ORIGINAL PAPER

APPLICATIONS OF QUATERNIONS IN GALILEAN SPACES

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Abstract. *In this paper, Galilean orthogonal matrices in G^5 and G_1^5 are obtained with the help of unit quaternions. Moreover, Galilean orthogonal matrices in G^4 and G_1^4 are acquired. These matrices produce Galilelan motions in Galilean spaces. We investigate the invariance of the plane where shear motion is acting in Galilean and pseudo-Galilean spaces. Additionally, related examples of matrices that belong to both spaces are provided. With a similar method, dual Galilean orthogonal matrices are obtained by using unit dual quaternions. Finally, we strengthen our work with examples and draw their figures to explore visual representations.*

Keywords: *quaternion; dual quaternion; dual transformation; Galilean space; pseudo-Galilean space.*

1. INTRODUCTION

Quaternions have a long history in many fields including kinematics, robotics, computer graphics, aerospace, quantum physics and other areas related. In recent years, quaternions have become more popular since rotation operations are simpler with quaternions than with matrices. In 1843, W.R. Hamilton first defined quaternions and generalized complex numbers to the higher dimensions. The set of quaternions is denoted by $\mathbb{H}$ in his honor, can be represented as

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k | x_0, x_1, x_2, x_3 \in \mathbb{R}\}.$$

Rotations in SO(3) can be represented by unit quaternions. In [1], the relationship between Euclidean and Lorentzian rotation matrices is examined with the help of dual transformation between $\mathrm{SO}(n) \backslash \{a_{nn} = 0\}$ and $\mathrm{SO}(n-1,1)$. We carried this work into dual spaces in [2] by investigating invariant axes in both spaces. Then, we gave a new method for obtaining a rotation matrix in Lorentzian space by using a unit quaternion in [3].

Galilean geometry is one of the non-Euclidean geometries, which is becoming more popular recently. In [4], the importance of Galilean geometry in mechanics and particularly in kinematics is mentioned. Kinematics deals with the motions of material points and bodies. In geometry, a motion is a certain type of point transformation which associates to each point A a definite point A'. In Galilean geometry, we investigate motions invariant under the Galilean transformation. In [5], the geometry of motions in Galilean and pseudo-Galilean spaces was studied. In the light of studies in Galilean geometry and dual transformations, we defined the transition from real Galilean space to pseudo-Galilean space in [6].

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The objective of this paper is to construct dual transformations by using quaternions to give the transition from Galilean space to pseudo-Galilean space. We denote the 5-dimensional Galilean space by G^5 and the 5-dimensional pseudo-Galilean space by G_1^5. We acquire Galilean orthogonal matrices in G^5 and G_1^5 with the help of unit quaternions. In other words, we will give the new representation of pseudo-Galilean matrix in G_1^5 obtained by using a unit real quaternion. Furthermore, Galilean orthogonal matrices in G^4 and G_1^4 are obtained. These matrices produce Galilelan motions in Galilean spaces. Shear motions come to mind when Galilean motions are mentioned. We also investigate the invariance of the plane where shear motion is acting in both spaces. Additionally, we provide applications to reinforce this transition and draw their figures to explore visual representations. Finally, with a similar method, dual Galilean orthogonal matrices are obtained by using unit dual quaternions. The difference of this work from [6] is that we construct the Galilean transformations using quaternions. We think that the idea of transferring quaternions and dual quaternions to Galilean and Pseudo-Galilean spaces with the help of a dual transformation may be a positive contribution to the literature.

2. BASIC CONCEPTS

In this section, fundamental concepts of real and dual quaternions will be given. A real quaternion q is an expression of the form

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

where x_0, x_1, x_2 and x_3 are real numbers, and i, j, k are real quaternionic units which satisfy the non-commutative multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

A real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is pieced into two parts with scalar piece $S_q = x_0$ and vectorial piece $\overrightarrow{V_q} = x_1 i + x_2 j + x_3 k$. We also write $q = S_q + \overrightarrow{V_q}$. The quaternionic conjugate of $q = S_q + \overrightarrow{V_q}$ is defined as $\bar{q} = S_q - \overrightarrow{V_q}$. The norm of a real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is

$$N_q = \sqrt{|q|} = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \geq 0,$$

where $q \in \mathbb{R}$. If $N_q = 1$ then q is called unit real quaternion.

Since quaternion algebra is associative they can be considered in terms of matrices. The map ϕ between the space of quaternions and the space of 4×4 matrices over real numbers defined as

$$\phi: (\mathbb{H}, +, \cdot) \to (\mathbb{M}_{(4,R)}, \oplus, \otimes)$$

$$\phi\{x_0 + x_1 i + x_2 j + x_3 k\} \mapsto \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix} \tag{1}$$

is an isomorphism.

The representation of the 3D rotation matrix corresponds to a unit real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is given by

$$R = \begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & -2x_0x_3 + 2x_1x_2 & 2x_0x_2 + 2x_1x_3 \\ 2x_1x_2 + 2x_3x_0 & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2x_2x_3 - 2x_1x_0 \\ 2x_1x_3 - 2x_2x_0 & 2x_1x_0 + 2x_2x_3 & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{bmatrix}. \quad (2)$$

For more details about concepts and properties of real quaternions, see [7-9]. Let us go through with some fundamental definitions and theorems in dual space.

Definition 2.1. If a and a^* are real numbers and $\epsilon^2 = 0$, the combination $\hat{a} = a + \epsilon a^*$ is called a dual number, where ϵ is the dual unit.

Definition 2.2. The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$. The set $\mathbb{D}^3 = \{\vec{\hat{a}} = (\hat{a}_1, \hat{a}_2, \hat{a}_3) | \hat{a}_i \in \mathbb{D}, 1 \leq i \leq 3\}$ is called a $\mathbb{D}$-module or dual space.

Definition 2.3. The elements of $\mathbb{D}^3$ are called dual vectors. A dual vector $\vec{\hat{a}}$ can be written $\vec{\hat{a}} = \vec{a} + \epsilon \vec{a}^*$ where $\vec{a}$ and $\vec{a}^*$ are real vectors in $\mathbb{R}^3$.

Definition 2.4. The norm of a dual vector $\vec{\hat{a}}$ is defined by $|\vec{\hat{a}}| = |\vec{a}| + \epsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{|\vec{a}|^2}, \vec{a} \neq 0$. See [10].

After recalling the basic definitions and theorems in dual space, we go through with concepts and properties of dual quaternions. The set of dual quaternions is denoted by $\mathbb{H}_D$ can be represented as

$$\mathbb{H}_D = \{\hat{q} = \widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k | \widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{x_3} \in \mathbb{D}\}.$$

The ring of dual quaternions is defined as the four-dimensional vector space over dual numbers $\mathbb{D}$ having a basis $\{1, i, j, k\}$ with the same multiplication property of the basis elements in real quaternions. A dual quaternion $\hat{q} = \widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k$ can be written as $\hat{q} = q + \epsilon q^*$, where q and q^* are real and pure dual quaternion components, respectively. We may consider a dual quaternion as $\hat{q} = x_0 + x_1 i + x_2 j + x_3 k + \epsilon(x_0^* + x_1^* i + x_2^* j + x_3^* k)$.

The map φ between the space of dual quaternions and the space of 4×4 matrices over dual numbers defined as

$$\varphi : (\mathbb{H}_D, +, \cdot) \to (\mathbb{M}_{(4,D)}, \oplus, \otimes)$$

$$\varphi\{\widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k\} \mapsto \begin{bmatrix} \widehat{x_0} & -\widehat{x_1} & -\widehat{x_2} & -\widehat{x_3} \\ \widehat{x_1} & \widehat{x_0} & -\widehat{x_3} & \widehat{x_2} \\ \widehat{x_2} & \widehat{x_3} & \widehat{x_0} & -\widehat{x_1} \\ \widehat{x_3} & -\widehat{x_2} & \widehat{x_1} & \widehat{x_0} \end{bmatrix} \quad (3)$$

is an isomorphism.

For more details about concepts and properties of dual quaternions, see [11-16]. In [16], the representation of dual orthogonal matrix corresponds to a unit dual quaternion $\hat{q} = \widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k$ was expressed as

$$\hat{R} = \begin{bmatrix} \widehat{x_0}^2 + \widehat{x_1}^2 - \widehat{x_2}^2 - \widehat{x_3}^2 & -2\widehat{x_0}\widehat{x_3} + 2\widehat{x_1}\widehat{x_2} & 2\widehat{x_0}\widehat{x_2} + 2\widehat{x_1}\widehat{x_3} \\ 2\widehat{x_1}\widehat{x_2} + 2\widehat{x_3}\widehat{x_0} & \widehat{x_0}^2 - \widehat{x_1}^2 + \widehat{x_2}^2 - \widehat{x_3}^2 & 2\widehat{x_2}\widehat{x_3} - 2\widehat{x_1}\widehat{x_0} \\ 2\widehat{x_1}\widehat{x_3} - 2\widehat{x_2}\widehat{x_0} & 2\widehat{x_1}\widehat{x_0} + 2\widehat{x_2}\widehat{x_3} & \widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2 \end{bmatrix}. \tag{4}$$

Let us introduce some basic definitions and theorems in Galilean space.

Definition 2.6. *Let* $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ *and* G^n *be Galilean space* $(\mathbb{R}^n, \|\|)$ *with*

$$\| X \| = \begin{cases} |x_1| & x_1 \neq 0 \\ \sqrt{{x_2}^2 + {x_3}^2 + \dots + {x_n}^2}, & x_1 = 0. \end{cases}$$

Theorem 2.7. *Let* A *be an* $n \times n$ *matrix,*

$$A = \begin{bmatrix} A_1 & C_1 \\ 0 & 1 \end{bmatrix},$$

where $A_1 \in \mathrm{SO}(n-1)$ and $C_1 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1n} \end{bmatrix} \in \mathbb{R}_1^{n-1}$.

f is a Galilean transformation, where

$$\begin{gathered} f: G^n \to G^n \\ X \mapsto f(X) = AX + C, \end{gathered} \tag{5}$$

where $C \in \mathbb{R}_1^n$.

In G^3, *shear movement* determined with the help of f with

$$\begin{gathered} f: G^3 \to G^3 \\ X \mapsto f(X) = AX. \end{gathered} \tag{6}$$

f is also called *motion in the meaning of Galilean*.

Definition 2.8. *Let* $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ *and* G_1^n *be pseudo-Galilean space* $(\mathbb{R}^n, \|\|)$ *with*

$$\| X \|_{PG} = \begin{cases} |x_1|, & x_1 \neq 0 \\ \sqrt{|{x_2}^2 + {x_3}^2 + \dots + {x_{n-1}}^2 - {x_n}^2|}, & x_1 = 0. \end{cases}$$

In G_1^3 pseudo-Galilean space,

$$\begin{gathered} f: G_1^3 \to G_1^3 \\ X \mapsto f(X) = AX \end{gathered} \tag{7}$$

is called shear motion.

Theorem 2.9. *Let A be an $n \times n$ matrix,*

$$A = \begin{bmatrix} A_1 & C_1 \\ 0 & 1 \end{bmatrix},$$

where $A_1 \in \mathrm{SO}(n-2,1)$, ${A_1}^{-1} = G{A_1}^T G$, $G^{-1} = G^T$ and

$$C_1 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1n} \end{bmatrix} \in \mathbb{R}_1^{n-1}.$$

f is a pseudo-Galilean transformation, where

$$\begin{aligned} f: G_1^n &\to G_1^n \\ X &\mapsto f(X) = AX + C, \end{aligned} \tag{8}$$

where $C \in \mathbb{R}_1^n$ [5] .

Recently, studies in Lorentzian space take an important place in the field of mathematical research that can be seen as part of differential geometry (see [17-20]) as well as kinematics, robotics, computer graphics, and in most areas of physics. Let us recall some fundamental definitions and properties in Lorentzian space that we use in this paper.

Definition 2.10. *Lorentzian metric $\langle , \rangle$ defined by*

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_{n-1} v_{n-1} - u_n v_n \tag{9}$$

in E_1^n will be used in this study. It is pointed out that $\langle , \rangle$ is a non-degenerate metric of index 1. It can also be written in the form:

$$\langle u, v \rangle = u^T \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{bmatrix} v = u^T G v. \tag{10}$$

Definition 2.11. *A vector $v \in E_1^n$ is called* spacelike if $\langle v, v \rangle > 0$ or $v = 0$; timelike if $\langle v, v \rangle < 0$; lightlike if $\langle v, v \rangle = 0$ and $v \neq 0$. See [20].

Definition 2.12. *An $n \times n$ matrix S is called* semi symmetric if $S^T = GSG$ or $S = GS^T G$; semi skew-symmetric if $S^T = -GSG$ or $S = -GS^T G$; semi-orthogonal if $S^T = GS^{-1} G$ or $S^{-1} = GS^T G$, where G is the sign matrix of Lorentzian space. See [21].

The dual transformation between $SO(n) \backslash \{a_{nn} = 0\}$ and $SO(n-1,1)$ will be used for obtaining semi-orthogonal matrices from orthogonal matrices.

Definition 2.13. *Dual transformation between* $\mathrm{SO}(n) \backslash \{a_{nn} = 0\}$ *and* $\mathrm{SO}(n-1,1)$ *is defined in [1]. Accordingly, two sets can be given by*

$$\mathrm{SO}(n) = \{A \in GL(n, \mathbb{R}) | A^T A = A A^T = I_n, \det A = 1\},$$

$$\mathrm{SO}(n-1,1) = \{A \in GL(n, \mathbb{R}) | A^T G A = A G A^T = G, \det A = 1\},$$

where $G = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ and I_n is $n \times n$ identity matrix. Let A be an $n \times n$ orthogonal matrix, thence it can be written in the block form as

$$A = \begin{bmatrix} B & C \\ D & a_{nn} \end{bmatrix},$$

where $a_{nn} \neq 0$. Here, B is an $(n-1) \times (n-1)$ square matrix, C is an $(n-1) \times 1$ column matrix and D is a row matrix. Since $a_{nn} \neq 0$, then we can use the following two sets given by

$$X = \{A \in \mathrm{SO}(n) | a_{nn} \neq 0\},$$

$$Y = \{A \in \mathrm{SO}(n-1,1) | a_{nn} \neq 0\}.$$

Thus, the dual transformation can be defined as

$$f: X \to Y$$

$$f: A \mapsto f(A) = \frac{1}{a_{nn}} \begin{bmatrix} a_{nn}(B^{-1})^T & C \\ -D & 1 \end{bmatrix}, \tag{11}$$

here T denotes transposition.

We now give the definition of dual transformation between $S\widehat{\mathrm{O}}(n) \backslash \{\hat{a}_{nn} = 0\}$ and $S\widehat{\mathrm{O}}(n-1,1)$. We will use it for obtaining dual semi-orthogonal matrices from dual orthogonal matrices.

Definition 2.14. *In order to define f dual transformation, we need the following sets given by*

$$\mathrm{S}\widehat{\mathrm{O}}(n) = \{\hat{A} \in GL(n, \mathbb{D}) | \hat{A}^T \hat{A} = \hat{A}\hat{A}^T = I_n, \det\hat{A} = 1\},$$

$$\mathrm{S}\widehat{\mathrm{O}}(n-1,1) = \{\hat{A} \in GL(n, \mathbb{D}) | \hat{A}^T G \hat{A} = \hat{A} G \hat{A}^T = G, \det\hat{A} = 1\},$$

where $G = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ and I_n is $n \times n$ identity matrix. We write an orthogonal $n \times n$ dual matrix $\hat{A}$ in the block form as

$$\hat{A} = \begin{bmatrix} \hat{B} & \hat{C} \\ \widehat{D} & \hat{a}_{nn} \end{bmatrix},$$

where $\hat{a}_{nn} \neq 0$. Since $\hat{a}_{nn} \neq 0$, then two sets can be written as

$$\hat{X} = \{\hat{A} \in \mathrm{S}\widehat{\mathrm{O}}(n) | \hat{a}_{nn} \neq 0\},$$

$$\hat{Y} = \{\hat{A} \in \mathrm{S}\widehat{\mathrm{O}}(n-1,1) | \hat{a}_{nn} \neq 0\}.$$

Now, f dual transformation can be defined as below

$$f: \hat{X} \to \hat{Y}$$

$$f:\hat{A} \mapsto f(\hat{A}) = \frac{1}{\hat{a}_{nn}}\begin{bmatrix} \hat{a}_{nn}(\hat{B}^{-1})^T & \hat{C} \\ -\hat{D} & 1 \end{bmatrix}. \tag{12}$$

See [2].

By using f dual transformation which is given in [1], we defined the dual transformation between Galilean and pseudo-Galilean spaces in [6].

Theorem 2.15. *Let $\tilde{A} \in G^n$ given by*

$$\tilde{A} = \begin{bmatrix} & & & a_{1n} \\ & A & & \vdots \\ & & & a_{n-1n} \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $A \in SO(n-1)$.

g defines a dual transformation,

$$g: G^n \to G_1^n$$

$$\tilde{A} \mapsto g(\tilde{A}) = \tilde{B} = \begin{bmatrix} & & & a_{1n} \\ & f(A) & & \vdots \\ & & & a_{n-1n} \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{13}$$

where f is the dual transformation given in Eq. (11), thus $f(A) \in \mathrm{SO}(n-2,1)$.

In the following, we give the dual transformation between dual Galilean space $\widehat{G^n}$ and dual pseudo-Galilean space $\widehat{G_1^n}$ (See [6]).

Theorem 2.16. *Let $\hat{\tilde{A}} \in \widehat{G^n}$ given by*

$$\hat{\tilde{A}} = \begin{bmatrix} & & & \hat{a}_{1n} \\ & \hat{A} & & \vdots \\ & & & \hat{a}_{n-1n} \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

where $\hat{A} \in S\hat{O}(n-1)$.

g defines a dual transformation,

$$g: \widehat{G^n} \to \widehat{G_1^n}$$

$$\hat{\tilde{A}} \mapsto g(\hat{\tilde{A}}) = \hat{\tilde{B}} = \begin{bmatrix} & & & \hat{a}_{1n} \\ & f(\hat{A}) & & \vdots \\ & & & \hat{a}_{n-1n} \\ 0 & \dots & 0 & 1 \end{bmatrix}, \tag{14}$$

where f is the dual transformation given in Eq. (12), thus $f(\hat{A}) \in \mathrm{S\widehat{O}}(n-2,1)$.

3. GALILEAN ORTHOGONAL MATRICES IN G^5 AND G_1^5

In this section, we acquire Galilean orthogonal matrices in G^5 and G_1^5 with the help of a unit quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$. In other words, we will give the new representation of pseudo-Galilean matrix in G_1^5 obtained by using a unit real quaternion.

Theorem 3.1. *Let $\tilde{Q} \in G^5$ given by*

$$\tilde{Q} = \begin{bmatrix} & & & a \\ & Q & & b \\ & & & c \\ & & & d \\ 0 & \dots & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 & a \\ x_1 & x_0 & -x_3 & x_2 & b \\ x_2 & x_3 & x_0 & -x_1 & c \\ x_3 & -x_2 & x_1 & x_0 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $Q \in SO(4)$ corresponds to a unit quaternion $q = (x_0, x_1, x_2, x_3)$ which is given in Eq.(1).

h defines a dual transformation,

$$h: G^5 \to G_1^5$$

$$\tilde{Q} \mapsto h(\tilde{Q}) = \tilde{Q}_G = \begin{bmatrix} & & & a \\ & f(Q) & & b \\ & & & c \\ & & & d \\ 0 & \dots & 0 & 1 \end{bmatrix}, \tag{15}$$

$$= \begin{bmatrix} \frac{x_0^2 + x_3^2}{x_0} & \frac{-x_0x_1 - x_2x_3}{x_0} & \frac{-x_0x_2 + x_1x_3}{x_0} & \frac{-x_3}{x_0} & a \\ \frac{x_0x_1 - x_2x_3}{x_0} & \frac{x_0^2 + x_2^2}{x_0} & \frac{-x_0x_3 - x_1x_2}{x_0} & \frac{x_2}{x_0} & b \\ \frac{x_0x_2 + x_1x_3}{x_0} & \frac{x_0x_3 - x_1x_2}{x_0} & \frac{x_0^2 + x_1^2}{x_0} & \frac{-x_1}{x_0} & c \\ \frac{-x_3}{x_0} & \frac{x_2}{x_0} & \frac{-x_1}{x_0} & \frac{1}{x_0} & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

where f is the dual transformation given in Eq. (11), therefore $f(Q) \in SO(3,1)$.

Proof: h is a dual transformation, since it satisfies

$$h^2(\tilde{Q}) = h(h(\tilde{Q}))$$

$$= h(\tilde{Q}_G), \quad f^2 = id.$$

$$= Q$$

$$h^2 = id.$$

We will give an example by using h dual transformation and a unit real quaternion.

Example 1. *Let* $q = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}})$ *a unit real quaternion can be expressed by the matrix* $\tilde{Q}$ *as follows*

$$\tilde{Q} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -2 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 2 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying the h dual transformation to the matrix $\tilde{Q}$, we obtain the pseudo-Galilean matrix $h(\tilde{Q}) = \tilde{Q}_L$ as below

$$\tilde{Q}_L = \begin{bmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -1 & -2 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -1 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 1 & 2 \\ -1 & 0 & 1 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So far, Galilean orthogonal matrices have been obtained using quaternions. These matrices produce Galilelan motions in Galilean spaces. When Galilean motions are refered, shear motions come to mind first. We now investigate the invariance of the plane where the shear motion is acting in Galilean and pseudo-Galilean spaces.

Theorem 3.2. *Let h is a dual transformation given by*

$$h: G^5 \to G_1^5$$

$$\tilde{Q} \mapsto h(\tilde{Q}) = \tilde{Q}_G = \begin{bmatrix} & & & a \\ & & & b \\ & f(Q) & & c \\ & & & d \\ 0 & \dots & 0 & 1 \end{bmatrix}, \tag{17}$$

where f is the dual transformation given in Eq. (11) and consequently $f(A) \in SO(3,1)$. h leaves invariant the plane where the shear motion is acting.

Proof: Since the shear motion is the same in Galilean and pseudo-Galilelan space (see Eq.(6) and Eq.(7)), then the plane where the shear motion is acting left invariant under h dual transformation.

4. GALILEAN ORTHOGONAL MATRICES IN G^4 AND G_1^4

In this section, we obtain Galilean orthogonal matrices in G^4 and G_1^4. We provide an application by using 3D rotation matrix corresponds to a unit real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$. So, we will give the new representation of pseudo-Galilean matrix in G_1^4 acquired by using a unit real quaternion.

First, we use the matrix R in Eq. (2.2) in order to obtain $\tilde{R} \in G^4$ as follows

$$\tilde{R} = \begin{bmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & -2x_0x_3 + 2x_1x_2 & 2x_0x_2 + 2x_1x_3 & a \\ 2x_1x_2 + 2x_3x_0 & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2x_2x_3 - 2x_1x_0 & b \\ 2x_1x_3 - 2x_2x_0 & 2x_1x_0 + 2x_2x_3 & x_0^2 - x_1^2 - x_2^2 + x_3^2 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we apply g dual transformation to $\tilde{R}$ in order to acquire the matrix $f(\tilde{R})$ given by

$$f(\tilde{R}) = \begin{bmatrix} \frac{x_0^2 - x_1^2 + x_2^2 - x_3^2}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{-2x_1x_2 - 2x_3x_0}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{2x_0x_2 + 2x_1x_3}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & a \\ \frac{2x_0x_3 - 2x_1x_2}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{x_0^2 + x_1^2 - x_2^2 - x_3^2}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{2x_2x_3 - 2x_1x_0}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & b \\ \frac{-2x_1x_3 + 2x_2x_0}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{-2x_1x_0 - 2x_2x_3}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & \frac{1}{x_0^2 - x_1^2 - x_2^2 + x_3^2} & c \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $x_0^2 - x_1^2 - x_2^2 + x_3^2 \neq 0$. It can be seen that $f(\tilde{R}) \in G_1^4$. Therefore, $f(\tilde{R})$ is the new representation of pseudo-Galilean matrix obtained by using a unit real quaternion.

Example 2. *Let us take the quaternion curve $\alpha(s) = (\alpha_0(s), \alpha_1(s), \alpha_2(s), \alpha_3(s)) = (\frac{\sqrt{2}}{\sqrt{3}} sin(s), 0, cos(s), \frac{1}{\sqrt{3}} sin(s))$. We can represent the Galilean matrix $\tilde{R}$ as*

$$\tilde{R} = \begin{bmatrix} \frac{1}{3}\sin(s)^2 - \cos(s)^2 & \frac{-2\sqrt{2}}{3}\sin(s)^2 & \frac{2\sqrt{2}}{\sqrt{3}}\sin(s)\cos(s) & 1 \\ \frac{\sqrt{2}}{3}\sin(s)^2 & \frac{1}{3}\sin(s)^2 + \cos(s)^2 & \frac{2}{\sqrt{3}}\sin(s)\cos(s) & -1 \\ \frac{-2\sqrt{2}}{\sqrt{3}}\sin(s)\cos(s) & \frac{2}{\sqrt{3}}\sin(s)\cos(s) & \sin(s)^2 - \cos(s)^2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\alpha_0(s)^2 - \alpha_1(s)^2 - \alpha_2(s)^2 + \alpha_3(s)^2 \neq 0$. We can also write the pseudo-Galilean matrix $f(\tilde{R})$ as follows

$$f(\tilde{R}) = \begin{bmatrix} \frac{\sin(s)^2+\cos(s)^2}{3(\sin(s)^2-\cos(s)^2))} & \frac{-2\sqrt{2}\sin(s)^2}{3(\sin(s)^2-\cos(s)^2))} & \frac{2\sqrt{2}\sin(s)\cos(s)}{\sqrt{3}(\sin(s)^2-\cos(s)^2))} & 1 \\ \frac{2\sqrt{2}\sin(s)^2}{3(\sin(s)^2-\cos(s)^2))} & \frac{\sin(s)^2-\cos(s)^2}{3(\sin(s)^2-\cos(s)^2))} & \frac{2\sqrt{3}\sin(s)\cos(s)}{3(\sin(s)^2-\cos(s)^2))}) & -1 \\ \frac{2\sqrt{2}\sin(s)\cos(s)}{\sqrt{3}(\sin(s)^2-\cos(s)^2))} & \frac{-2\sin(s)\cos(s)}{\sqrt{3}(\sin(s)^2-\cos(s)^2))} & \frac{1}{\sin(s)^2-\cos(s)^2} & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\alpha_0(s)^2 - \alpha_1(s)^2 - \alpha_2(s)^2 + \alpha_3(s)^2 \neq 0$.

5. DUAL GALILEAN ORTHOGONAL MATRICES IN $\widehat{G^5}$ AND $\widehat{G_1^5}$

In this section, we will define a dual transformation to obtain dual pseudo-Galilean matrix from a matrix corresponds a unit dual quaternion $\hat{q} = \widehat{x_0} + \widehat{x_1}i + \widehat{x_2}j + \widehat{x_3}k$.

Theorem 5.1. *Let $\hat{\hat{Q}} \in \widehat{G^5}$ given by*

$$\hat{\hat{Q}} = \begin{bmatrix} & & & \hat{a} \\ & \hat{Q} & & \hat{b} \\ & & & \hat{c} \\ & & & \hat{d} \\ 0 & \dots & 0 & 1 \end{bmatrix} = \begin{bmatrix} \widehat{x_0} & -\widehat{x_1} & -\widehat{x_2} & -\widehat{x_3} & \hat{a} \\ \widehat{x_1} & \widehat{x_0} & -\widehat{x_3} & \widehat{x_2} & \hat{b} \\ \widehat{x_2} & \widehat{x_3} & \widehat{x_0} & -\widehat{x_1} & \hat{c} \\ \widehat{x_3} & -\widehat{x_2} & \widehat{x_1} & \widehat{x_0} & \hat{d} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\hat{Q} \in S\hat{O}(4)$ corresponds to a unit quaternion $\hat{q} = (\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{x_3})$.

h_D defines a dual transformation,

$$h_D : \widehat{G^5} \to \widehat{G_1^5}$$

$$\hat{\hat{Q}} \mapsto h_D(\hat{\hat{Q}}) = \hat{\hat{Q}}_G = \begin{bmatrix} & & & \hat{a} \\ & f(\hat{Q}) & & \hat{b} \\ & & & \hat{c} \\ & & & \hat{d} \\ 0 & \dots & 0 & 1 \end{bmatrix}, \tag{18}$$

$$= \begin{bmatrix} \frac{\widehat{x_0}^2+\widehat{x_3}^2}{\widehat{x_0}} & \frac{-\widehat{x_0}\widehat{x_1}-\widehat{x_2}\widehat{x_3}}{\widehat{x_0}} & \frac{-\widehat{x_0}\widehat{x_2}+\widehat{x_1}\widehat{x_3}}{\widehat{x_0}} & \frac{-\widehat{x_3}}{\widehat{x_0}} & \hat{a} \\ \frac{\widehat{x_0}\widehat{x_1}-\widehat{x_2}\widehat{x_3}}{\widehat{x_0}} & \frac{\widehat{x_0}^2+\widehat{x_2}^2}{\widehat{x_0}} & \frac{-\widehat{x_0}\widehat{x_3}-\widehat{x_1}\widehat{x_2}}{\widehat{x_0}} & \frac{\widehat{x_2}}{\widehat{x_0}} & \hat{b} \\ \frac{\widehat{x_0}\widehat{x_2}+\widehat{x_1}\widehat{x_3}}{\widehat{x_0}} & \frac{\widehat{x_0}\widehat{x_3}-\widehat{x_1}\widehat{x_2}}{\widehat{x_0}} & \frac{\widehat{x_0}^2+\widehat{x_1}^2}{\widehat{x_0}} & \frac{-\widehat{x_1}}{\widehat{x_0}} & \hat{c} \\ \frac{-\widehat{x_3}}{\widehat{x_0}} & \frac{\widehat{x_2}}{\widehat{x_0}} & \frac{-\widehat{x_1}}{\widehat{x_0}} & \frac{1}{\widehat{x_0}} & \hat{d} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{19}$$

here f is the dual transformation expressed in Eq. (2.12), thence $f(\hat{Q}) \in S\hat{O}(3,1)$.

Proof: h_D is a dual transformation, since it satisfies

$$h_D^2(\hat{\hat{Q}}) = h_D(h_D(\hat{\hat{Q}}))$$

$$= h_D(\hat{\hat{Q}}_G), \quad f^2 = id.$$

$$= \hat{\hat{Q}}$$

$$h_D^2 = id.$$

6. DUAL GALILEAN ORTHOGONAL MATRICES IN $\widehat{G^4}$ AND $\widehat{G_1^4}$

We can provide an application by using the dual matrix corresponds to a unit dual quaternion $\hat{q} = \widehat{x_0} + \widehat{x_1}i + \widehat{x_2}j + \widehat{x_3}k$ given in Eq. (2.3). We will give the new representation of dual pseudo-Galilean matrix in $\widehat{G_1^4}$ obtained by using a unit dual quaternion.

First, we use the matrix $\hat{R}$ in Eq. (2.4). in order to obtain $\hat{\hat{R}} \in \widehat{G^4}$ as follows

$$\hat{\hat{R}} = \begin{bmatrix} \widehat{x_0}^2 + \widehat{x_1}^2 - \widehat{x_2}^2 - \widehat{x_3}^2 & -2\widehat{x_0}\widehat{x_3} + 2\widehat{x_1}\widehat{x_2} & 2\widehat{x_0}\widehat{x_2} + 2\widehat{x_1}\widehat{x_3} & \hat{a} \\ 2\widehat{x_1}\widehat{x_2} + 2\widehat{x_3}\widehat{x_0} & \widehat{x_0}^2 - \widehat{x_1}^2 + \widehat{x_2}^2 - \widehat{x_3}^2 & 2\widehat{x_2}\widehat{x_3} - 2\widehat{x_1}\widehat{x_0} & \hat{b} \\ 2\widehat{x_1}\widehat{x_3} - 2\widehat{x_2}\widehat{x_0} & 2\widehat{x_1}\widehat{x_0} + 2\widehat{x_2}\widehat{x_3} & \widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2 & \hat{c} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, we apply g dual transformation to $\hat{\hat{R}}$ to get the matrix $f(\hat{\hat{R}})$ given by

$$f(\hat{\hat{R}}) = \begin{bmatrix} \frac{\widehat{x_0}^2 - \widehat{x_1}^2 + \widehat{x_2}^2 - \widehat{x_3}^2}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{-2\widehat{x_1}\widehat{x_2} - 2\widehat{x_3}\widehat{x_0}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{2\widehat{x_0}\widehat{x_2} + 2\widehat{x_1}\widehat{x_3}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \hat{a} \\ \frac{2\widehat{x_0}\widehat{x_3} - 2\widehat{x_1}\widehat{x_2}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{\widehat{x_0}^2 + \widehat{x_1}^2 - \widehat{x_2}^2 - \widehat{x_3}^2}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{2\widehat{x_2}\widehat{x_3} - 2\widehat{x_1}\widehat{x_0}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \hat{b} \\ \frac{-2\widehat{x_1}\widehat{x_3} + 2\widehat{x_2}\widehat{x_0}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{-2\widehat{x_1}\widehat{x_0} - 2\widehat{x_2}\widehat{x_3}}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \frac{1}{\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2} & \hat{c} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\widehat{x_0}^2 - \widehat{x_1}^2 - \widehat{x_2}^2 + \widehat{x_3}^2 \neq 0$.

Example 3. *Let $\hat{\alpha}(s) = (\hat{\alpha}_0(s), \hat{\alpha}_1(s), \hat{\alpha}_2(s) + \hat{\alpha}_3(s)) = (\frac{\sqrt{2}}{\sqrt{3}} sin(\hat{s}), 0, cos(\hat{s}), \frac{1}{\sqrt{3}} sin(\hat{s}))$ be a dual quaternion curve corresponds to the dual orthogonal matrix $\hat{R}$ for $\hat{s} = s + \epsilon$. We can represent the dual Galilean matrix as follows*

$$\hat{\tilde{R}} = \begin{bmatrix} \frac{1}{3}\sin^2(s+\epsilon) - \cos^2(s+\epsilon) & \frac{-2\sqrt{2}}{3}\sin^2(s+\epsilon) & \frac{2\sqrt{2}}{\sqrt{3}}\sin(s+\epsilon)\cos(s+\epsilon) & 1+\epsilon \\ \frac{\sqrt{2}}{3}\sin^2(s+\epsilon) & \frac{1}{3}\sin^2(s+\epsilon) + \cos^2(s+\epsilon) & \frac{2}{\sqrt{3}}\sin(s+\epsilon)\cos(s+\epsilon) & -\epsilon \\ \frac{-2\sqrt{2}}{\sqrt{3}}\sin(s+\epsilon)\cos(s+\epsilon) & \frac{2}{\sqrt{3}}\sin(s+\epsilon)\cos(s+\epsilon) & \sin^2(s+\epsilon) - \cos^2(s+\epsilon) & \sqrt{2}+\epsilon \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\hat{\alpha}_0(s)^2 - \hat{\alpha}_1(s)^2 - \hat{\alpha}_2(s)^2 + \hat{\alpha}_3(s)^2 \neq 0$. Next, we write the dual pseudo-Galilean matrix $f(\hat{\tilde{R}})$ as below

$$f\left(\hat{\tilde{R}}\right) = \begin{bmatrix} \frac{\sin^2(s+\epsilon) + \cos^2(s+\epsilon)}{3(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & \frac{-2\sqrt{2}\sin^2(s+\epsilon)}{3(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & \frac{2\sqrt{2}\sin(s+\epsilon)\cos(s+\epsilon)}{\sqrt{3}(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & 1+\epsilon \\ \frac{2\sqrt{2}\sin(s)^2}{3(\sin(s)^2 - \cos(s)^2))} & \frac{\sin^2(s+\epsilon) - \cos^2(s+\epsilon)}{3(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & \frac{2\sqrt{3}\sin(s+\epsilon)\cos(s+\epsilon)}{3(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))}) & -\epsilon \\ \frac{2\sqrt{2}\sin(s+\epsilon)\cos(s+\epsilon)}{\sqrt{3}(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & \frac{-2\sin(s+\epsilon)\cos(s+\epsilon)}{\sqrt{3}(\sin^2(s+\epsilon) - \cos^2(s+\epsilon))} & \frac{1}{\sin^2(s+\epsilon) - \cos^2(s+\epsilon)} & \sqrt{2}+\epsilon \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

7. APPLICATIONS

In this section, we do applications and draw their figures to explore visual representations. h dual transformation leaves invariant the hyperplane H, which is $z = 1$. Hence, we give examples by taking a unit quaternion curve $\alpha(t) \in S^3$.

Example 4. *Let us take a unit quaternion curve* $\alpha(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t)) = (\frac{1}{\sqrt{2}}cos(t), \frac{1}{\sqrt{2}}cos(t), 0, sin(t))$ *in the hyperplane* H, *which can be represented by the Galilean orthogonal matrix* $\tilde{R}$ *as follows*

$$\tilde{R} = \begin{bmatrix} \cos(t)^2 - \sin(t)^2 & -\sqrt{2}\cos(t)\sin(t) & \sqrt{2}\cos(t)\sin(t) & t^2 \\ \sqrt{2}\cos(t)\sin(t) & -\sin(t)^2 & -\cos(t)^2 & \sin(t) \\ \sqrt{2}\cos(t)\sin(t) & \cos(t)^2 & \sin(t)^2 & \cos(t) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\alpha_0(t)^2 - \alpha_1(t)^2 - \alpha_2(t)^2 + \alpha_3(t)^2 \neq 0$. Let us go through with obtaining surfaces from Galilean matrices by multiplying $\phi(s) = (\cos(s), 0, \sin(s), 1)$. First, we obtain a surface from Galilean orthogonal matrix $\tilde{R}$ by multiplying $\phi(s)$.

The elements of matrix $\tilde{R}.\phi(s)$ can be represented by a surface $M_1 = \Psi(t,s) = ((\cos(t)^2 - \sin(t)^2)\cos(s) + \sqrt{2}\sin(t)\cos(t)\sin(s) + t^2, \sqrt{2}\sin(t)\cos(t)\cos(s) - \cos(t)^2\sin(s) + \sin(t), \sqrt{2}\sin(t)\cos(t)\cos(s) + \sin(t)^2\sin(s) + \cos(t))$ in the hyperplane H. See Fig. 1.

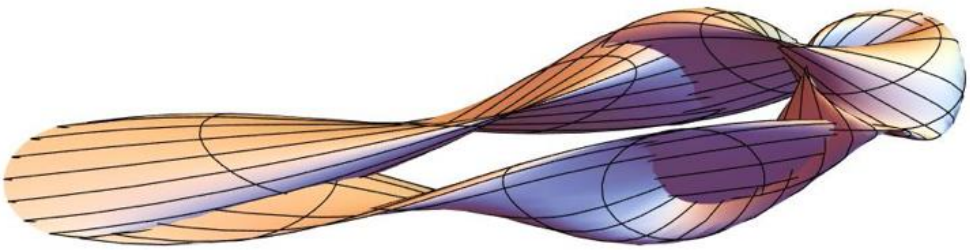

Figure 1. The surface M_1

Second, we can examine the pseudo-Galilean orthogonal matrix $f(\tilde{R})$ with the help of h dual transformation as we did in the previous sections of this paper. The elements of matrix $f(\tilde{R}).\phi(s)$ can be represented by a surface $M_2 = \Psi(t,s) = (-\cos(s) + \frac{\sqrt{2}\cos(t)\sin(s)}{\sin(t)} + t^2, \frac{\sqrt{2}\cos(t)\cos(s)}{\sin(t)} - \frac{\cos(t)^2\sin(s)}{\sin(t)^2} + \sin(t), \frac{-\sqrt{2}\cos(t)\cos(s)}{\sin(t)} - \frac{\sin(s)}{\sin(t)^2} + \cos(t))$.

See Fig. 2.

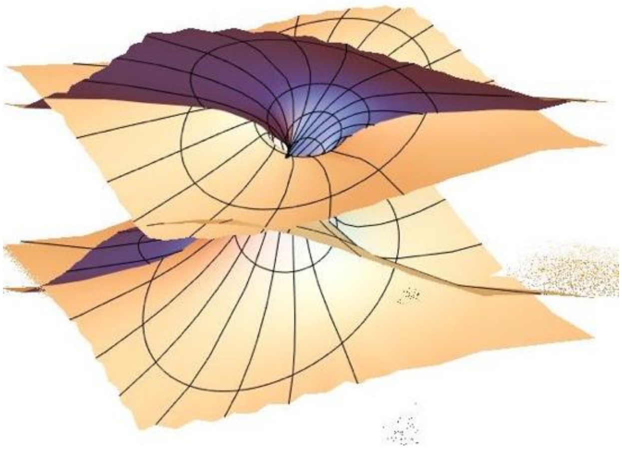

Figure 2. The surface M_2

Example 5. *Let us take a unit quaternion curve* $\alpha(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t)) = (\frac{1}{\sqrt{3}}cos(t), \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}sin(t), \frac{1}{\sqrt{3}})$ *in* H. *Therefore, we can write the Galilean orthogonal matrix* $\tilde{R}$ *as follows*

$$\tilde{R} = \begin{bmatrix} \frac{1}{3}(\cos(t)^2 - \sin(t)^2) & \frac{-2}{3}\cos(t) + \frac{2}{3}\sin(t) & \frac{2}{3}\cos(t)\sin(t) + \frac{2}{3} & \cos(t) \\ \frac{2}{3}\cos(t) + \frac{2}{3}\sin(t) & \frac{1}{3}\sin(t)^2 + \frac{1}{3}\cos(t)^2 - \frac{2}{3} & \frac{-2}{3}\cos(t) + \frac{2}{3}\sin(t) & t \\ \frac{-2}{3}\cos(t)\sin(t) + \frac{2}{3} & \frac{2}{3}\cos(t) + \frac{2}{3}\sin(t) & \frac{1}{3}(\cos(t)^2 - \sin(t)^2) & \sin(t) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\alpha_0(t)^2 - \alpha_1(t)^2 - \alpha_2(t)^2 + \alpha_3(t)^2 \neq 0$. Then, we get the pseudo-Galilean orthogonal matrix $f(\tilde{R})$ by using the h dual transformation. After, we obtain a surface from Galilean orthogonal matrix $\tilde{R}$ by multiplying $\phi(s) = (\cos(s), 0, \sin(s), 1)$.

The elements of matrix $\tilde{R}.\phi(s)$ can be represented by a surface $M_3 = \Psi(t,s) = (\frac{1}{3}(\cos(t)^2 - \sin(t)^2)\cos(s) + \frac{2}{3}(\cos(t)\sin(t) + 1)\sin(s) + \cos(t), \frac{2}{3}(\cos(t) +$

$\sin(t))\cos(s) + \frac{2}{3}(\sin(t) - \cos(t))\sin(s) + t, \frac{2}{3}(1 - \cos(t)\sin(t))\cos(s) + \frac{1}{3}(\cos(t)^2 - \sin(t)^2)\sin(s) + \sin(t))$ in the hyperplane H. See Fig. 3.

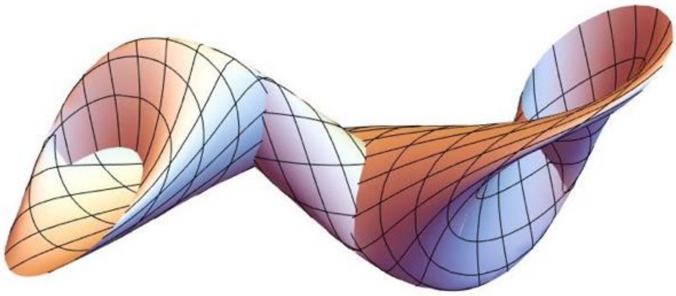

Figure 3. The surface M_3

Next, we obtain a surface from $f(\tilde{R})$ by multiplying $\phi(s)$. With a similar method, we can express the surface as

$$M_4 = \Psi(t,s) = \left(\frac{-\cos(s)}{\cos(t)^2-\sin(t)^2} + \frac{2(\cos(t)\sin(t)+1)\sin(s)}{\cos(t)^2-\sin(t)^2} + \cos(t), \frac{2(\cos(t)-\sin(t))\cos(s)}{\cos(t)^2-\sin(t)^2} + \frac{2(\sin(t)-\cos(t))\sin(s)}{\cos(t)^2-\sin(t)^2} + t, \frac{2(\cos(t)\sin(t)-1)\cos(s)}{\cos(t)^2-\sin(t)^2} + \frac{3\sin(s)}{\cos(t)^2-\sin(t)^2} + \sin(t)\right).$$ See Fig. 4.

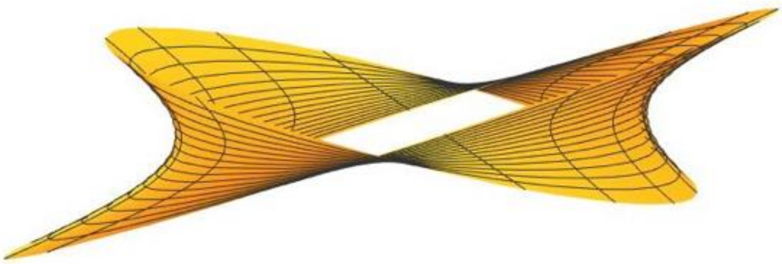

Figure 4. The surface M_4

CONCLUSIONS

We obtained surfaces M_1, M_2, M_3 and M_4 in H with the help of a unit quaternion in S^3. Drawing these figures allowed us to visualize the applications. We have defined dual transformations in Galilean and pseudo-Galilean spaces with the help of unit quaternions. Through these visuals, we have the opportunity to compare the surfaces drawn with the help of these motions in both spaces. If we examine the figures in the examples carefully, we can imagine the figures obtained in pseudo-Galilean spaces as the more opened forms of those obtained in Galilean spaces. In our previous studies (see [2] and [3]), we compared the images we obtained in Euclidean space and Lorentz space. We have also captured similar views in Galilean and pseudo-Galilean spaces and considered it appropriate to include them in this study.

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