ORIGINAL PAPER

THE EXISTENCE AND NON-EXISTENCE OF CONVEX OR CONVEX-CONCAVE SOLUTIONS OF A DIFFERENTIAL EQUATION RESULTING FROM FLUID MECHANICS

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Abstract. *A similar problem was recently considered with $f'(t) \to \lambda$ as $t \to T_c$, (T_c is the maximum value for the existence or no existence of f_c), $\lambda \in \{0, 1, \pm\infty\}$ and $0 < \beta < 1$. In this paper, we study same the equation (with $\beta < 0$) and we establish the existence of solutions satisfying the boundary conditions $f(0) = a \in \mathbb{R}, f'(0) = b < 0$ and $f'(+\infty) \in \{0, 1\}$. In order to do that, we utilize the shooting technique, then we consider the initial value problem consisting of the differential equation such that $f(0) = a \in \mathbb{R}, f'(0) = b < 0$ and $f''(0) = c \geq 0$. We accompany our results by some numerical illustrations.*

Keywords: *Convex; concave; solution, shooting.*

1. INTRODUCTION

Consider the nonlinear autonomous differential equation:

$$f''' + ff'' + \beta f'(f' - 1) = 0 \tag{1}$$

on [0, +∞) with the boundary conditions:

$$f(0) = a < 0,\ f'(0) = b < 0 \text{ and } f'(t) \to \lambda \text{ as } t \to +\infty \tag{2}$$

Equation (1) ($b < \beta < 0$) has already been considered in [1-3]. Such problems arise from the study of free convection and of mixed convection boundary layer flows over a vertical surface embedded in a porous medium in [4-8]. Our goal here is to study, as in [9] and [1], the existence or nonexistence and uniqueness of solution of the following boundary problem ($P_{a,b,\lambda}$) with $b < 0$ and $\lambda \in \{0,1\}$:

$$\begin{cases} f''' + ff'' + \beta f'(f' - 1) = 0 \ \ on\ [0, +\infty) \\ f(0) = a \\ f'(0) = b \\ f'(+\infty) = \lambda \end{cases} \qquad (P_{a,b,\lambda})$$

We will focus our attention on convex and convex-concave solutions. In what follows, convex and concave will mean strictly convex and strictly concave. Recall that a function $f: I \to \mathbb{R}$ of class C^3on the interval I is convex (resp-concave) if and only if f' is increasing

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(resp. decreasing). If the function f is of class C^3 such that $f'' > 0$ (resp. $f'' < 0$) on I, then f is convex (resp. concave). The converse, however, is not true.

Remark 1.1. The solutions of the problem $P\ (a, b, 1)$ are called similarity solutions (a similarity solution is a particular type of solution which reflects the invariance of the properties of equation (1.1) the latter arises from a system of partial differential equations in certain situations where simplifying assumptions have been made) and the famous example is the Blasius equation (1907) which corresponds to $\beta = 0$ and arises in the study of the laminar boundary layer on a flat plate. Note here that many papers have been published about the Blasius equation, see [3, 5]. To solve the boundary value problem $P\ (a, b, 1)$ we investigate the results of [2] and we will use the shooting technique. Let f_c be the solution of the initial value problem $P_i\ (a, b, c)$ consisting of Equation (1) and the conditions $f(0) = a \in \mathbb{R}, f'(0) = b < 0$ and $f''(0) = c \geq 0$. Let $[0, T_c)$ be the right maximal interval of existence of f_c. Obtaining a solution of $(P(a, b, 1))$ is equivalent to finding a value of a c such that $T_c = +\infty$ and ${f'}_c(t) \to 1$ as $t \to +\infty$.

We examine in Section 2 some preliminary results about the equation:

$$f''' + ff'' + \beta f'(f' - 1) = 0$$

In Sections 3 and 4, we investigate the problem $P_i\ (a, b, c)$ and the convex, convex-concave solutions of the problem $P\ (a, b, 0)$ and $P\ (a, b, 1)$ for $a < 0$ and $a > 0$. Finally, we accompany our results by some numerical illustrations for the problem $P_i\ (a, b, c)$ using shooting algorithm mathematic.

2. MATERIALS AND METHODS

2.1. MATERIALS

In this section, we recall some results about sub-solutions and super-solutions of the Blasius equation (see [2, 10, 11]). The Blasius equation is the third order ordinary differential equation:

$$f''' + ff'' = 0 \tag{2.1}$$

obtained from (1) with $\beta = 0$.

Definition 2.1.1. Let $I \subset \mathbb{R}$ be an interval. Say that a function $f: I \to \mathbb{R}$ is a sub-solution (resp. a super-solution) of the Blasius equation, if f is of class C^3 and if

$$f''' + ff'' \leq 0 \ \ (resp. f''' + ff'' \geq \ 0)\ on\ I.$$

Proposition 2.1.2. Let $t_0 \in \mathbb{R}$. There does not exist a nonpositive convex super-solution of the Blasius equation on the interval $[t_0, +\infty)$.

Proposition 2.1.3. Let $t_0 \in \mathbb{R}$. There does not exist a nonpositive concave sub-solution of the Blasius equation on the interval $[t_0, +\infty)$.

Proposition 2.1.4. Let f be a solution of (1) on some maximal interval $I = (T_-, T_+)$. The following properties are satisfied:

1) $(f''(t)+f(t)(f'(t)-1))'=(1-\beta)f'(t)(f'(t)-1)$ on I
2) If F is any primitive function of f on I

$$(f''e^F)'=-\beta f'(f'-1)e^F.$$

3) If $\alpha\epsilon\{0,1\}$ and if there exists a point $t_0\in I$ such that $f''(t_0)=0$ and $f'(t_0)=\alpha$, then

$$f(t)=\alpha(t-t_0)+f(t_0) \text{ for all } t\in I.$$

4) If $T_+=+\infty, f'(t)\to\lambda\in\mathbb{R}$ as $t\to+\infty$, and if f is of constant sign at infinity, then

$$f''(t)\to 0 \; as \; t\to+\infty.$$

5) $T_+=+\infty$ And if $f'(t)\to\lambda\in\mathbb{R}$ as $t\to+\infty$, then $\lambda\in\{0,1\}$.

6) $T_+<+\infty$, then f'' and f' are unbouned near T_+.

Proof: Let's prove point 4:
Let us assume that f is nonnegative at infinity. we have

$$(f''^2+2\beta(\frac{f'^3}{3}-\frac{f'^2}{2}))'=2f'''f''+2\beta f''f'(f'-1)$$
$$=2(-ff''-\beta f'(f'-1))f''+2\beta f''f'(f'-1)=-2ff''^2.$$

Thus $(f''^2+2\beta(\frac{f'^3}{3}-\frac{f'^2}{2}))$ is nonincreasing at infinity, and since $(\frac{f'^3}{3}-\frac{f'^2}{2})\to(\frac{\lambda^3}{3}-\frac{\lambda^2}{2})$ as $t\to+\infty$, we see that $f''^2(t)$ has a limit as $t\to+\infty$, and this limit necessarily equals 0. Since $f'(t)$ has a finite limit as $t\to+\infty$.

2.2. METHODS

We have used the shooting method and maple, Mathematica (Wolfram Mathematica 10, Version Number: 10.0.1.0.) for numerical results.

3. RESULTS AND DISCUSSION

3.1. RESULTS

The $a<0$ CASES:
In order to obtain solutions of (1), (2), we consider for $c\geq 0,\ \beta<0,\ b<0$ and the following initial value problem on $[0,T_c)$:

$$\begin{cases} f'''+ff''+\beta f'(f'-1)=0 \;\; on\; [0,+\infty) \\ f(0)=a \\ f'(0)=b \\ f''(0)=\mathrm{c} \end{cases} \qquad P_i\;(a,b,c)$$

Lemma 3.1.1. If f_c is a solution of the initial value problem $P_i\,(a,b,c)$, then there exists a point t_0 satisfying $f'_c(t_0)=0$ and $f''_c(t_0)>0$.

Proof: Let us assume that f_c is a convex solution of $P_i\,(a,b,c)$ on $[0,T_c)$. Hence, from $f_c'(0)=b<0$ and thanks to Proposition 2.1.4 (Item (2)), we have $f''_c>0$ on $[0,T_c)$. Thus f_c' is bounded on $[0,T_c)$. Hence from Proposition 2.1.4 (Items (5) and (6)), we have $T_c=+\infty$ and $f'(t)\to 0$ as $t\to +\infty$. Consequently f_c is a non-positive convex solution of $P(a,b,0)$ on $[0,+\infty)$, and

$$f'''_c+f_cf''_c=-\beta\, f'_c(f'_c-1)>0,$$

So, f_c is non-positive convex super-solution of the Blasius equation which contradicts Proposition 2.1.2.

Proposition 3.1.2. The boundary value problem $\mathrm{P(a,b,0)}$ has no convex solution.

Proof: This results from the previous lemma.

Proposition 3.1.3. The boundary value problem $P(a,b,0)$ has no non-positive convex-concave solution.

Proof: Let us assume that f_c is a non-positive convex-concave solution, let t_1 be the point of $[0,+\infty)$ such that $f''_c>0$ on $[0,t_1)$, and $f''_c<0$ on $(t_1,+\infty)$. Hence, f'_c is strictly increasing on $[0,t_1)$, strictly decreasing on $(t_1,+\infty)$. And since $f'_c(+\infty)=0$ and $f'_c(0)=b<0$, for all $t>t_1$, $0<f'_c(t)<1$ because $f'_c(t_1)<1$ by Proposition 2.1.4 (Item 2). We have

$$f'''_c+f_cf''_c=-\beta\, f'_c(f'_c-1)<0 \text{ on } (t_1,+\infty).$$

Then f_c is non-positive concave sub-solution of the Blasius equation, and this contradicts Proposition 2.1.3.

Lemma 3.1.4. Let us assume that $\mathrm{b}<\beta<0$, $\mathrm{a}<0$ and let f_c be a solution on $[0,\mathrm{T_c})$ of $\mathrm{P_i}\,(\mathrm{a,b,c})$. If there exists $\mathrm{t_1}\in[0,\mathrm{T_c})$ such that $f''_c(\mathrm{t_1})=0$, then $f_c(\mathrm{t_1})>0$.

Proof: From

$$H_c(t)=f''_c(t)+f_c(t)\big(f'_c(t)-\beta\big),$$

we have

$$H'_c(t)=(1-\beta)f'^{\,2}_c(t)\geq 0.$$

Then H is nondecreasing on $[0,\mathrm{T_c})$ and

$$H(0)=c+a(b-\beta)>0.$$

So $\mathrm{H_c(t)}>0$. We have

$$H_c(t_1)=f_c(t_1)(f'_c(t_1)-\beta).$$

by Proposition 2.1.4 (Item 2), we have $0<f'_c(t_1)<1$ and $\mathrm{b}<\beta<0$, then

$$\big(f'_c(t_1)-\beta\big)>0.$$

So $f_c(t_1)>0$.

Proposition 3.1.5. Let $\alpha = -\sqrt{\frac{1-b^2}{\beta-2b}}$ if $b \in (-\infty,-1], 2b < \beta < 0$ and $a < 0$ (resp.$b < \beta < 0$, $b \in (-1,0]$ and $a \in (-\infty,\alpha]$). Then there does not exist c, such that f_c is a solution of the problem $P(a,b,1)$.

Proof: Let us assume that f_c is a solution of $P_i\ (a,b,c)$. Then there exists $t_* \in [0,T_c)$ such that $f_c(t_*) = 0$. Let

$$K_c(t) = f_c(t)f''_c(t) - \frac{1}{2}f'^2_c(t) + f_c^2(t)\left(f'_c(t) - \frac{1}{2}\beta\right).$$

Then

$$K'_c(t) = (2-\beta)f_c(t)f'^2_c(t).$$

For all $t \in [0.t_*)$, we get $K'_c(t) < 0$, and then

$$2ac - b^2 + a^2(2b-\beta) > -f'^2_c(t).$$

We put:

$$p(a) = a^2(\beta - 2b) + b^2 - 1.$$

If $b \in (-\infty,-1], p(a) > 0$, then $f'_c(t_*) > 1$. The same results are obtained where $b \in (-1,0]$ and $a \in (-\infty,\alpha]$.

Theorem 3.1.6 . Let $2b < \beta < 0$ and $a < 0$.
1) The boundary value problem $P(a,b,0)$ has no convex solution on $[0,+\infty)$.
2) If $b \le -1$ or $b \in (-1,0]$ and $a \in (-\infty,\alpha]$, then any solution of the initial problem $P_i\ (a,b,c)$ with $c \ge 0$ is a convex solution of the boundary value problem $P(a,b,+\infty)$.

Proof: The first result follows from Lemma 3.1.1, from which we have obtained that f_c is a solution of the initial value problem $P_i\ (a,b,c)$ such that f'_c tends to 0, 1 or $+\infty$, as $t\to +\infty$. Also, we have shown that f_c is not a solution convex of $P(a,b,0)$.

The second result follows from Lemma 3.1.1, and Lemma 3.1.3 gives that the boundary value problem $P(a,b,0)$ has no non-positive convex-concave solution, and Proposition 3.1.5 shows that f_c is not a solution of the problem $P(a,b,1)$. So, we have that f_c is a solution convex of $P(a,b,+\infty)$.

The $a > 0$ CASES:

Let $a,b \in \mathbb{R}$ with $b < 0$ and $a > 0$, and consider the solution f_c of the initial value problem $P_i\ (a,b,c)$ on the right maximal interval of existence $[0,T_c)$.

Proposition 3.1.7. Any solution of $P_i\ (a,b,c)$ changes monotony.

Proof: See [1].

Lemma 3.1.8. If $c > \frac{b^2}{2a} - a\left(b - \frac{\beta}{2}\right)$, let $t_1 \in (0.+\infty)$ be the point where $f_c$changes monotony at then $f_c(t_1) > 0$.

Proof: Let

$$K_c(t) = f_c(t)f''_c(t) - \frac{1}{2}f'^2_c(t) + f_c^2(t)\left(f'_c(t) - \frac{1}{2}\beta\right),$$

let $t_* \in [0,T_c)$ and let $t_* < t_1$ such that $f_c(t_*) = 0$, $f_c > 0$ on $[0,t_*)$. Then

$$K'_c(t) = (2-\beta) f_c(t) f'^2_c(t) > 0 \text{ on } [0, t_*).$$

We have

$$K(0) = ac - \frac{b^2}{2} + a^2\left(b - \frac{\beta}{2}\right) > 0.$$

Then $\mathrm{K} > 0$ on $[0, \mathrm{t}_*)$ and

$$K(t_*) = -\frac{1}{2} f'^2_c(t_*) < 0,$$

a clear contradiction.

Proposition 3.1.9. If $\mathrm{c} > \frac{\mathrm{b}^2}{2\mathrm{a}} - \mathrm{a}\left(\mathrm{b} - \frac{\beta}{2}\right)$, then the change of the convexity is above the axis of ox.

Proof: This results from the previous Lemma and by Proposition 2.1.4 (Item 2).

Lemma 3.1.10. If $b \in [a^2 - \sqrt{a^4 - a^2\beta}, 0)$ and suppose that f_c is a solution of the problem $P_i\ (a, b, c)$ which changes the monotony at the point $t_1 \in [0, +\infty)$. Then $f_c(t_1) > 0$.

Proof: Let

$$K_c(t) = f_c(t) f''_c(t) - \frac{1}{2} f'^2_c(t) + f_c{}^2(t)\left(f'_c(t) - \frac{1}{2}\beta\right).$$

Let $t_* \in [0, T_c)$ and let $t_* < t_1$ such that $f_c(t_*) = 0$, $f_c > 0$ on $[0, t_*)$. Then

$$K'_c(t) = (2-\beta) f_c(t) f'^2_c(t) > 0. \text{ on } [0, t_*).$$

$$K(0) = ac - \frac{b^2}{2} + a^2\left(b - \frac{\beta}{2}\right) > 0$$

because

$$p(a) = ac - \frac{b^2}{2} + a^2\left(b - \frac{\beta}{2}\right) > 0.$$

If $b \in \left[a^2 - \sqrt{a^4 - a^2\beta}, 0\right)$, then $K > 0$ on $[0, t_*)$ and

$$K(t_*) = -\frac{1}{2} f'^2_c(t_*) < 0,$$

which is a contradiction.

Proposition 3.1.11. If $\mathrm{b} \in \left[\mathrm{a}^2 - \sqrt{\mathrm{a}^4 - \mathrm{a}^2\beta}, 0\right)$, then the change of the convexity is above the axis of Ox.

Proof: As a consequence of the Lemma 3.1.10, there is f_c is a solution of the problem $P_i\ (a, b, c)$ which changes the monotony to the point $t_1 \in [0, +\infty)$, and from Proposition 2.1.4 (Item 2)

$$(f_c{}''e^F)' = -\beta f_c{}'(f_c{}' - 1)e^F < 0,$$

because $\forall t > t_1$, $0 < f_c{}'(t) < 1$. Thus, we have found the change of the convexity.

The following results for $\mathrm{b} \in \left[\mathrm{a}^2 - \sqrt{\mathrm{a}^4 - \mathrm{a}^2\beta}, 0\right)$.

Lemma 3.1.12. There is no value of c such that f_c is a solution of the problem $P(a, b, 1)$.

Proof: Suppose that f_c is the convex-concave solution of $P_i\ (a,b,c)$, then $\exists t_0 \in [0,+\infty)$ such that, $f_c{}'' > 0$ on $[0,t_0), f_c{}'$ increasing on $[0,t_0), f_c{}'' < 0$ on $[t_0,+\infty], f_c{}'$ decreasing on $[t_0,+\infty]$ and $f_c{}''(t_0) = 0$. On the other hand, we know that $0 < f_c{}'(t_0) < 1$ because

$$(f_c{}''e^F)' = -\beta f_c{}'(f_c{}' - 1)e^F < 0$$

on [0.1], so $f_c{}''e^F$ decreasing on [0.1], and $f_c{}'$ decreasing on $(t_0,+\infty]$. Thus, $f_c{}'$ does not tend to 1 as $t \to +\infty$.

Remark 3.1.13. If f_c is a convex-concave solution of $P_i\ (a,b,c)$, then $f_c{}'$ tend to 0 or -∞ as $t \to +\infty$.

Proposition 3.1.14. The problem $P_i\ (a,b,c)$ does not admit a convex-concave solution such that $f_c{}'$ tends to $l \in [-\infty,0)$ as $t \to +\infty$.

Proof: Suppose that f_c is a convex-concave solution, then $\exists t_1 \in [0,+\infty), f_c{}'\ (t_1) = 0$ and $f_c{}''(t_1) > 0.\ \exists\ t_3 > t_1 \in [0,+\infty), f_c\ (t_3) = 0$ and $f_c{}''(t_3) < 0$.

Let

$$H(t) = f_c{}''(t) + f_c(t)(f_c{}'(t) - \beta)$$

and

$$H'(t) = (1-\beta){f_c{}'}^2\ (t) > 0, H(t_1) = f_c{}''(t_1) - \beta\, f_c(t_1) > 0,$$

then H >0 on $[t_1,+\infty)$ but

$$H(t_3) = f_c{}''(t_3) < 0,$$

which is a contradiction.

Lemma 3.1.15. Any convex-concave solution f_c of the initial value problem $P_i\ (a,b,c)$ on the right maximal interval of existence $[0;T_c)$ is a convex-concave solution of the boundary value problem $P(a,b,0)$ on the right maximal interval of existence [0, +∞).

Proof: A mere consequence of the previous proposition.

Theorem 3.1.16. Let $b \in [a^2 - \sqrt{a^4 - a^2\beta}, 0)$ and $c > \frac{b^2}{2a} - a\left(b - \frac{\beta}{2}\right)$.

1) The boundary value problem $P(a,b,1)$has no convex-concave solution on $[0,+\infty)$.

2) All solution of the initial problem $P_i\ (a,b,c)$ with $c \geq 0$ is a convex-concave solution of the boundary value problem $P(a,b,0)$.

Proof: Let $b \in [a^2 - \sqrt{a^4 - a^2\beta}, 0)$ and $c > \frac{b^2}{2a} - a\left(b - \frac{\beta}{2}\right)$.

1)The first result follows from Lemma 3.1.8, Proposition 3.1.9, Lemma 3.1.10, and Proposition 3.1.11 which give the change of the convexity, and Lemma 3.1.12 gives that $f_c$is not a convex-concave solution of Problem $P(a,b,1)$.

2) The second result follows from Lemma 3.1.14 and Lemma 3.1.15. Indeed, Lemma 3.1.14 shows that the problem $P_i\ (a,b,c)$ does not admit a convex-concave solution such that $f_c{}'$ tends to $l \in [-\infty,0)$ as $t \to +\infty$, and Lemma 3.1.15 then demonstrates that any solution of the initial problem $P_i\ (a,b,c)$ with $c \geq 0$ is a convex-concave solution of the boundary value problem $P(a,b,0)$.

Lemma 3.1.17. Let $b \in [a^2 - \sqrt{a^4 - a^2\beta}, 0)$ and $c > \frac{b^2}{2a} - a\left(b - \frac{\beta}{2}\right)$. All solutions of the initial problem $P_i\ (a,b,c)$ are positive.

Proof: From Lemma 3.1.10, we have shown that f_c is a solution of Problem $\mathrm{P_i}\ (\mathrm{a,b,c})$ which changes the monotony at the point $\mathrm{t_1} \in [0,+\infty)$. Then $f_c(\mathrm{t_1}) > 0$, and from Lemma 3.1.12, Proposition 3.1.14, Lemma 3.1.15, and Theorem 3.1.16 we have that f_c is a convex-concave solution of the boundary value problem $\mathrm{P(a,b,0)}$. So, $f_c$always remains above the axis of Ox. Accordingly, $f_c$is positive.

3.2. NUMERICAL SOLUTIONS

In this section, Problem $P_i\ (a,b,c)$ is numerically solved using a shooting algorithm of Mathematica and Maple. In Table 1, we give some selected values of the initial conditions and β. We find that the numerical results for various values of the shooting parameter are compatible with the results of Theorem 3.1.6 (see Figs. 1-4).

Table 1. Related values of $c > 0$.

a	b	β
-4	-1.5	-2.5
-4	-0.05	-0.01
-0.9	-0.02	-0.01
-7	-0.05	-0.01

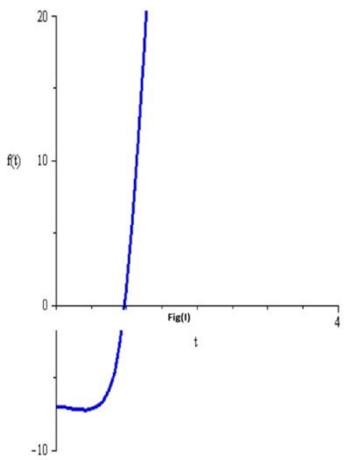

Figure 1. Solution with $[a = -7, b = -0.9,\ \beta = -0.01]$

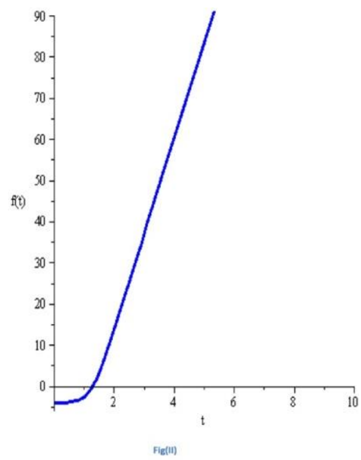

Figure 2. Solution with $[\mathrm{a} = -4, \mathrm{b} = -0.05,\ \mathrm{c} = -0.01]$

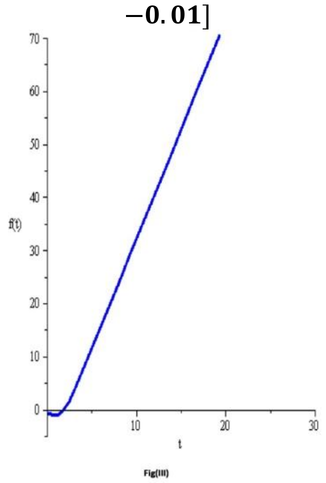

Figure 3. Solution with $[a = -0.9, b = -0.5,\ \beta = -0.01]$

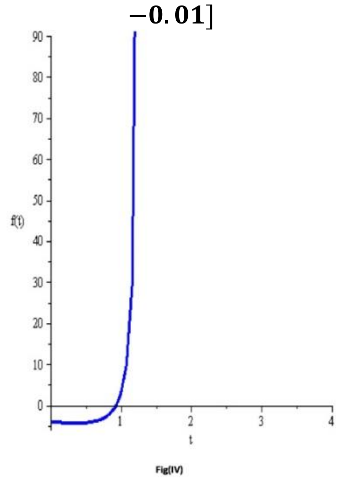

Figure 5. Solution with $[a = -4, b = -1.5,\ c = 0.5, \beta = -2.5]$

3.3. DISCUSSION

From what we have obtained in our results, we establish two essential results with $a < 0$:

A - 1) The boundary value problem $\mathrm{P}(\mathrm{a}, \mathrm{b}, 0)$ has no convex solution on $[0, +\infty)$.
2) If $\mathrm{b} \leq -1$ or $\mathrm{b} \in (-1,0]$ and $\mathrm{a} \in (-\infty, \alpha]$ then all solutions of the initial problem $\mathrm{P_i}\ (\mathrm{a}, \mathrm{b}, \mathrm{c})$ with $\mathrm{c} \geq 0$ are convex solutions of the boundary value problem $\mathrm{P}(\mathrm{a}, \mathrm{b}, +\infty)$.
B - For a>0, we have obtained:
Any solution of $\mathrm{P_i}\ (\mathrm{a}, \mathrm{b}, \mathrm{c})$ change monotony and for some associated conditions on b, c we have:
1 - The change the monotone above the axis ox ($\mathrm{c} > \frac{\mathrm{b}^2}{2\mathrm{a}} - \mathrm{a}\left(\mathrm{b} - \frac{\beta}{2}\right)$).
2 - If $\mathrm{b} \in \left[\mathrm{a}^2 - \sqrt{\mathrm{a}^4 - \mathrm{a}^2\beta}, 0\right)$, then the change of the convexity is above the axis of ox.
3 - If $\mathrm{b} \in \left[\mathrm{a}^2 - \sqrt{\mathrm{a}^4 - \mathrm{a}^2\beta}, 0\right)$, then f_c is a convex-concave solution of $\mathrm{P_i}\ (\mathrm{a}, \mathrm{b}, \mathrm{c})$, then f_c' tends to 0 as $\mathrm{t} \to +\infty$.
C – With the algortithm of Mathematica and maple, we find that the numerical results for various values of the shooting parameter are compatible with the results of Theorem 3.1.6.

4. CONCLUSIONS

For a and b sufficiently small. In Table 2, we give some selected values of the initials conditions for $a < 0, b < \beta < 0$ (see Figs. 5-8).

Table 2. Related values of b,c and $\boldsymbol{\beta = -0.01, a = -10^{11}}$.

c	b
1.5	-0.02
1.5	-0.05
-0.4863	-0.05
-0.49	-0.05
0.6	-0.05
1.46	-0.05

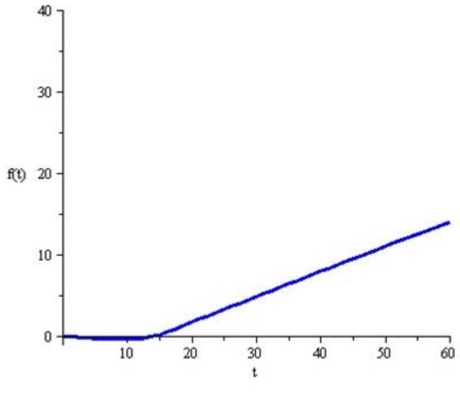

Figure 5. Solution with $\left[\boldsymbol{a = -\frac{1}{100000000000}, b = -0.05, c\ assez\ petit\ positif, \beta = -0.01}\right]$

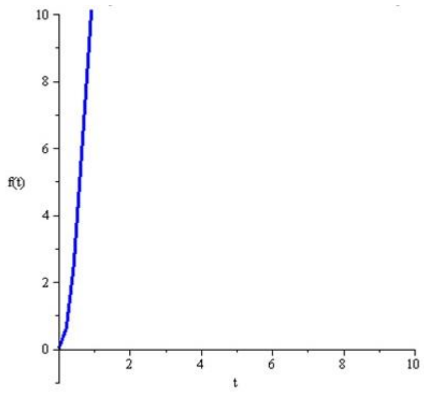

Figure 6. Solution with $\left[\boldsymbol{a = -\frac{1}{100000000000}, b = -0.05, c = 30, \beta = -0.01}\right]$

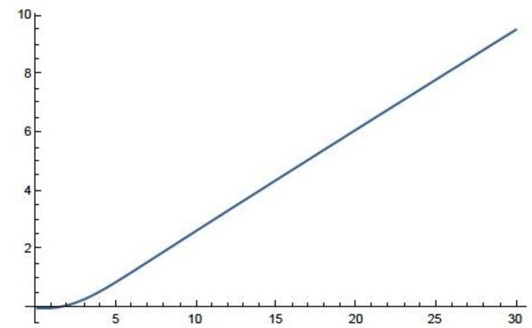

Figure 7. Interpretation of the graph of $f_c{}'$. Plot $[y^1[t]\ V.ss, \{t, 0, 30\}]$

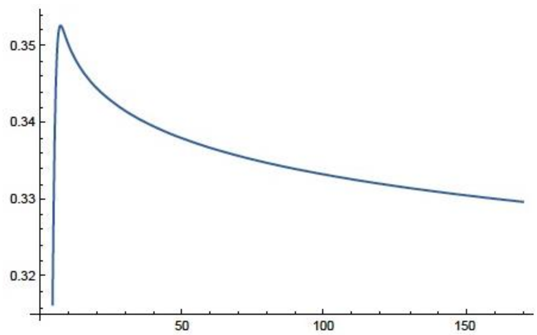

Figure 8. Interpretation of the graph of $f_c{}'$. Plot $[y^2[t]\ V.ss, \{t, 0, 170\}]$

The graph of the function $f_c{}'$ shows that the solution f_c becomes convex-concave afterwards.

According to the graphs of Table 2, one concludes that:

- If a, b, β are sufficiently small negative numbers, and if :
  - c is sufficiently small positive, then the convex-concave function f_c and $f_c{}'$ tends to 0 as t tends to $T_c = +\infty$.
  - c is sufficiently large positive, then the convex-concave function f_c and $f_c{}'$ tends to +∞ as t tends to $T_c \leq +\infty$.
  - c is satisfiable, then the convex-concave function f_c and $f_c{}'$ tends to 0 as t tends to $T_c = +\infty$.
- If b, β are sufficiently large negative numbers, a is sufficiently small negative, c is sufficiently large positive, then f_c convex and $f_c{}'$ tends to +∞ as t tends to $T_c < +\infty$.

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