class of (n,m) power-hyponormal operators under the direct sum and tensor product.

Theorem 4.1. Let $T_1, T_2, ..., T_k$ be (n,m) power hyponormal operators, then

- $(T_1 \oplus T_2 \oplus .... \oplus T_k)$ is (n,m) power -hyponormal operator
- $(T_1 \otimes T_2 \otimes .... \otimes T_k)$ is (n,m) power hyponormal operator.

Proof 4.2:

- ***The direct sum***

$$(T_1 \oplus T_2 \oplus ... \oplus T_k)^n (T_1 \oplus T_2 \oplus ... \oplus T_k)^{\#m} =$$

$$= (T_1^n \oplus T_2^n \ldots \oplus T_k^n)(T_1^{\#m} \oplus T_2^{\#m} \ldots \oplus T_k^{\#m}) \quad (1)$$

$$= T_1^n T_1^{\#m} \oplus T_2^n T_2^{\#m} \oplus \ldots \oplus T_k^n T_k^{\#m} \quad (2)$$

$$\leq T_1^{\#m} T_1^n \oplus T_2^{\#m} T_2^n \oplus \ldots \oplus T_k^{\#m} T_k^n \quad (3)$$

$$= (T_1^{\#m} \oplus T_2^{\#m} \oplus \ldots \oplus T_k^{\#m})(T_1^n \oplus T_2^n \oplus \ldots \oplus T_k^n) \quad (4)$$

$$= (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{\#m} (T_1 \oplus T_2 \oplus \ldots \oplus T_k)^n \quad (5)$$

Then $(T_1 \oplus T_2 \oplus \ldots \oplus T_m)$ is (n,m) power -hyponormal operator.

- ***The tensorproduct***

$$(T_1 \otimes T_2 \otimes \ldots \otimes T_k)^n (T_1 \otimes T_2 \otimes \ldots \otimes T_k)^{\#m} (x_1 \otimes x_2 \otimes \ldots \otimes x_k) \quad (1)$$

$$= (T_1^n \otimes T_2^n \otimes \ldots \otimes T_k^n)(T_1^{\#m} \otimes T_2^{\#m} \otimes \ldots \otimes T_k^{\#m})(x_1 \otimes x_2 \otimes \ldots \otimes x_k) \quad (2)$$

$$= (T_1^n T_1^{\#m} x_1 \otimes T_2^n T_2^{\#m} x_2 \otimes \ldots \otimes T_k^n T_k^{\#m} x_k) \quad (3)$$

$$\leq (T_1^{\#m} T_1^n x_1 \otimes T_2^{\#m} T_2^n x_2 \otimes \ldots \otimes T_k^{\#m} T_k^n x_k) \quad (4)$$

$$= (T_1^{\#m} \otimes T_2^{\#m} \otimes \ldots \otimes T_k^{\#m})((T_1^n \otimes T_2^n \otimes \ldots \otimes T_k^n)(x_1 \otimes x_2 \otimes \ldots \otimes x_k) \quad (5)$$

$$=(T_1 \otimes T_2 \otimes \ldots \otimes T_k)^{\#m} (T_1 \otimes T_2 \otimes \ldots \otimes T_k)^n (x_1 \otimes x_2 \otimes \ldots \otimes x_k) \quad (6)$$

5. CONCLUSIONS

It may be concluded that, if an operator (n,m) power-hyponormal is isometrically equivalent to an operator S then S is (n,m) power hyponormal operator in a semi hilbertian space .We gave also the conditions for product and sum of two (n,m) power hyponormal operators to be (n,m) power hyponormal,so we proved the stability of the class of (n,m) power-hyponormal operators under the direct sum and tensor product.

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