*ORIGINAL PAPER*

# THE EFFECT OF ALLEE FACTOR ON A NONLINEAR DELAYED POPULATION MODEL WITH HARVESTING

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*Manuscript received: 28.10.2021; Accepted paper: 02.03.2022;*
*Published online: 30.03.2022.*

***Abstract.*** *The Allee effect may have a stabilizing or destabilizing effect on population dynamics, or reaching stability may be delayed. Also this effect can push populations to extinction, especially under sharp harvesting effect. Compensatory or overcompensatory population maps are situations that change the effect of the Allee factor on population dynamics. The harvesting factor has also the effect of changing the equilibrium levels differently in theoretical population models. This study includes the local stability of the equilibrium point of the delayed discrete-time population model exposed to the harvest effect with and without Allee effect. We also review the effect of Allee factor on the local stability of equilibrium point of the discrete-time population model involving delay [1]. The dynamics of this population models are given together with the enriched dynamics.*

***Keywords:*** *stability; allee effect; discrete-time population models; time delay; harvesting.*

## 1. INTRODUCTION

Most of the nonlinear models offer a more realistic approach than linear models [2, 3]. When the parameters affecting the population are included in the model, the results obtained on the population are more valid in terms of biological reality. Allee effect which is first described by Allee [4] has a significant effect on the local stability analysis of population models. Some biological causes, defined as the Allee effect, change the dynamics of populations [5-7]. Former studies demonstrate that the Allee effect can have different effect on population dynamics [4, 8-22]. The more realistic approach is provided with this effect. Thus, we have more accurate information about the future of the population.

Mathematical models are widely used to predict population dynamics. Harvesting effect is another important factor affecting population dynamics. Populations can be successfully managed with harvesting strategies that are sustainable to a certain extent [3, 23-25].

While a difference equation is considered a discrete version of a differential equation; a differential equation is considered a continuous version of a difference equation. It's good to be familiar with continuous structures to better understand population patterns [26-28].

In this study, our aim is to examine the local stability of the delayed model exposed to harvest with and without Allee effect and to present rich dynamics showing theoretical results. At the same time, it is to see the effect of the harvest on the population, thanks to the

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dynamic behaviours based on the theoretical results of the low-density model with and without Allee effect without the harvest effect.

The remainder of the article is organised as follows. In section 2, we present the effect of Allee factor on the local stability of equilibrium point of the discrete-time population model involving delay [1] together with the enriched dynamics, again. In section 3, we give the local stability of the equilibrium point of the delayed discrete-time population model exposed to the harvest effect with and without the Allee effect. Also, the some numerical simulations concerning with our results are presented. The last section is devoted to conclusion.

Now, let us recall definition of the equilibrium point and Jury criterion (see, for instance [2]) which will be useful in the stability analysis of population models.

Let us consider the difference equation $x(n+1) = f(x(n))$. A point $x^*$ in the domain of $f$ is said to be an equilibrium point of this equation such that $f(x^*) = x^*$.

If it is considered the second–order difference equation $x(n+2) + px(n+1) + qx(n) = 0$ then the roots of the characteristic polynomial $P(\lambda) = \lambda^2 + p\lambda + q = 0$ of this equation inside the unit disk if and only if $P(1) > 0, P(-1) > 0, |q| < 1$.

In this study, the derivative according to the population density at time $t-1$ will be demonstrated by $' = \frac{\partial}{\partial N_{t-1}}$.

## 2. THE LOCAL STABILITY ANALYSIS OF EQUILIBRIUM POINTS OF THE DELAY POPULATION MODEL WITH AND WITHOUT ALLEE EFFECT

In this section, we review the local stability results of the equilibrium point of the model given in [1] and present enriched numerical examples. This model which is often used to study the whale populations is introduced as follows:

$$N_{t+1} = sN_t + f(N_{t-1}) = F(s, N_t, N_{t-1}), \qquad s > 0. \tag{1}$$

Here, $N_t$ is the adult breeding populations, $f(N_{t-1})$ is the adult stage population with delay of "$one$" year and $s$ is the survival coefficient such that $s \in (0,1]$. Assume that $F$ has a unique positive equilibrium point $N^*$.

The following general assumptions on the function $f$ are given as follows:

**(i)** $\frac{\partial}{\partial N_{t-1}}(f(N_{t-1})) < 0$ for $N_t \in (0, \infty)$; i.e., as the density increases, function $f$ reduces continuously.

**(ii)** $f(0)$ has a finite positive value.

The following assumptions on the Allee function are derived from biological facts :

**(iii)** The Allee function is zero when the population density is zero.

**(iv)** The derivatives of the Allee function are always positive for all positive values.

**(v)** Limit of the Allee function approaches to 1 as the population size increases.

If the delay population model in Eq.(1) is subject to the Allee effect at time $t$, we have the following nonlinear population model

$$N_{t+1} = s^*\alpha(N_t)N_t + f(N_{t-1}) = F_{\alpha,t}(s^*, \alpha, N_t, N_{t-1}), \qquad s^* > 0 \tag{2}$$

where the function $f$ satisfies the conditions (i) and (ii). Note that since $s^*$ is the normalized per capita growth rate by $s/a(N^*)$, Eq.(1) and Eq.(2) have the same equilibrium point. Also, $a(N_t)$ is the Allee function at time $t$.

If the Allee effect at time $t-1$ into Eq.(1) is added, we get

$$N_{t+1} = s^*\alpha(N_{t-1})N_t + f(N_{t-1}) = F_{\alpha,t-1}(s^*,\alpha,N_t,N_{t-1}), \qquad s^* > 0. \tag{3}$$

Similarly, note that the equilibrium point of Eq.(1) is also an equilibrium point of Eq.(3). Then we can give the following theorems.

**Theorem 1. (*Local stability analysis of Eq.(1) without Allee effect*)** *Let $N^*$ be a positive equilibrium point of Eq.(1). Then $N^*$ is locally stable if the following inequality*

$$f'(N^*) > -1 \tag{4}$$

holds.

*Proof:* By considering (i)-(ii), we say that $F$ is a continuous function. If Eq.(1) is linearized in neighbourhood of $N^*$, we can write

$$u_{t+1} = su_t + f'(N^*)u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial of the last equality will be

$$p(\lambda) = \lambda^2 - s\lambda - f'(N^*).$$

If $|p| < 1 - q < 2$, then $N^*$ is locally stable [see, 2], and we get

$$-1 < f'(N^*) < 1 - s.$$

It is clear that $1 - s \geq 0$ for $s \in (0,1]$. Then the Ineq.(4) is true for $f'(N^*) < 0$.

**Theorem 2. (*Local stability analysis of Eq.(1) with Allee effect at time t*)** *Let $N^*$ be a positive equilibrium point of Eq.(2) with respect to $s^*$. Then $N^*$ is locally stable if the following inequality holds:*

$$-1 < f'(N^*) < 1 - [s^*a'(N^*)N^* + s^*a(N^*)]. \tag{5}$$

*Proof:* If Eq.(3) is linearized in a neighbourhood of $N^*$, we get

$$u_{t+1} = [s^*a'(N^*)N^* + s^*a(N^*)]u_t + f'(N^*)u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial from the last equality will be

$$p(\lambda) = \lambda^2 - [s^*a'(N^*)N^* + s^*a(N^*)]\lambda - f'(N^*).$$

Here, $s^*a'(N^*)N^* + s^*a(N^*) > 0$ and $\alpha(N^*) < 1$. $N^*$ is locally stable such that $|p| < 1 - q < 2$. From this, Ineq.(5) is easily seen.

**Corollary 3.** *The Allee effect at time $t$ doesn' t change the local stability of an equilibrium point of Eq.(1), if*

$$0 < [s^*a'(N^*)N^* + s^*a(N^*)] \leq 1 \tag{6}$$

and the Allee effect at time $t$ decreases the local stability of an equilibrium point of Eq.(1), if

$$1 < [s^* a'(N^*)N^* + s^* a(N^*)] < 2. \tag{7}$$

*Proof:* From (6), it is seen that Ineq.(4) is equivalent to Ineq.(5). So, the stability of equilibrium point of Eq.(1) doesn't change. On the other hand, from $1 - [s^* a'(N^*)N^* + s^* a(N^*)] < 0$, we easily see that the stability of equilibrium point of Eq.(1) decreases.

**Theorem 4. (Local *stability analysis of Eq.(1) with Allee effect at time $t - 1$*)** *Let $N^*$ be a positive equilibrium point of Eq.(3) with respect to $s^*$. Then $N^*$ is locally stable if the following inequality*

$$-1 - s^* a'(N^*)N^* < f'(N^*) < 1 - [s^* a'(N^*)N^* + s^* a(N^*)] \tag{8}$$

holds.

*Proof:* If Eq.(3) is linearized in a neighbourhood of $N^*$, then we have

$$u_{t+1} = s^* a(N^*)u_t + [s^* a'(N^*)N^* + f'(N^*)]u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial from the last equality will be

$$p(\lambda) = \lambda^2 - s^* a(N^*)\lambda - [s^* a'(N^*)N^* + f'(N^*)].$$

Here, $s^* a(N^*) > 0$ and $\alpha(N^*) < 1$. Ineq. (8) is obtained by using $|p| < 1 - q < 2$ as desired.

**Corollary 5.** *The Allee effect at time $t - 1$ increases the local stability of an equilibrium point of Eq.(1), if*

$$0 < [s^* a'(N^*)N^* + s^* a(N^*)] \leq 1.$$

and also the Allee effect at time $t - 1$ decreases the local stability of an equilibrium point of Eq.(1), if

$$1 < [s^* a'(N^*)N^* + s^* a(N^*)] < 2 + s^* a'(N^*)N^*.$$

*Proof:* The proof is similar to proof of Corollary 3.

**Remark 6. (i)** *The local stability of equilibrium point of Eq.(2) is stronger than the local stability of an equilibrium point of Eq.(3) in terms of reducing impact of stability.*

**(ii)** Let us convert Eq. (1) with $f(N_t) := \frac{1}{N_t + a}$ to the following system without the effect of Allee

$$\begin{aligned} N_{t+1} &= sN_t + \frac{1}{Y_t + a} \\ Y_{t+1} &= N_t. \end{aligned} \tag{9}$$

Then the following numerical simulations are given according to some parameter values.

## *2.1. NUMERICAL SIMULATIONS*

In this section, we graph 2D trajectories of the delay population model in Eq.(1) with and without Allee effect at time $t$ and $t-1$.

**Example 7.** For $s=0.4,\ a=0.75$, the following system

$$N_{t+1}=0.4N_t+\frac{1}{Y_t+0.75}$$
$$Y_{t+1}=N_t \tag{10}$$

has the fixed point $(N^*,Y^*)=(0.969355, 0.969355)$. The stability of the system (10) with initial value $(0.94, 0.98)$ is ensured numerically by calculating the eigen values from the Jacobian matrix which is $|\lambda_{1,2}|=|0.20\pm\ \text{i}\ 0.546144|=0.581613<1$.

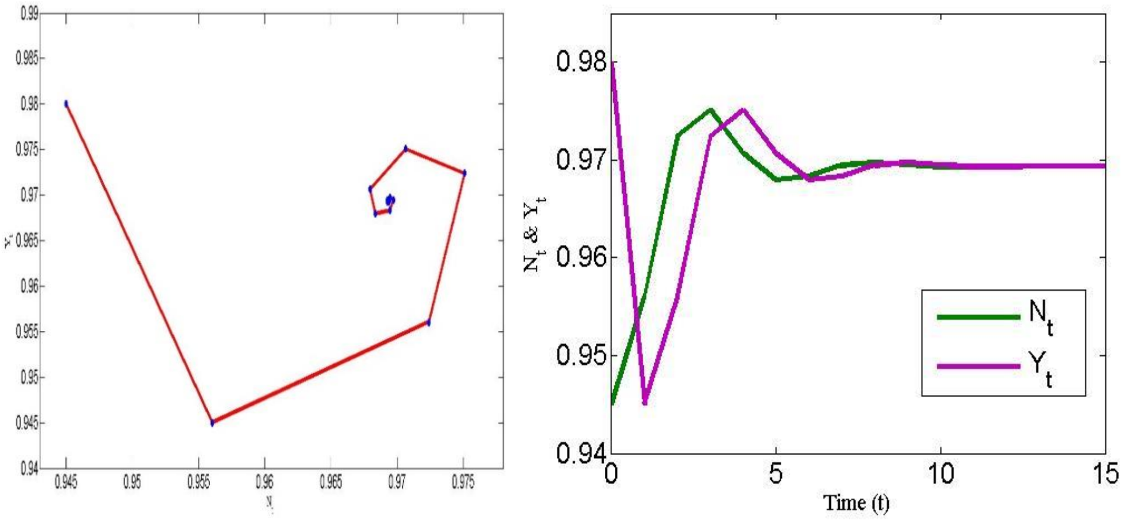


**Figure 1. Phase Portrait and Time Series plot of System (10).**

The numerical calculation is supported by simulation of phase portrait that demonstrates the inward motion of the system (10) towards the equilibrium point and time plots after the initial oscillations attain stability at 0.969355 with increasing time and they are presented in Figure 1.

**Example 8.** With Allee effect at time t, for $b=0.7$ and $s=0.4$, the following system is

$$N_{t+1}=\frac{0.688851722N_t^2}{N_t+0.7}+\frac{1}{Y_t+0.75}$$
$$Y_{t+1}=N_t \tag{11}$$

where $s^*=0.688851722$. Note that $(N^*,Y^*)=(0.969355, 0.969355)$. For the above parameter values and the initial condition $(0.965, 0.96)$, the eigen values are $|\lambda_{1,2}|=|0.283864\pm\ \text{i}\ 0.507636|=0.581613<1$.

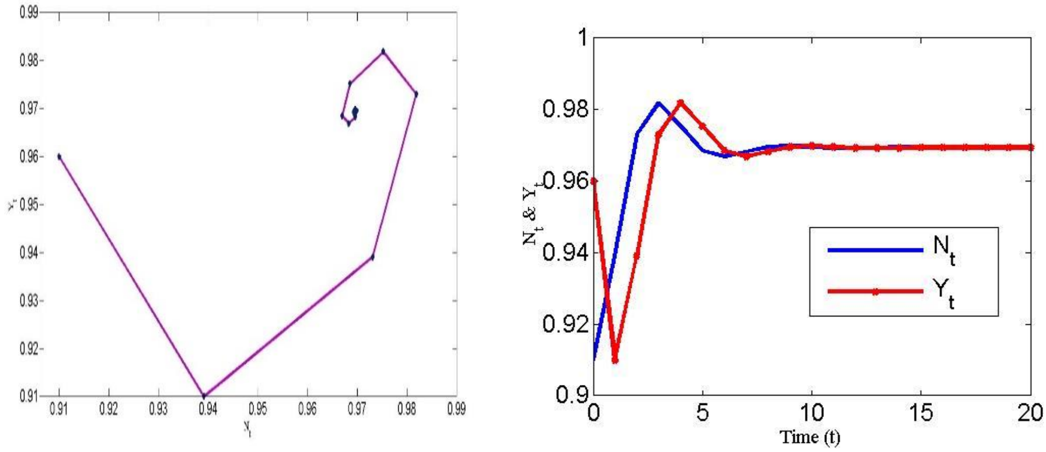


**Figure 2. Phase Portrait and Time Series plot of System (11).**

Figure 2 illustrates the stability of the system (11) considered in the example with effect of Allee. The spiral motion of the system (11) towards the equilibrium point is explained by the phase portrait diagram and respective time series plot is also presented.

**Example 9.** With Allee effect at time t-1 $\alpha(N_{t-1}) = \alpha(Y_t) = \dfrac{Y_t}{Y_t + b}$ for $b = 0.7$ and $s = 0.4$, the following system is

$$
\begin{aligned}
N_{t+1} &= \frac{0.688851722 N_t Y_t}{Y_t + 0.7} + \frac{1}{Y_t + 0.75} \\
Y_{t+1} &= N_t
\end{aligned}
\qquad (12)
$$

where $s^* = 0.688851722$ and $(N^*, Y^*) = (0.969355, 0.969355)$.

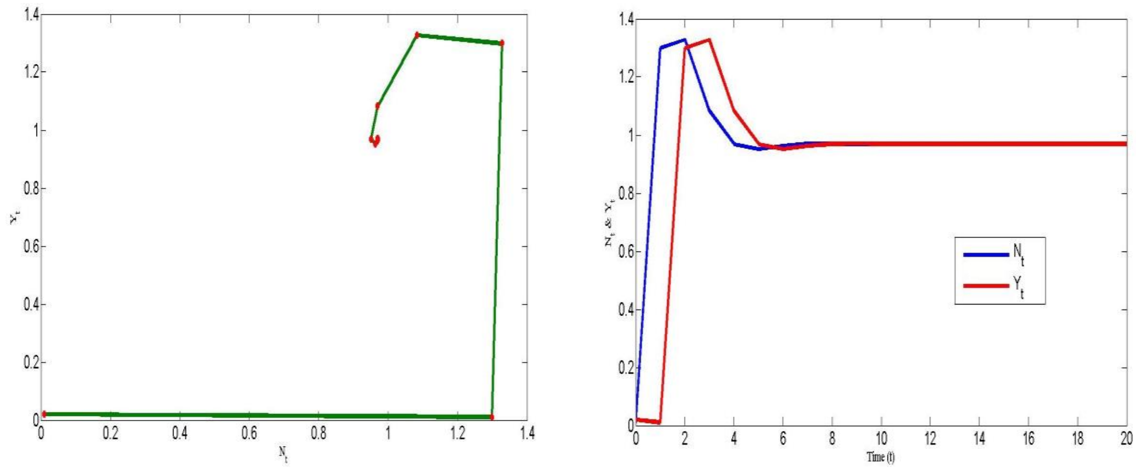


**Figure 3. Phase Portrait and Time Series plot of System (12).**

The eigen values $|\lambda_{1,2}| = |0.19999 \pm 0.361309| = 0.412970 < 1$, obtained for the above considered parameters value and initial condition (0.9, 0.92) for the system (12) confirm the stable behaviour of the system (12) which is well supported by the phase plane diagram and time series plot presented in Figure 3.

**Example 10.** For $s=0.1$ and $s=0.6$ such that $a=b=0.5$, respectively, we have the following systems

$$\begin{aligned} N_{t+1} &= 0.1N_t + \frac{1}{Y_t+0.5} \\ Y_{t+1} &= N_t \end{aligned} \tag{13}$$

such that $(N^*,Y^*)=(0.833333,0.833333)$ , and

$$\begin{aligned} N_{t+1} &= 0.6N_t + \frac{1}{Y_t+0.5} \\ Y_{t+1} &= N_t \end{aligned} \tag{14}$$

such that $(N^*,Y^*)=(1.35078,1.35078)$.

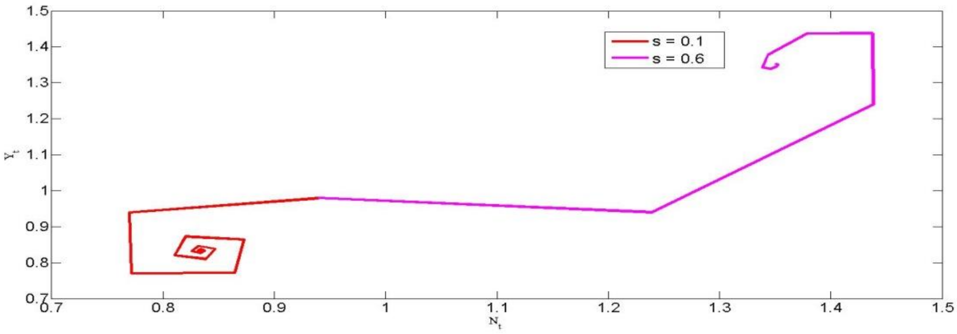


**Figure 5 : Phase Portrait of Systems (13) and (14).**

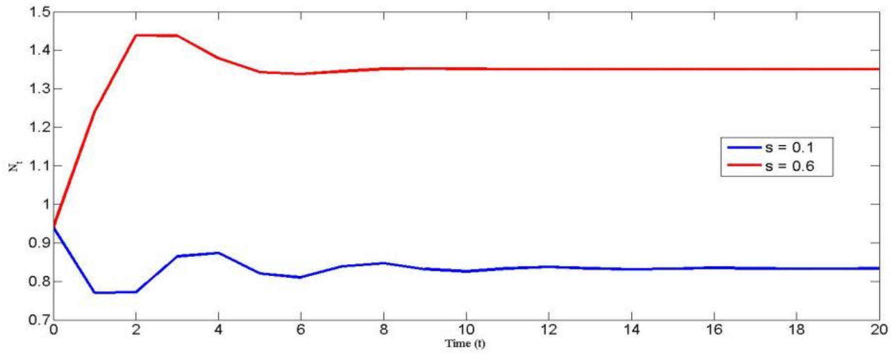


**Figure 6. Time Series plot for $N_t$ of Systems (13) and (14).**

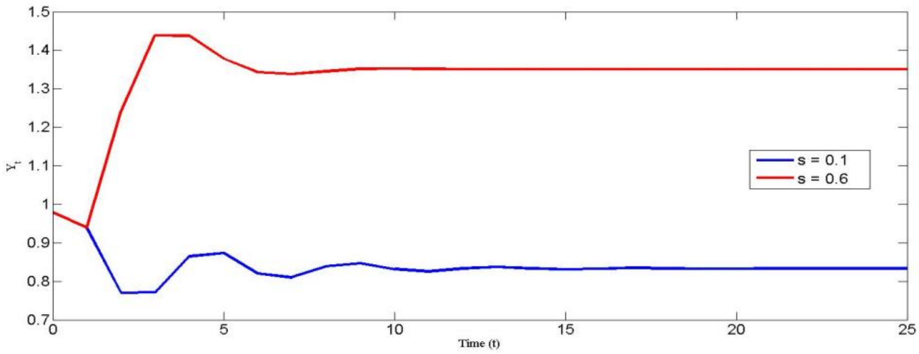


**Figure 7. Time Series plot for $Y_t$ of Systems (13) and (14).**

This example exaplains the impact of the parameter $s$ on the systems (13) and (14) with initial condition $(0.94, 0.98)$. The change in behaviour of the systems (13) and (14) is clearly visible from the phase plane diagram with system (13) for $s = 0.1$ forms a spiral motion towards the fixed point, though system (14) for $s = 0.6$ moves toward the fixed point and the time taken for attaining stability matters. This time factor taken by systems (13) and (14) to attain stability is illustrated by the time series plots presented in Figures 6 and 7. Based on the simulations the following conclusion can be arrived at, smaller the value of $s$ greater the time taken for the systems (13) and (14) to achieve stability.

**Example 11.** The impact of Allee effect and time delayed Allee effect for fixed parameter values are analysed in this example. For the parameters $s = 0.1,\ a = b = 0.5$, with Allee effect at time t, we consider the following system

$$N_{t+1} = \frac{0.160000024 N_t^2}{N_t + 0.5} + \frac{1}{Y_t + 0.5} \qquad (15)$$
$$Y_{t+1} = N_t$$

and with Allee effect at time t-1, the following system is written as

$$N_{t+1} = \frac{0.160000024 N_t Y_t}{Y_t + 0.5} + \frac{1}{Y_t + 0.5} \qquad (16)$$
$$Y_{t+1} = N_t$$

where $(N^*, Y^*) = (0.833333, 0.833333)$ and $s^* = 0.160000024$.

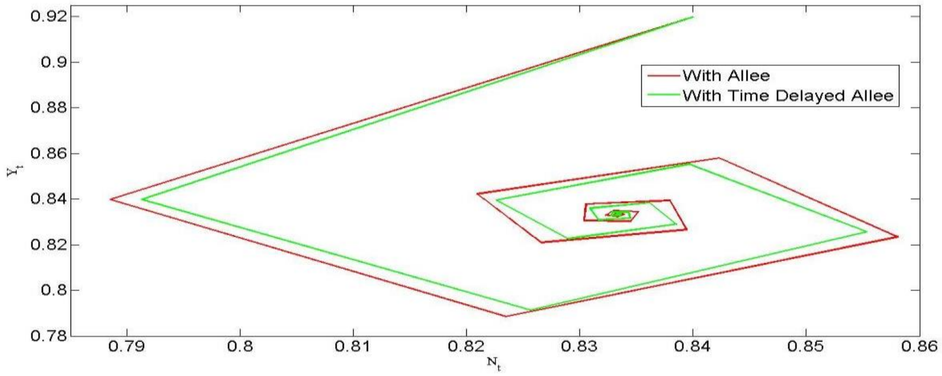


**Figure 8. Phase Portrait of Systems (15) and (16).**

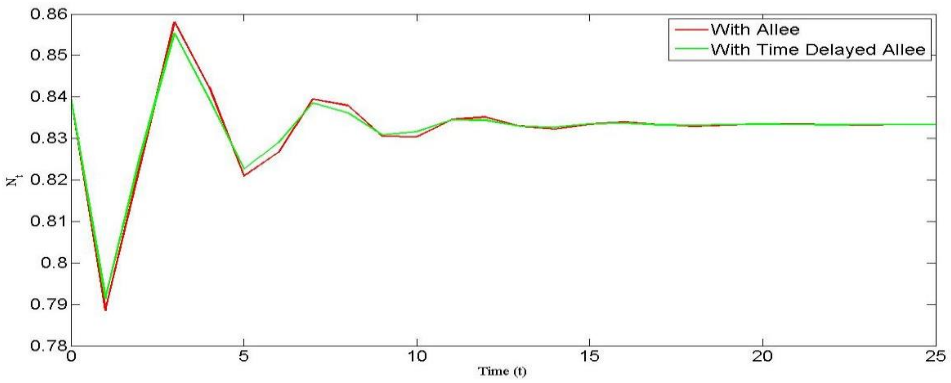


**Figure 9. Time Series plot for $N_t$ of Systems (15) and (16).**

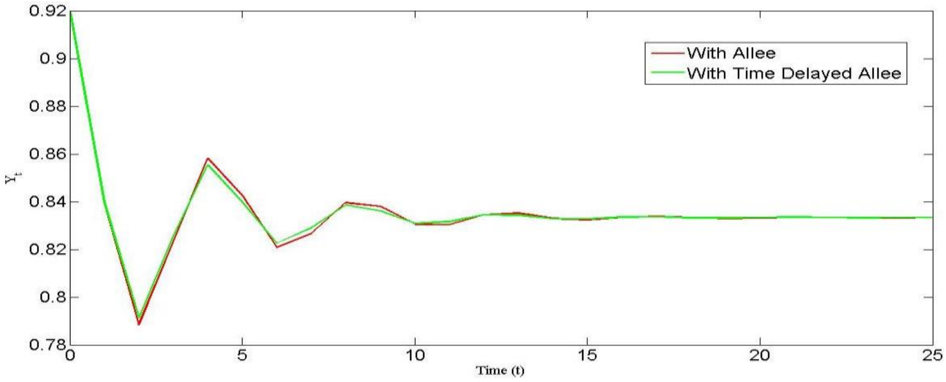


**Figure 10. Time Series plot for $Y_t$ of Systems (15) and (16).**

Impact of introducing the Allee effect at time t in system (15) and Allee effect at time $t-1$ in system (16) at initial values $(0.84, 0.92)$ are explained by the Figures 8, 9 and 10. Observing Figure 8, it is understood that the spiral motion of system (15) towards the fixed takes longer time than of the spiral motion formed by system (16). This is explained with time series plots for $N_t$ and $Y_t$. In order to arrive at general conclusion on the impact of Allee effect at time t and delayed Allee effect at time t-1 for change in values of $s$, we now perform the simulations for value of $s = 0.6$.

Then, with Allee effect at time t, the following system is

$$N_{t+1} = \frac{0.822093902 N_t^2}{N_t + 0.5} + \frac{1}{Y_t + 0.5} \qquad (17)$$
$$Y_{t+1} = N_t$$

and with Allee effect at time t-1, the following system is

$$N_{t+1} = \frac{0.822093902 N_t Y_t}{Y_t + 0.5} + \frac{1}{Y_t + 0.5} \qquad (18)$$
$$Y_{t+1} = N_t$$

where $s^* = 0.822093902$ and $(N^*, Y^*) = (1.35078, 1.35078)$.

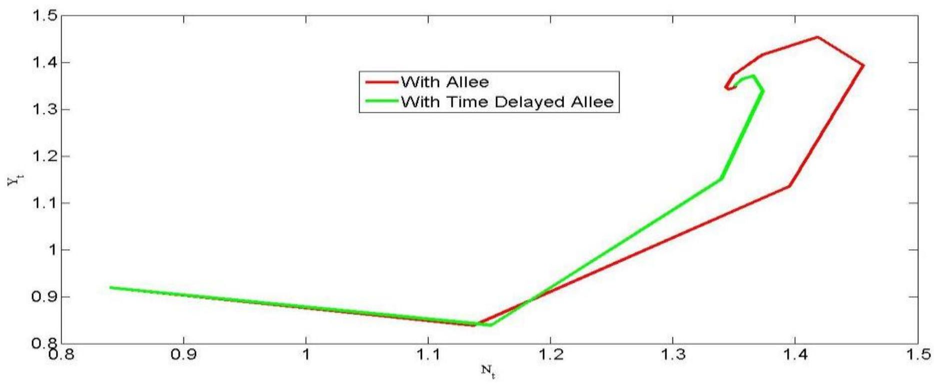


**Figure 11. Phase Portrait of Systems (17) and (18).**

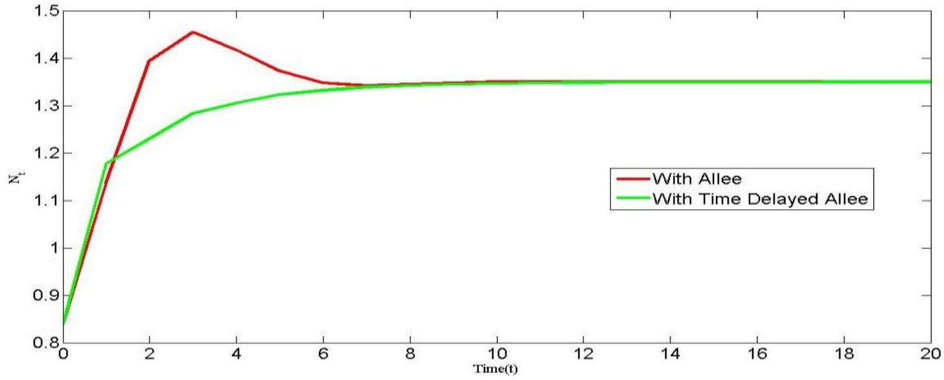


**Figure 12. Time Series plot for $N_t$ of Systems (17) and (18).**

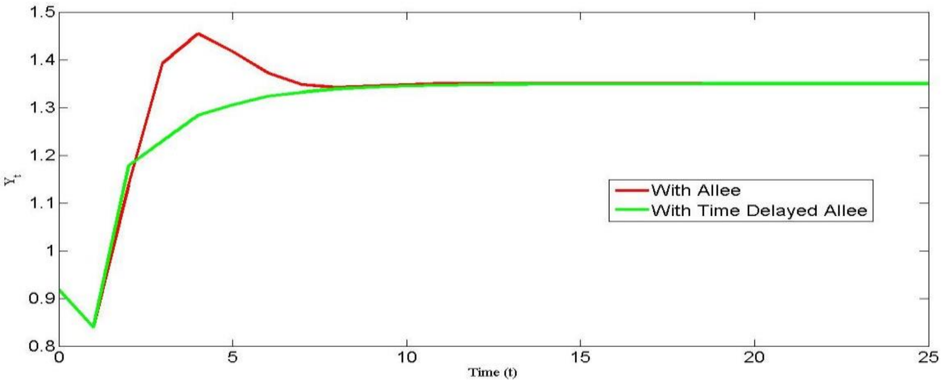


**Figure 13. Time Series plot for $Y_t$ of Systems (17) and (18).**

Comparison of impact of Allee effect at time t in system (17) and Allee effect at time $t-1$ in system (18) at initial values $(0.84, 0.92)$ is provided in Figures 11, 12 and 13. Thus, the comparison of impact of Allee effects at $s = 0.1$ and $s = 0.6$ which yields a similar pattern in terms of phase plane diagrams and time series plots. The introduction of time delayed Allee effect to a system attains stability faster when compared to the system with Allee effect at time t. The initial oscillations that are obtained in the respective time series plots of the systems and the time taken for the oscillations to converge to the fixed point give the required explanation on the impact created by the Allee effects on the systems.

## 3. THE LOCAL STABILITY ANALYSIS OF EQUILIBRIUM POINTS OF THE DELAY POPULATION MODEL INVOLVING HARVESTING WITH AND WITHOUT ALLEE EFFECT

In [29], a model which is often used to study the whale populations was introduced as follows:

$$N_{t+1} = rN_t + N_{t-1}f(N_{t-1}) - eN_t = F(r, e, N_t, N_{t-1}), \tag{19}$$

where $e$ is the harvesting effect and $r, e > 0$ and $r > e$. Here, $N_t$ is the adult breeding populations, $G(N_{t-\tau}) = N_{t-1}f(N_{t-1})$ is the recruitment to the adult stage with delay of $\tau$ years, and $r$ is the survival coefficient such that $r \in (0,1]$. Note that the function $f$ satisfies the general assumptions (i)-(ii).

*3.1. LOCAL STABILITY ANALYSIS OF EQUILIBRIUM POINT OF EQ.(19) WITHOUT ALLEE EFFECT*

In this section, we obtain a condition on the local stability of the positive equilibrium point of Eq.(19). Assume that $F$ has a unique positive equilibrium point $N^*$. So, we can give the following theorem.

**Theorem 12** *Let $N^*$ be a positive equilibrium point of Eq.(19). Then $N^*$ is locally stable if the following inequality holds:*

$$N^* f'(N^*) > r - e - 2. \tag{20}$$

*Proof:* By considering (i)-(ii), we say that $F$ is a continuous function. If Eq.(19) is linearized in neighbourhood of $N^*$, we can write

$$u_{t+1} = [r-e]u_t + [f(N^*) + N^* f'(N^*)]u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial of the last equality will be

$$p(\lambda) = \lambda^2 - [r-e]\lambda - [1 - r + e + N^* f'(N^*)]$$

by using definition of the equilibrium point [2]. If $|p| < 1 - q < 2$ from Jury criterion [2], then $N^*$ is locally stable, and we get

$$r - e - 2 < N^* f'(N^*) < 2(r-e).$$

It is clear that $N^* f'(N^*) < 0$. Then the Ineq.(20) is true.

*3.2. LOCAL STABILITY ANALYSIS OF EQUILIBRIUM POINT OF EQ.(19) WITH ALLEE EFFECT*

In this section, we investigate stability conditions of the positive equilibrium point of Eq.(19) with Allee effect at time $t$ and $t-1$.

**3.2.1. Stability equilibrium point of Eq.(19) with Allee effect at time $t$**

If the delay population model in Eq.(19) is subject to the Allee effect at time $t$, we have the following nonlinear population model

$$N_{t+1} = r^* \alpha(N_t) N_t + N_{t-1} f(N_{t-1}) - eN_t = F_{\alpha,t}(r^*, \alpha, N_t, N_{t-1}), \tag{21}$$

such that $r^* > 0$ and $r^* > e$. Also, the function $f$ satisfies the conditions (i) and (ii). Note that since $r^*$ is the normalized per capita growth rate by $r/\alpha(N^*)$, Eq.(19) and Eq.(21) have the same equilibrium point.

We then obtain the following theorem.

**Theorem 13.** *Let $N^*$ be a positive equilibrium point of Eq.(21) with respect to $r^*$. Then $N^*$ is locally stable if the following inequality holds:*

$$r - e - 2 < f'(N^*)N^* < -\frac{ra'(N^*)N^*}{a(N^*)}. \tag{22}$$

*Proof:* If Eq.(21) is linearized in a neighbourhood of $N^*$, we get

$$u_{t+1} = [r^*a(N^*) + r^*a'(N^*)N^* - e]u_t + [f(N^*) + f'(N^*)N^*]u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial from the last equality will be

$$p(\lambda) = \lambda^2 - [r + \frac{ra'(N^*)N^*}{a(N^*)} - e]\lambda - [1 - r + e + f'(N^*)N^*]$$

by considering definition of the equilibrium point [2] and $r^* = r/a(N^*)$. It is known that $N^*f'(N^*) < 0$. From this, Ineq.(22) is easily seen from Jury criterion.

**Corollary 14.** *The Allee effect at time $t$ decreases the local stability of equilibrium point $N^*$of Eq.(19).*

*Proof:* It is seen from (20) and (22).

### 3.2.2. Stability of Eq.(19) with Allee effect at time $t-1$

Let's add Allee effect at time $t-1$ into Eq.(19). So, we get

$$N_{t+1} = r^*\alpha(N_{t-1})N_t + N_{t-1}f(N_{t-1}) - eN_t = F_{\alpha,t-1}(r^*, \alpha, N_t, N_{t-1}), \tag{23}$$

such that $r^* > 0$ and $r^* > e$. The equilibrium point of Eq.(19) is also an equilibrium point of Eq.(23). Then we can give the following theorem.

**Theorem 15.** *Let $N^*$ be a positive equilibrium point of Eq.(23) with respect to $r^*$. Then $N^*$ is locally stable if the following inequality holds:*

$$r - e - 2 - \frac{ra'(N^*)N^*}{a(N^*)} < N^*f'(N^*) < -\frac{ra'(N^*)N^*}{a(N^*)} \tag{24}$$

*Proof:* If Eq.(23) is linearized in a neighbourhood of $N^*$, then we have

$$u_{t+1} = [r - e]u_t + [r^*a'(N^*)N^* + f(N^*) + N^*f'(N^*)]u_{t-1}$$

such that $u_t = N_t - N^*$. The characteristic polynomial from the last equality will be

$$p(\lambda) = \lambda^2 - [r^*a(N^*) - e]\lambda - [\frac{ra'(N^*)N^*}{a(N^*)} + 1 - r + e + N^*f'(N^*)].$$

By considering definition of the equilibrium point [2] and $r^* = r/a(N^*)$. it is known that $N^*f'(N^*) < 0$. Ineq. (24) is obtained by using Jury criterion as desired.

**Corollary 16.** *The Allee effect at time $t-1$ decreases the local stability of equilibrium point $N^*$of Eq.(19).*

*Proof:* It is seen from (20) and (24).

**Remark 17.** Let us convert the Eq. (19) with $f(N_t) := \frac{1}{N_t + a}$ to following system without the effect of Allee

$$\begin{aligned} N_{t+1} &= rN_t + \frac{Y_t}{Y_t + a} - eN_t \\ Y_{t+1} &= N_t. \end{aligned} \tag{25}$$

### 3.2.3. Numerical Simulations

In this section we elaborate the numerical results on stability of the discrete population model (19) with harvesting and Allee effect. The numerical simualtions with suitable value of parameters are performed and the effect of Allee factor with time (t) and delay (t-1) are presented.

**Example 18.** Let $r = 0.4, a = 0.75, e = 0.2$ be the value of the parameters in the system (25) without Allee effect. The interior equilibrium point of the system for the assumed values is $(0.5, 0.5)$. The mathematical calculations are performed to obtain the eigen values $|\lambda_1| = |-0.6| < 1$ and $|\lambda_2| = |0.8| < 1$ satisfying the Jury condition and confirms the stability of the system (25). The simulations for the system (25) with the above assumed values and initial position $(0.94, 0.98)$ are presented in Figures 14 and 15. Figure 14 explains the time series plot and Figure 15 presents the phase plane diagram ensuring stability.

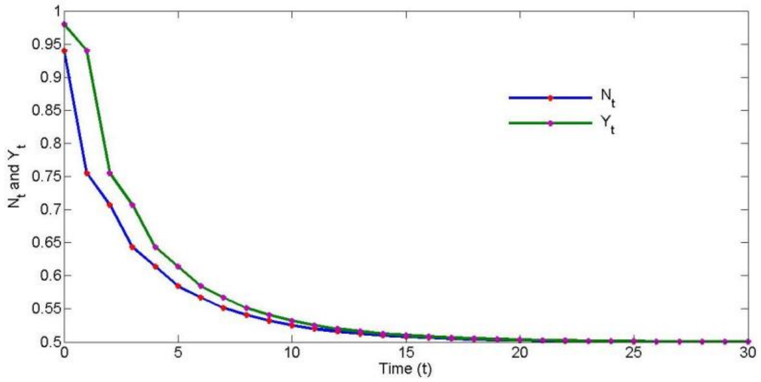


**Figure 14. Time Series plot for $N_t$ and $Y_t$ of System (25).**

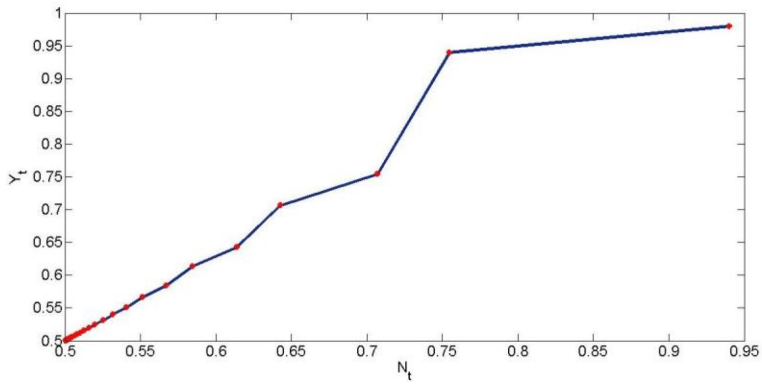


**Figure 15. Phase plane diagram for $N_t$ and $Y_t$ of System (25).**

**Example 19.** Consider the system representation of the Equation (21) with the Allee effect term given by

$$\begin{aligned} N_{t+1} &= \frac{rN_tN_t}{N_t+b}+\frac{Y_t}{Y_t+a}-eN_t \\ Y_{t+1} &= N_t. \end{aligned} \tag{26}$$

The parameters of the system (26) takes the following value as $r = 0.14$, $a = 0.7$, $b = 0.85$, $e = 0.13$. The non trivial equilibrium point for the system (26) is $(0.206957, 0.206957)$ and the eigen values obtained at the non trivial equilibrium point $|\lambda_1| = |-0.963640| < 1$ and $|\lambda_2| = |0.883098| < 1$ follow the Jury condition.

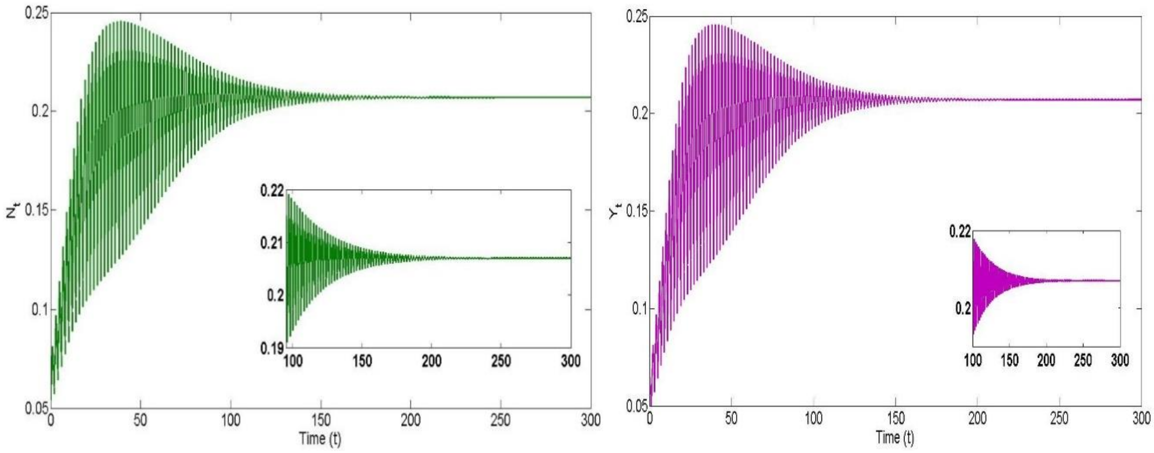

**Figure 16. Time Series plot for $N_t$ and $Y_t$ of System (26).**

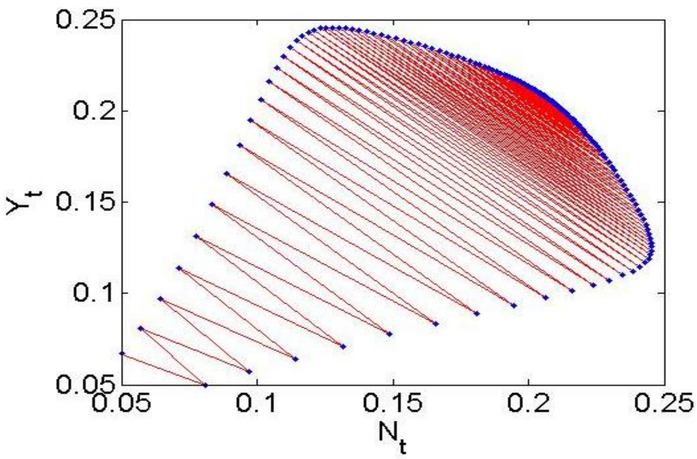

**Figure 17. Phase plane diagram for $N_t$ and $Y_t$ of System (26).**

Thus, the system (26) is stable for the given values and is supported with plots of $N_t$ *and* $Y_t$ against time and phas plane presentation in Figures 16 and 17 respectively.

**Example 20.** Let

$$\begin{aligned} N_{t+1} &= \frac{rN_tY_t}{Y_t+b}+\frac{Y_t}{Y_t+a}-eN_t \\ Y_{t+1} &= N_t. \end{aligned} \tag{27}$$

be the system derived from equation (23). The stability of the system is mathematically evaluated for the following value of paramters $r = 0.2$, $a = 0.5337$, $b = 0.69506$, $e = 0.18$.

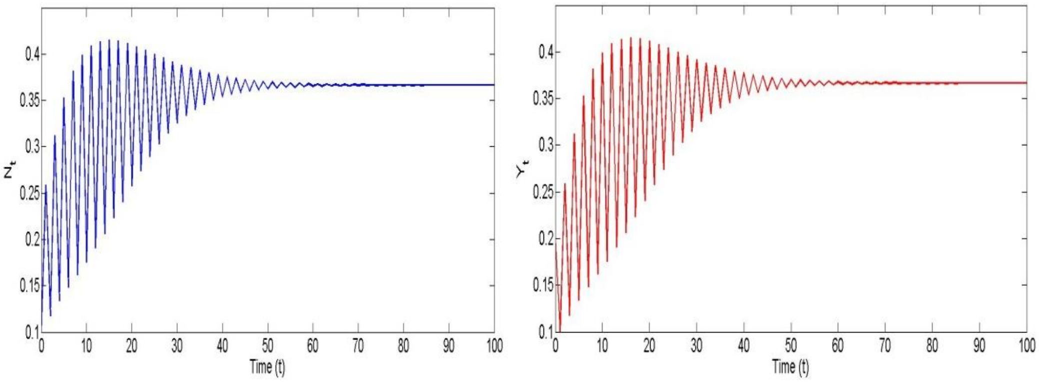


**Figure 18. Time Series plot for $N_t$ and $Y_t$ of System (27).**

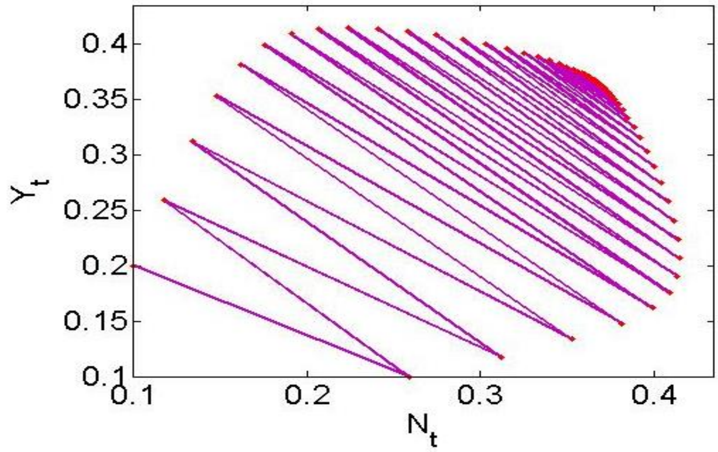


**Figure 19. Phase plane diagram for $N_t$ and $Y_t$ of System (27).**

The eigen values obtained from the Jacobian matrix are $|\lambda_1| = |-0.896310| < 1$ and $|\lambda_2| = |0.785350| < 1$. From Jury condition, it is clear that the system is stable and the motion of the time series plots to the non trivial equilibrium point $(0.366422, 0.366422)$ supports the results obtained by mathematical calculations.

**Example 21.** For $r = 0.4$, $a = 0.75$, $e = 0.2$ and $b = 0.7$ the following system is

$$\begin{aligned} N_{t+1} &= 0.96 N_t \frac{N_t}{N_t + 0.7} + \frac{Y_t}{Y_t + 0.75} - 0.2 N_t \\ Y_{t+1} &= N_t \end{aligned} \tag{28}$$

with Allee effect at time t. The system has the fixed point $(N^*, Y^*) = (0.5, 0.5)$. Initial position is $(0.94, 0.98)$ and $r^* = 0.96$.

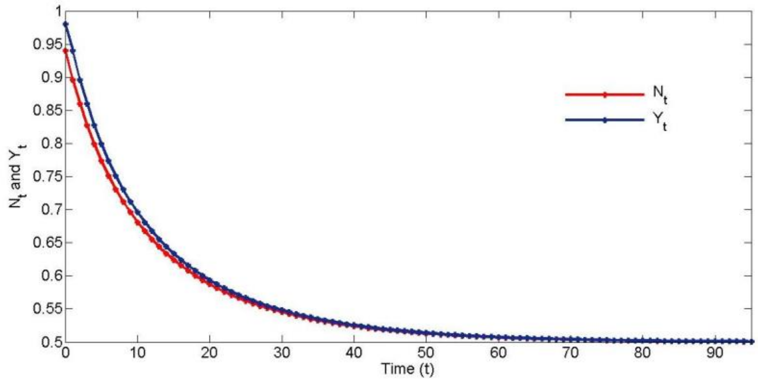


**Figure 20. Time Series plot for $N_t$ and $Y_t$ of System (28).**

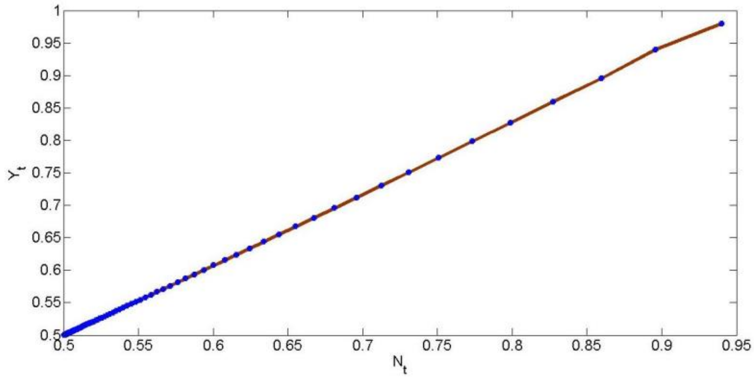


**Figure 21. Phase plane diagram for $N_t$ and $Y_t$ of System (28)**

**Example 22.** For $r = 0.4$, $a = 0.75$, $e = 0.2$ and $b = 0.7$ the following system is

$$\begin{aligned} N_{t+1} &= 0.96 N_t \frac{Y_t}{Y_t + 0.7} + \frac{Y_t}{Y_t + 0.75} - 0.2 N_t \\ Y_{t+1} &= N_t \end{aligned} \tag{29}$$

with Allee effect at time t -1. The system has the fixed point $(N^*, Y^*) = (0.5, 0.5)$. Initial position is $(0.94, 0.98)$ and $r^* = 0.96$.

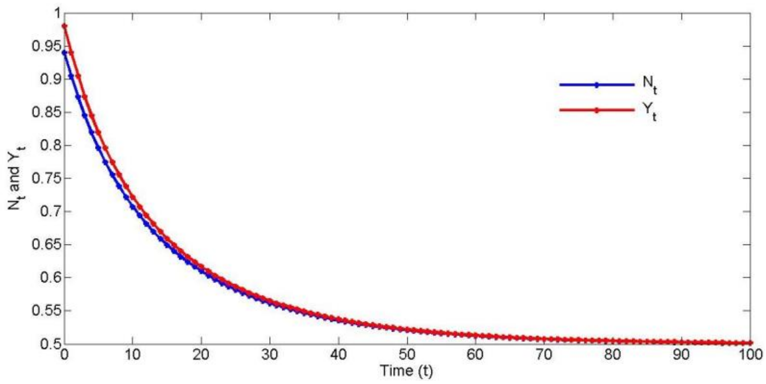


**Figure 22. Time Series plot for $N_t$ and $Y_t$ of System (29).**

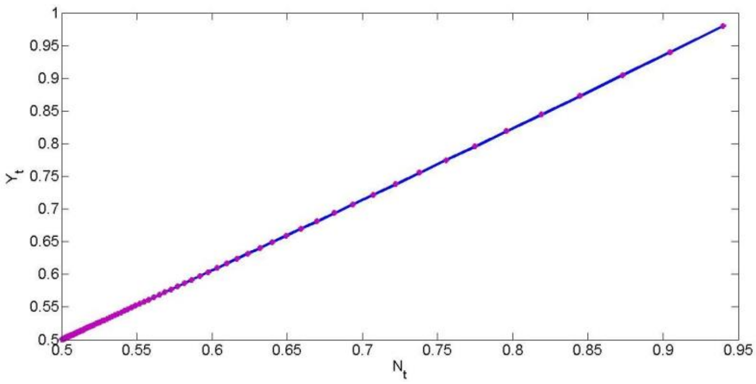


**Figure 23. Phase plane diagram for $N_t$ and $Y_t$ of System (29).**

Phase plane diagrams for the suitable value of the parameters in Examples 18, 19, 20, 21, 22 present the oscillatory behaviour of the system, which after certain period of time move towards the equilibrium point. The evidence of the system approaching equilibrium point can be clearly understood from the time series plots given in Figures 14, 16, 18, 20, 22 which attains the equilibrium point and remains consistent at that particular point for a long time without any change. The impact of the Allee effect on the dynamics can be examined from the Examples 18, 21, 22. The population model with Allee term at time (t) and (t-1) takes much longer time to attain stability. Examples 19 and 20 include the population behavior of the model with Allee factor at time (t) and (t-1) for different parameter values, respectively.

## 4. CONCLUSION

The analysis of stability for a population model with harvesting and Allee effect in discrete time population model are carried out in this article. Mathematical calculations are perfomed considering three cases as model without Allee term and model with presence of Allee term at time (t) and time delayed Allee term. The delayed population model was further analysed with the introduction of harvesting term and the stability results are presented. The numerical evaluations in this articles are supported with suitable simulations.

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