ORIGINAL PAPER

A STUDY OF MAGNETOHYDRODYNAMIC POUISILLE FLOW OF A THIRD-GRADE FLUID IN NON-DARCIAN POROUS PLATES WITH SLIP EFFECT AND HEAT TRANSFER

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Abstract. *In this paper, the MHD Pouisille flow of a third-grade fluid in a non-Darcian porous plate with a slip effect has been studied. The flow flux and the heat transfer analysis were carried out. The governing non-linear ordinary differential equations (ODE) were non-dimensionalize and the resulting ODE is solved by perturbation techniques. The result obtained was illustrated and presented in form of graphs. The effect of various governing flow parameters on velocity and temperature profile was taken into account. It is found that the effect of cross Renold's number, the porosity of the medium, and the Magnetic parameter are to slow down the velocity profile of the fluid while it increases with an increase in Darcy's number and slip parameter. Also, it is noticed that the temperature profile of the fluid increases with the increase in the value of the Prandtl number.*

Keywords: *porous medium; MHD Pouisille flow; third-grade fluid; slip effects; heat transfer*.

1. INTRODUCTION

The flow of non-Newtonian fluid has caught the attention of researchers because of its industrial and technological application. Specifically, the MHD flow of a third-grade fluid filling the porous space in a channel with slip effect has a wider application in soil physics, metal casting, biological fluid, geophysics, hydrology, and petroleum engineering.

Several constitutive equations have been suggested to model such flow. Since this study describes the flow of third-grade fluid in a porous channel, a modified Darcy law is used to show the relationship between the directions of the local pressure gradient to the superficial velocity of the fluid. This was done through fluid viscosity and channel permeability.

Some great significant literature that deals with the flow of third-grade fluid in a porous channel are identified [1] in which the unsteady flow of a third-grade fluid in a porous space is analyzed using modified Darcy law and solution using similarity transformation with a numerical method is obtained. MHD flow and heat transfer of a third-grade fluid in a non-Darcian horizontal porous channel is studied [2] and the flow along its direction is induced by the constant pressure gradient and a uniform magnetic field acted perpendicular to the flow direction.

The influence of slip boundary condition on unsteady MHD flow of Newtonian fluid induced by an accelerated plate is studied by applied the Laplace transformation method to

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obtained exact solutions for constant and variable accelerated flow [3]. In [4] the influence of third grade, partial slip, and other parameters on the steady flow and mass transfer of third-grade fluid in an infinite vertical insulated porous plate is investigated by employ a numerical scheme with Richardson's extrapolation to solve the governing system that arose from the problem. Thermodynamics analysis of hydromagnetic third-grade fluid flow through a channel filled with a porous medium was studied in [5] by assumed that the flow is induced by a constant pressure gradient and uniform magnetic field.

In all the above works of literature, the effects of the slip on the MHD flow of a third-grade fluid in a porous have not much be considered. The present study is to investigate MHD Pouisille flow of a third-grade fluid in a non-Darcian porous plate with a slip effect. Also, the heat transfer analysis of the problem was carried out. The governing non-linear ordinary differential equations (ODE) were non-dimensionalize and the resulting ODE is solved by perturbation techniques. The result obtained was illustrated and presented in form of a graph. The effects of various governing flow parameters on velocity and temperature profile were discussed.

2. MATERIALS AND METHODS

2.1. BASIC EQUATIONS

The equation that governing MHD flow of incompressible fluid are given by continuity equation

$$\nabla \cdot \vec{u} = 0 \tag{1}$$

Momentum equation

$$\rho \frac{D\vec{u}}{Dt} = \nabla \cdot \vec{\tau} + \vec{J} \times \vec{B} \tag{2}$$

where $\vec{J} = \sigma(\vec{u} \times \vec{B} + \vec{E})$ and energy equation

$$\rho C_p \frac{D\vec{T}}{Dt} = K\nabla^2 \vec{T} + \vec{\tau} \cdot \nabla \vec{u} \tag{3}$$

where $\vec{u}$ is the velocity of the fluid, T represent fluid temperature, ρ is the fluid density, $\vec{J}$ represent the current density, $\vec{B} = \vec{B}_0 + \vec{b}$ is the magnetic induction in which $\vec{B}_0$ and $\vec{b}$ are applied and induced magnetic fields respectively. $\vec{E} = 0$ is the electric field, C_p is the specific heat capacity, K represent the thermal conductivity while $\vec{\tau}$ is the Cauchy stress tensor and for third grade fluid, it is given as

$$\vec{\tau} = -\vec{P}I + \sum_{i=1}^{3} C_i \tag{4}$$

where $c_1 = \mu A_1$, $c_2 = \alpha_1 A_2 + \alpha_2 A_1^2$, $c_3 = \beta_1 A_3 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta_3 (trA_2) A_1$,

μ, α_i and β_j are material constant $(i = 1,2 \text{ and } j = 1,2,3)$, μ is the dynamic viscosity, I represent the identity tensor and $A_i\,(i = 1,2,3)$ are called Rivlin Ericksen tensors given by

$$A_0 = 1\,,\ \ A_1 = \nabla \vec{u} + (\nabla \vec{u})^T\,,\ \ \ A_n = \left(\frac{\partial}{\partial t} + (\nabla \vec{u})\right) A_{n-1} + (\nabla \vec{u})^T A_{n-1},\ \ \ n \geq 2$$

From the reference [2], we have the following

$$\mu \geq 0,\ \alpha_1 = 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1, \beta_2 = 0,\ \beta_3 \geq 0$$

Since the flow of the fluid is considered in porous medium, the modified Darcian law was implemented and the following term is added to Cauchy's momentum equation

$$N = -\frac{\varphi}{\kappa}\left[\mu + 2\beta_3 \left(\frac{d\vec{u}}{dy}\right)^2\right]\vec{u} \tag{5}$$

where ϕ and κ are porosity and intrinsic permeability of the porous medium

2.2. FORMULATION OF THE PROBLEM

We considered the steady MHD Pouiselle flow of a third-grade fluid between two parallel rigid porous plates distant apart. The lower plate (at) and the upper plate (at) are fixed. The flow is set into motion by a constant pressure gradient in the flow direction and a uniform magnetic field applied perpendicular to the direction of the flow. The slip condition is considered in terms of shear stress. The two plates are kept at a constant temperature with the velocity field for the flow to be $\vec{u} = (\vec{u}(y), 0, 0)$. Since the continuity equation is satisfied, then equation (1) - (5) combined gives the governing equation of the flow as:

$$\mu \frac{d^2\vec{u}}{dy^2} - \rho v_0 \frac{d\vec{u}}{dy} + 6\beta_3 \left(\frac{d\vec{u}}{dy}\right)^2 \frac{d^2\vec{u}}{dy^2} - \frac{\phi}{\kappa}\left[\mu + 2\beta_3\left(\frac{d\vec{u}}{dy}\right)^2\right]\vec{u} - \sigma B_0^2 \vec{u} = \frac{d\vec{P}}{dx} \tag{6}$$

where the modified pressure is

$$\vec{P} = p - (2\alpha_1 + \alpha_2)\left(\frac{d\vec{u}}{dy}\right)^2$$

$$\frac{d\vec{P}}{dy} = 0$$

The suitable boundary conditions are

$$\vec{u} - \gamma\left[\frac{d\vec{u}}{dy} + \frac{2\beta_3}{\mu}\left(\frac{d\vec{u}}{dy}\right)^3\right] = 0 \ \text{ at } \ y = 0 \tag{7}$$

$$\vec{u}+\gamma\left[\frac{d\vec{u}}{dy}+\frac{2\beta_3}{\mu}\left(\frac{d\vec{u}}{dy}\right)^3\right]=0 \text{ at } y=h \tag{8}$$

γ = 0 implies no slip condition.

$$k\frac{d^2\vec{T}}{dy^2}-\rho c_p v_0\frac{d\vec{T}}{dy}+\mu\left(\frac{d\vec{u}}{dy}\right)^2+2\beta_3\left(\frac{d\vec{u}}{dy}\right)^4+\frac{\phi}{\kappa}\left[\mu+2\beta_3\left(\frac{d\vec{u}}{dy}\right)^2\right]\vec{u}^2+\sigma B_0^2\vec{u}^2=0 \tag{9}$$

with

$$\vec{T}=0 \quad \text{at } y=0 \tag{10}$$

$$\vec{T}=T_w \quad \text{at} \quad y=h \tag{11}$$

To perform the non-dimensional analysis, we introduced the following non-dimensional quantities and variables

$$\vec{x}=\frac{x}{h},\ \vec{y}=\frac{y}{h},\ \vec{u}=\frac{u}{U},\ \vec{T}=\frac{T}{T_w},\ \bar{\gamma}=\frac{\gamma}{h}, \tag{12}$$

Substituting these into equation (6) – (11) and remove the "- ", we obtain

$$\frac{d^2u}{dy^2}-R_e\frac{du}{dy}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-\left[\frac{\phi}{D_a}\left[1+2\beta\left(\frac{du}{dy}\right)^2\right]+M\right]u+P=0 \tag{13}$$

$$y=0: \qquad u-\gamma\left[\frac{du}{dy}+\frac{2\beta_3}{\mu}\left(\frac{du}{dy}\right)^3\right]=0 \tag{14}$$

$$y=1: \qquad u+\gamma\left[\frac{du}{dy}+\frac{2\beta_3}{\mu}\left(\frac{du}{dy}\right)^3\right]=0 \tag{15}$$

$$\frac{d^2T}{dy^2}-P_rR_e\frac{dT}{dy}+B_r\left[\left(\frac{du}{dy}\right)^2+2\beta\left(\frac{du}{dy}\right)^4+\left[\frac{\phi}{D_a}\left[1+2\beta\left(\frac{du}{dy}\right)^2\right]+M\right]u^2\right]=0 \tag{16}$$

$$y=0: \quad T=0, \qquad y=1: \quad T=1 \tag{17}$$

where, $\beta=\frac{6\beta_3U^2}{\mu h^2}$ -Third grade parameter, $\mathrm{Re}l=\frac{\rho h v_0}{\mu}$ -Cross Renold's number, $D_a=\frac{\kappa}{h^2}$ -Darcy's number, $M=B_0h\sqrt{\frac{\sigma}{\mu}}$ -Hatmann number, $P=\frac{-h^2}{\mu}\frac{d\vec{P}}{dx}$ -Dimensionless Pressure gradient, $P_r=\frac{\mu c_p}{k}$, -Prandtl number, $B_r=\frac{\mu}{kT_w}$ -Brinkman number.

2.3. SOLUTION OF THE PROBLEM

In order to solve equation (13) and (16) with corresponding boundary condition (14), (15) and (17), we use the perturbation techniques

$$u(y) = u_0(y) + \beta u_1(y) + \mathrm{O}(\beta^2) \tag{18}$$

$$T(y) = T_0(y) + \beta T_1(y) + \mathrm{O}(\beta^2) \tag{19}$$

Substituting equation (18) and (19) into equation (13) – (17), collecting the like term base on the power of β and neglecting term in $+\mathrm{O}(\beta^2)$ to obtain the following equation for u_0, u_1, T_0 and T_1.

The Zeroth-order Equation

$$\frac{d^2u_0}{dy^2} - R_e\frac{du_0}{dy} - \left[\frac{\phi}{D_a} + M\right]u_0 + P = 0 \tag{20}$$

$$y = 0: \qquad u_0 - \gamma\frac{du_0}{dy} = 0 \tag{21}$$

$$y = 1: \qquad u_0 + \gamma\frac{du_0}{dy} = 0 \tag{22}$$

The solution of equation (20) with boundary condition (21) and (22) is obtain as

$$u_0 = e^{a_1 y}c_1 + e^{a_2 y}c_2 + CDa\left(\phi + MDa\right)^{-1} \tag{23}$$

The First-order Equation

$$\frac{d^2u_1}{dy^2} - R_e\frac{du_1}{dy} - \left[\frac{\phi}{D_a} + M\right]u_1 + 6\left(\frac{du_0}{dy}\right)^2\frac{d^2u_0}{dy^2} - 2\frac{\phi}{D_a}\left(\frac{du_0}{dy}\right)^2 u_0 = 0 \tag{24}$$

$$y = 0: \qquad u_1 - \gamma\frac{du_1}{dy} - 2\gamma\left(\frac{du_0}{dy}\right)^3 = 0 \tag{25}$$

$$y = 1: \qquad u_1 + \gamma\frac{du_1}{dy} + 2\gamma\left(\frac{du_0}{dy}\right)^3 = 0 \tag{26}$$

The solution of equation (24) with boundary condition (25) and (26) is given by

$$\begin{aligned} u_1 = {} & c_3 e^{a_1 y} + c_4 e^{a_2 y} + a_{23}e^{(a_1+2a_2)y} - a_{24}e^{(2a_1+a_2)y} - a_{25}e^{(a_1+2a_2)y} \\ & - a_{26}e^{2a_1 y} - a_{27}e^{3a_1 y} - a_{28}e^{2a_2 y} - a_{29}e^{3a_2 y} \end{aligned} \tag{27}$$

The Zeroth-order Equation

$$\frac{d^2T_0}{dy^2} - P_r R_e \frac{dT_0}{dy} + B_r\left[\left(\frac{du_0}{dy}\right)^2 + \left[\frac{\phi}{D_a} + M\right]u_0{}^2\right] = 0 \tag{28}$$

$$y = 0: \qquad T_0 = 0, \qquad\qquad y = 1: T_0 = 1 \tag{29}$$

The solution of equation (28) with boundary condition (29) is obtain as

$$T_0 = a_{47}e^{2a_1 y} + a_{48}e^{a_2 y} + a_{52}e^{(a_1+a_2)y} + a_{53}y + c_5 a_{51}e^{\Pr \mathrm{Re} ly} + a_{49}e^{2a_2 y} + a_{50}e^{a_2 y} + c_6 \tag{30}$$

The First-order Equation

$$\frac{d^2T_1}{dy^2} - P_r R_e \frac{dT_1}{dy} + 2B_r\left[\frac{du_0}{dy}\frac{du_1}{dy} + \left(\frac{du_0}{dy}\right)^4 + \frac{\phi}{D_a}\left(\frac{du_0}{dy}\right)^2 u_0{}^2 + \left(\frac{\phi}{D_a} + M\right)u_0 u_1\right] = 0 \tag{31}$$

$$y = 0: \qquad T_0 = 0, \qquad\qquad y = 1: T_0 = 0 \tag{32}$$

The solution of equation (31) with boundary condition (32) is given by

$$\begin{aligned} T_1 &= a_{95}e^{4a_2 y} + a_{96}e^{(3a_2+a_1)y} + a_{97}e^{3a_2 y} + a_{98}e^{2(a_2+a_1)y} + a_{99}e^{(2a_2+a_1)y} \\ &+ a_{100}e^{2a_2 y} + a_{101}e^{(a_2+a_1)y} + a_{102}e^{(a_2+2a_1)y} + a_{103}e^{(a_2+3a_1)y} + a_{104}e^{a_2 y} \\ &+ a_{105}e^{4a_1 y} + a_{106}e^{3a_1 y} + a_{107}e^{a_1 y} + a_{108}e^{2a_1 y} + c_7 a_{109}e^{\Pr \mathrm{Re} ly} + c_7 \end{aligned} \tag{33}$$

Finally using (23), (27), (30) and (33) with (18) and (19), we obtained the needed solution. The flow flux of the fluid within the channel is given as

$$Q = \int_0^1 u\, dy \tag{34}$$

while rate of heat transfer is given in terms of Nusselt number (N_u) at the surface of the plates is given as

$$N_u = \left[\frac{dT}{dy}\right]_{y=0} \text{ and } N_u = \left[\frac{dT}{dy}\right]_{y=1} \tag{35}$$

3. RESULTS AND DISCUSSION

In Fig. 1, the effect of Cross Renold's number (Rel) on the flow of the fluid was illustrated. It is observed that an increase in Rel decreases the velocity of the fluid. This implication of this phenomenon is that the flow behaves like laminar at the initial stage and turbulent as the Rel increases due to the increase in loss of viscosity.

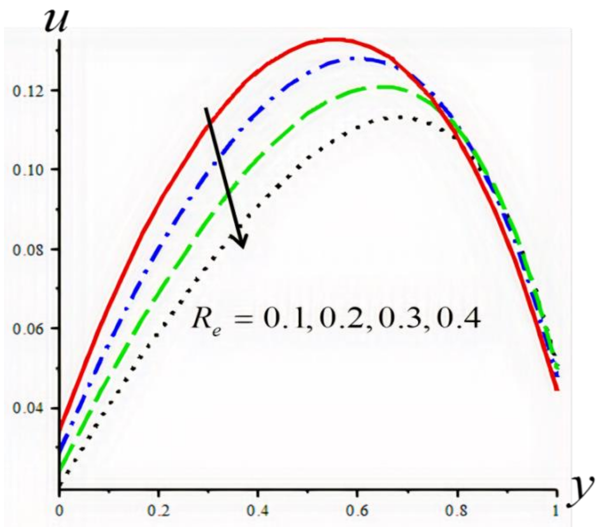

Figure 1. Influence of Reynold's number (Re) on the velocity profile when $\beta = 0.001, \phi = 0.5,$ $Da = 0.5, P = 1, M = 1$ **and** $\gamma = 0.1$

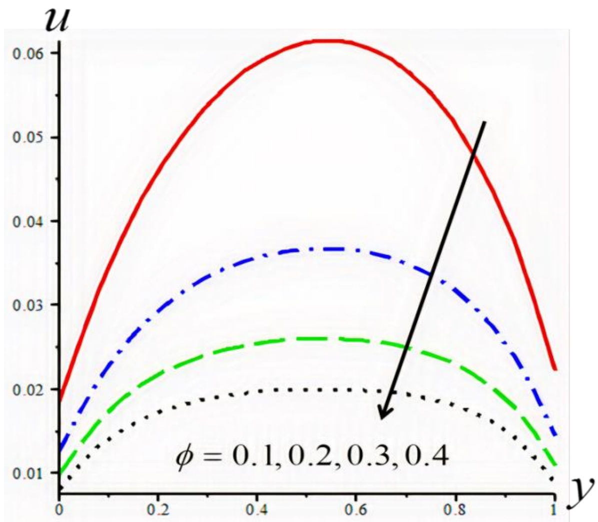

Figure 2. Influence of Porosity of medium (ϕ **) on the velocity profile when** $\beta = 0.001, \mathrm{Re}\, l = 1,$ $D_a = 0.5, P = 1, M = 1$ **and** $\gamma = 0.1$

Fig. 2 shows the effect of porosity of the medium (ϕ) on the velocity profile. It is observed that increase ϕ reduces the fluid velocity along the surface of the plates due to the increase in the thickness of the boundary layer.

Increase Darcy's number (Da) increases the flow of the fluid as illustrated in Fig. 3. This means that increase in the porous permeability of the plates show that the homogeneity of the medium can be reached. The influence of Magnetic parameter (M) on the velocity of fluid is graphically represented in Fig. 4. It is found that increase in M reduce the fluid velocity. This is due to the presence of magnetic field (since the fluid is electrically conductive) which introduces Lorentz force and then acts perpendicular to the direction of the fluid thereby slow down the velocity of the fluid.

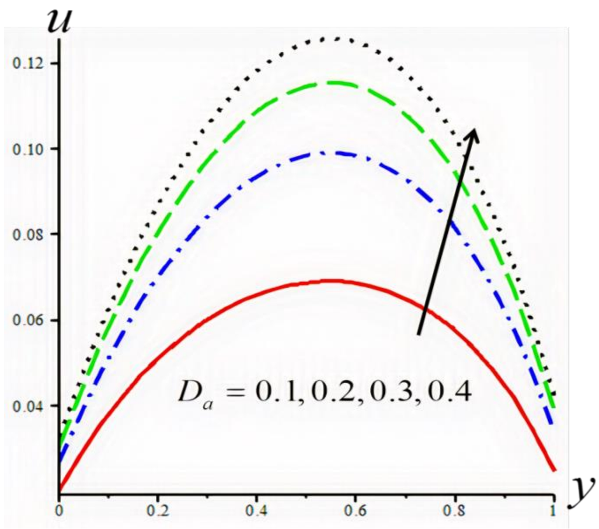

Figure 3. Influence of Darcy's number (D_a) on the velocity profile when $\beta = 0.001, \phi = 1, M = 1,$ $\mathrm{Re}\,l = 1, P = 1$ **and** $\gamma = 0.1$

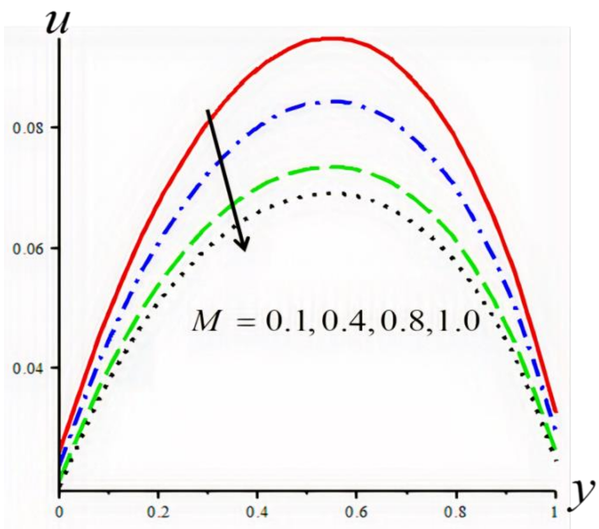

Figure 4. Influence of Hartmann number (M) on the velocity profile when $\beta = 0.001, \phi = 1, D_a = 0.1,$ $\mathrm{Re}\,l = 1, P = 1$ **and** $\gamma = 0.1$

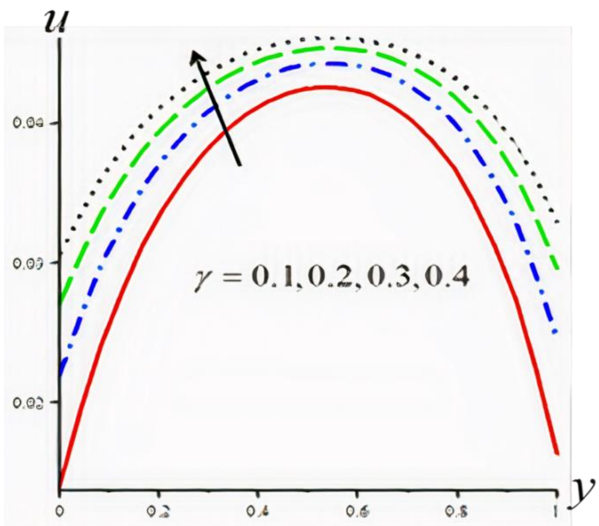

Figure 5. Influence of Slip parameter (γ) on the velocity profile when $\beta = 0.001, \phi = 1, D_a = 0.1,$ $\mathrm{Re}\,l = 1, P = 1$ **and** $M = 1$

Fig. 5 is plotted to shows the impact of slip parameter (γ) on the fluid flow. It is found that as γ increases, there is corresponding increase in the velocity of the fluid due to the decrease in shearing force which also driving the flow.

As Prandtl number (Pr) increases, the fluid temperature increases as presented in Fig. 6 due to increase in the rate of viscous diffusivity.

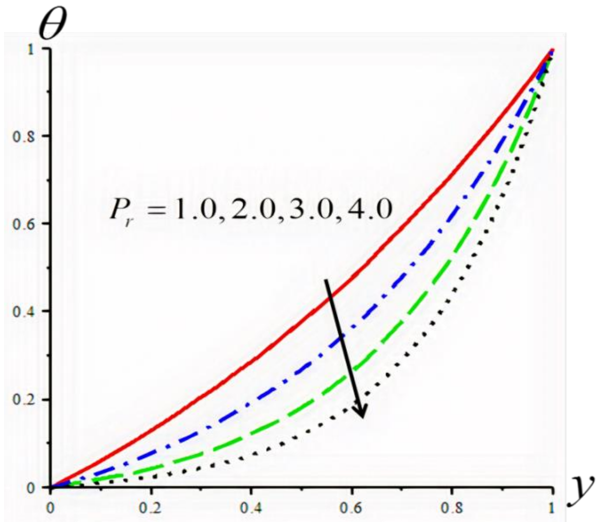

Figure 6. Influence of Prandtl number (Pr) on the velocity profile when $\beta = 0.001, \phi = 1, \gamma = 0.1$ $D_a = 0.5,\ \mathrm{Re}l = 1, P = 1,\ M = 5,$ **and** $Br = 0.07$

Table 1. Influence of parameters M and γ **on the flow flux and Nusselt's number**

$\beta = 0.001,\ \phi = P = \mathrm{Re} = 1,\ Da = \gamma = 0.1,\ \mathrm{Pr} = 0.7$				$\beta = 0.001,\ \phi = P = \mathrm{Re} = M = 1,\ Da = 0.1,\ \mathrm{Pr} = 0.7$			
M	Q	$N_u(0)$	$N_u(1)$	γ	Q	$N_u(0)$	$N_u(1)$
5	0. 0424031	0. 691653	1. 389147	0.1	0. 0516325	0. 691876	1. 388915
10	0. 0346844	0. 691459	1. 389359	0.2	0. 0590177	0. 692002	1. 388724
15	0. 0293609	0. 691322	1. 389513	0.3	0. 0640533	0. 692131	1. 388551
20	0. 0254657	0. 691219	1. 389631	0.4	0. 0677107	0. 692245	1. 388400

Table 2. Influence of parameters Da and ϕ **on the flow flux and Nusselt's number**

$\beta = 0.001,\ \phi = P = \mathrm{Re} = M = 1,\ \gamma = 0.1,\ \mathrm{Pr} = 0.7$				$\beta = 0.001,\ P = \mathrm{Re} = M = 1,\ Da = \gamma = 0.1,\ \mathrm{Pr} = 0.7$			
Da	Q	$N_u(0)$	$N_u(1)$	ϕ	Q	$N_u(0)$	$N_u(1)$
0.1	0. 0516325	0. 691876	1. 388915	1	0. 0516325	0. 691876	1. 388915
0.2	0. 0710527	0. 692309	1. 388488	2	0. 0334689	0. 691428	1. 389397
0.3	0. 081281	0. 692518	1. 388298	3	0. 0248086	0. 691201	1. 389651
0.4	0. 087595	0. 692639	1. 388194	4	0. 0197327	0. 691065	1. 389811

The influence of Hartmann's number, slip parameter, Darcy's number and porosity of the medium on the volumetric rate of flow (i.e. flow flux) and heat transfer (i.e. Nusselt's number) are computed and shown in the tables above. It is noticed that the flow flux reduces with increase in Hartmann's number and porosity of the medium while slightly increases with increase in slip parameter and Darcy's number. Also, from the tables, it is discovered that the rate of heat transfer increases from the lower plate to the upper plate as magnetic field and porosity of the medium increase while there exists a reduction (as shown in Tables 1-2) in heat transfer from lower plate to the upper plate as the value of slip parameter and Darcy's number increase.

4. CONCLUSIONS

The MHD Pouiselle flow of a third-grade fluid in non-Darcian porous plates and slip effect has been studied with volumetric flow flux and heat transfer analysis being carried out. It is observed that the increase in Cross Reynold's number, the porosity of the medium, and the Magnetic parameter decelerate the velocity of the fluid. It is also showed that the velocity profile of the fluid increases with an increase in Darcy's number and slip parameter. The increase in the value of the Prandtl number increases the temperature profile of the fluid. It is illustrated that the axial velocity and temperature of the fluid in the non-Darcian porous plates reach its highest value at the centerline and reduces gradually as its approaches the plates.

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