ORIGINAL PAPER

AN INTEGRO DIFFERENTIAL LANE EMDEN EQUATION INVOLVING THREE PHI-CAPUTO DERIVATIVES

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Abstract. *We study an integro differential problem of Lane-Emden type that involves three phi Caputo derivatives. We begin by proving an existence results by means of Schauder theorem. Then, we investigate the niqueness of solution using Banach contraction principle. At the end, one example is discussed.*

Keywords: *Lane-Emden type* equation; φ-Caputo derivative; existence of solution.

1. INTRODUCTION

The fractional calculus has shown to be extremely useful in a variety of scientific domains, as evidenced by [1-6]. However, the majority of these studies have taken into account the fractonal techniques of Riemann-Liouville, Hadamard, Katugampola, and Caputo. However, the approach of "functions with regard to another function" appears to be missing in the studies mentioned above. [7] is an example of such an approach; it differs from the others in that its kernel is treated as a term of another function φ. Some recent results relating to this approach have been discussed in [8-13]. The Schauder, Krasnoselskii, Darbo, or Monch theories have been employed in the majority of the current works to verify the existence of solutions to nonlinear fractional differential equations with some restrictive conditions [14-16]. Bahous and Dahmani [17] have considered a Lane Emden type problem involving both Caputo derivative and Riemann-Liouville integral in its nonlinearities. The problem is the following:

$$\begin{cases} \mathtt{D}^{\beta}\left(\mathtt{D}^{\alpha}+\frac{\mu}{t^{\lambda}}\right)y(t)+f\left(t,y(t),\mathtt{D}^{\delta}y(t)\right)+g(t,y(t),\mathtt{I}^{\rho}y(t))=h(t),t\in(0,1), \\ \qquad y(0)=v,\, y(1)=b\int_0^{\eta}q(s)y(s)ds, 0<\alpha,\beta,\eta\le 1, \mu>0, \lambda>0, \end{cases}$$

where $\mathtt{D}^{\alpha}$ is in the sense of Caputo, $\mathtt{I}^{\rho}$ is the Riemann-Liouville integral of order ρ, the functions $g,f\ :[0,1]\times R\times R\to R$ are continuous and h,q are continuous over $[0,1]$. The authors have investigated the existence and uniqueness of solutions. Then, they have studied the Ulam-Hyers stability.

Gouari et al. [18] have proposed the following nonlinear singular integrodifferential equation of Lane Emden type with nonlocal multi point integral conditions:

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$$\begin{cases} \mathrm{D}^{\beta}\left(\mathrm{D}^{\alpha}+\frac{\mu}{t^{\lambda}}\right)y(t)+\Delta_1 f\left(t,y(t),\mathrm{D}^{\delta}y(t)\right)+\Delta_2 g(t,y(t),\mathrm{I}^{\rho}y(t))+h(t,y(t))=l(t),t\in(0,1), \\ y(0)=0, y(1)=b\int_0^{\eta}y(s)ds, 0<\eta\leq 1, \\ \mathrm{I}^{\rho}y(u)=y(1), 0<u<1 \\ \mu>0,\ \ 0<\lambda\leq 1,\ \ 1\leq\beta\leq 2,\ \ 0\leq\alpha,\ \ \delta\leq 1 \end{cases}$$

where $\Delta_1>0,\Delta_2>0, J=[0,1]$ the derivatives of the problem are in the sense of Caputo, I^{ρ} denotes the Riemann-Liouville integral of order ρ, f and g are two given functions defined on $J\times\mathrm{R}^2$, and h,l are two given functions defined over J.

Very recently, Beddani and Dahmani [19] investigated the following new nonlinear Langevin equation that involves the $\varphi-$Caputo derivatives:

$$\begin{cases} {}^c\mathrm{D}_{a^+}^{\alpha_1;\varphi}\left({}^c\mathrm{D}_{a^+}^{\alpha_2;\varphi}\left({}^c\mathrm{D}_{a^+}^{\alpha_3;\varphi}+\mu\right)\right)u(t)=g(t,u(t),{}^c\mathrm{D}_{a^+}^{\alpha_4;\varphi}u(t)),t\in J=(a,b) \\ u(a)=\left({}^c\mathrm{D}_{a^+}^{\alpha_4;\varphi}\right)u(b)=0,\ \ u(b)=\rho\sum_{i=1}^{n}u(\zeta_i), \\ \mu,\rho>0,\ \ 0\leq a<\zeta_i<b<\infty, \text{ and } \varphi(b)-\varphi(a)=M>0, \end{cases}$$

where ${}^c\mathrm{D}_{a^+}^{\alpha_i;\varphi}, i=\overline{1,4}$ are the $\varphi-$Caputo fractional derivative of orders α_i, $0<\alpha_i<1$, $\alpha_4<\alpha_3$, and $a,\mu,\rho\in\mathrm{R}_+^*$, and $\varphi : J\to\mathrm{R}$ is an increasing function with $\varphi'\neq 0$. The two other functions of our problem are to be defined later.

In this work we study the existence and uniqueness of solutions for the following problem:

$$(1.1)\quad \begin{cases} {}^c\mathrm{D}_{a^+}^{\alpha_1;\varphi}\left({}^c\mathrm{D}_{a^+}^{\alpha_2;\varphi}\left({}^c\mathrm{D}_{a^+}^{\alpha_3;\varphi}+\frac{\mu}{t^{\lambda}}\right)\right)u(t)=d_1 g(t,u(t),{}^c\mathrm{D}_{a^+}^{\alpha_4;\varphi}u(t))+d_2 f(t,u(t),\mathrm{I}_{a^+}^{\alpha_5;\varphi}u(t)) \\ +h(t,u(t)+x(t) \\ t\in J=(a,b),\ u(a)={}^c\mathrm{D}_{a^+}^{\alpha_4;\varphi}u(b)=0,\ \ \mathrm{I}_{a^+}^{\alpha_5;\varphi}u(b)=\sum_{i=1}^{n}\rho_i u(\zeta_i), \\ \mu,\rho_i>0,\ \ 0\leq a<\zeta_i<b<\infty, \text{ and } \varphi(b)-\varphi(a)=M>0, \\ \mu>0,\ \ 0<\lambda\leq 1,\ \ d_1>0,d_2>0 \end{cases}$$

where ${}^c\mathrm{D}_{a^+}^{\alpha_i;\varphi}, i=\overline{1,4}$ are the $\varphi-$Caputo fractional derivative of orders $\alpha_i,\alpha_4<\alpha_3$, and $\mathrm{I}_{a^+}^{\alpha_5;\varphi}$ the left-sided $\varphi-$Riemann Liouville fractional integral of order $\alpha_{5,}$ and $0<\alpha_i<1, i=\overline{1,5}, a,\mu,\rho_i\in\mathrm{R}_+^*$, and $\varphi : J\to\mathrm{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$, to be defined later, $g,f : J\times R\times R\to R^+$ is a given function satisfying some assumptions that will be specified later, and h,x are two given functions defined over J.

2. PRELIMINARIES

Let $\varphi : J\to\mathrm{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$, and let $C(J,R)$ be the Banach space

Definition 1 *([4]). For $\alpha > 0$, the left-sided $\varphi-$ Riemann Liouville fractional integral of order α for an integrable function $u : J \to R$ with respect to another function $\varphi : J \to R$ that is an increasing differentiable function such that $\varphi'(t) \neq 0$, for all $t \in J$ is defined as follows*

$$(2.1) \qquad \mathbf{I}_{a^+}^{\alpha;\varphi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{\alpha-1} u(s)ds,$$

where Γ is the gamma function. Note that equation (2.1) is reduced to the Riemann Liouville and Hadamard fractional integrals when $\varphi(t) = t$ and $\varphi(t) = \ln t$, respectively.

Definition 2 *([4]). Let $n \in \mathrm{N}$ and let $\varphi, u \in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0,$ for all $t \in J$. The left-sided $\varphi-$Riemann Liouville fractional derivative of a function u of order α is defined by*

$$\begin{aligned} \mathbf{D}_{a^+}^{\alpha;\varphi} u(t) &= \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \mathbf{I}_{a^+}^{n-\alpha;\varphi} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} u(s)ds, \end{aligned}$$

where $n = [\alpha]+1.$

Definition 3 *([4]). Let $n \in \mathrm{N}$ and let $\varphi, u \in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0,$ for all $t \in J$. The left-sided $\varphi-$ Caputo fractional derivative of a function u of order α is defined by*

$${}^c\mathbf{D}_{a^+}^{\alpha;\varphi} u(t) = I_{a^+}^{n-\alpha;\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t),$$

where $n = [\alpha]+1$ for $\alpha \notin \mathrm{N}, n = \alpha$ for $\alpha \in \mathrm{N}.$

To simplify notation, we will use the abbreviated symbol

$$u_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t).$$

From the definition, it is clear that,

$$(2.2) \qquad {}^c\mathbf{D}_{a^+}^{\alpha;\varphi} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} u_\varphi^{[n]}(s)ds, & \text{if } \alpha \notin \mathrm{N}, \\ u_\varphi^{[n]}(t) & \text{if } \alpha \in \mathrm{N}. \end{cases}$$

This generalization (2.2) yields the Caputo fractional derivative operator when $\varphi(t) = t$. Moreover, for $\varphi(t) = \ln t$, it gives the Caputo Hadamard fractional derivative.

Lemma 1 *([4]). Let* $\alpha,\beta>0$, *and* $u\in L^1(J)$. *Then*

$$\mathbf{I}_{a^+}^{\alpha;\varphi}\mathbf{I}_{a^+}^{\beta;\varphi}u(t)=\mathbf{I}_{a^+}^{\alpha+\beta;\varphi}u(t),\quad \text{a.e. } t\in J.$$

In particular, if $u\in C(J)$. *Then* $\mathbf{I}_{a^+}^{\alpha;\varphi}\mathbf{I}_{a^+}^{\beta;\varphi}u(t)=\mathbf{I}_{a^+}^{\alpha+\beta;\varphi}u(t),\quad t\in J.$

Next, we recall the property describing the composition rules for fractional $\varphi-$ *integrals and* $\varphi-$*derivatives.*

Lemma 2 *([4]). Let* $\alpha>0$ *The following holds:*

If $u\in C([a,b])$ *then*

$${}^{c}\mathbf{D}_{a^+}^{\alpha;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)=u(t), t\in[a,b].$$

If $u\in C^n(J)$, $n-1<\alpha<n.$ *Then*

$$\mathbf{I}_{a^+}^{\alpha;\varphi c}\mathbf{D}_{a^+}^{\alpha;\varphi}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u_\varphi^{[k]}(a)}{k!}\left[\varphi(t)-\varphi(a)\right]^k,$$

for all $t\in[a,b]$. *In particular, if* $0<\alpha<1$, *we have*

$$\mathbf{I}_{a^+}^{\alpha;\varphi c}\mathbf{D}_{a^+}^{\alpha;\varphi}u(t)=u(t)-u(a).$$

Lemma 3 *([4, 20]). Let* $t>a$, $\alpha\geq 0$ *; and* $\beta>0$. *Then*

- $\mathbf{I}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left[\varphi(t)-\varphi(a)\right]^{\beta+\alpha-1},$
- ${}^{c}\mathbf{D}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left[\varphi(t)-\varphi(a)\right]^{\beta-\alpha-1},$
- ${}^{c}\mathbf{D}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{k}=0,$ *for all* $k\in\{0,...,n-1\}, n\in\mathrm{N}$.

Lemma 4 ***([18, 20])**. Let* $\alpha>0, n\in\mathrm{N},$ *such that* $n-1<q\leq n.$ *Then:*

- ${}^{c}\mathbf{D}_{a^+}^{q;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)={}^{c}\mathbf{D}_{a^+}^{q-\alpha;\varphi}u(t),$ *if* $q>\alpha.$
- ${}^{c}\mathbf{D}_{a^+}^{q;\varphi}\mathbf{I}_{a^+}^{\alpha;\varphi}u(t)=\mathbf{I}_{a^+}^{\alpha-q;\varphi}u(t),$ *if* $\alpha>q.$

Lemma 5 *([20]) Given a function* $u\in C^n[a,b]$ *and* $0<q<1,$ *we have*

$$\left|\mathbf{I}_{a^+}^{q;\varphi}u(t_2)-\mathbf{I}_{a^+}^{q;\varphi}u(t_1)\right|\leq\frac{2\|u\|}{\Gamma(q+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^q.$$

Lemma 6. *For a given* $\Upsilon\in L^1(J,R,R)$, *the unique solution of the linear fractional initial value problem*

$$\begin{cases} {}^{c}\mathbf{D}_{a^+}^{\alpha_1;\varphi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_2;\varphi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_3;\varphi}+\frac{\mu}{t^\lambda}\right)\right)u(t)=\Upsilon(t)\\ t\in J=(a,b),\ u(a)={}^{c}\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(b)=0,\ \ \mathbf{I}_{a^+}^{\alpha_5;\varphi}u(b)=\sum_{i=1}^{n}\rho_i u(\zeta_i),\\ \mu,\rho_i>0,\ \ 0\leq a<\zeta_i<b<\infty,\ \text{and}\ \varphi(b)-\varphi(a)=M>0,\\ \mu>0,\ \ 0<\lambda\leq 1,\ \ d_1>0, d_2>0 \end{cases}$$

is given by

$$u(t)=\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\varphi}\Upsilon(t)-\mu\mathbf{I}_{a^+}^{\alpha_3;\varphi}\left(\frac{u(t)}{t^\lambda}\right)$$
$$+\left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}-\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}\right)\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(b)$$
$$+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$-\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Delta\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$+\frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$-\frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$

where

$$\Delta=\frac{M^{\alpha_2+\alpha_3}}{\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}-\frac{M^{\alpha_2+\alpha_3}}{\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}.$$

and if $e\in\{\alpha_3,\alpha_3-\alpha_4,\alpha_3+\alpha_5,\alpha_1+\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3-\alpha_4,\alpha_1+\alpha_2+\alpha_3+\alpha_5\}$, $y\in\{t,b\}$, then

$$\mathbf{I}_{a^+}^{e;\varphi}\left(\frac{u(y)}{y^\lambda}\right)=\frac{1}{\Gamma(e)}\int_a^y\varphi'(s)(\varphi(y)-\varphi(s))^{e-1}\frac{u(s)}{s^\lambda}ds,$$
$$\mathbf{I}_{a^+}^{e;\varphi}(\Upsilon(y))=\frac{1}{\Gamma(e)}\int_a^y\varphi'(s)(\varphi(y)-\varphi(s))^{e-1}(\Upsilon(s))ds.$$

Proof: For $0<\alpha_i<1, i=\overline{1,3}$, and Lemma 2 we get

$$\left({}^c\mathbf{D}_{a^+}^{\alpha_2;\phi}\left({}^c\mathbf{D}_{a^+}^{\alpha_3;\phi}+\frac{\mu}{t^\lambda}\right)\right)u(t)=\mathbf{I}_{a^+}^{\alpha_1;\phi}\Upsilon(t)+c_1$$

and

$$\left({}^c\mathbf{D}_{a^+}^{\alpha_3;\phi}+\frac{\mu}{t^\lambda}\right)u(t)=\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\phi}\Upsilon(t)+\mathbf{I}_{a^+}^{\alpha_2;\phi}c_1+c_2,$$

so

$$u(t)=\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\phi}\Upsilon(t)-\mu\mathbf{I}_{a^+}^{\alpha_3;\phi}\left(\frac{u(t)}{t^\lambda}\right)+\mathbf{I}_{a^+}^{\alpha_2+\alpha_3;\phi}c_1+\mathbf{I}_{a^+}^{\alpha_3;\phi}c_2+c_3,$$

where $c_1,c_2,c_3\in R$, by conditions $u(a)=0$, we get

$$u(t)=\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\varphi}\Upsilon(t)-\mu\mathbf{I}_{a^+}^{\alpha_3;\varphi}\left(\frac{u(t)}{t^\lambda}\right)+\mathbf{I}_{a^+}^{\alpha_2+\alpha_3;\varphi}c_1+\mathbf{I}_{a^+}^{\alpha_3;\varphi}c_2, \tag{2.3}$$

by integrating we find

$$u(t) = \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\varphi}\Upsilon(t) - \mu\mathbf{I}_{a^+}^{\alpha_3;\varphi}\left(\frac{u(t)}{t^{\lambda}}\right) + c_1\frac{(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + c_2\frac{(\varphi(t)-\varphi(a))^{\alpha_3}}{\Gamma(\alpha_3+1)}. \tag{2.4}$$

For $0<\alpha_4,\alpha_5<1$, and by (2.3), and Lemma 2 we have

$$^{c}\mathbf{D}_{a^+}^{\alpha_4;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\phi}\Upsilon(t) - \mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}\left(\frac{u(t)}{t^{\lambda}}\right) + \mathbf{I}_{a^+}^{\alpha_2+\alpha_3-\alpha_4;\phi}c_1 + \mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}c_2,$$

and

$$\mathbf{I}_{a^+}^{\alpha_5;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\phi}\Upsilon(t) - \mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}\left(\frac{u(t)}{t^{\lambda}}\right) + \mathbf{I}_{a^+}^{\alpha_2+\alpha_3+\alpha_5;\phi}c_1 + \mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}c_2.$$

So,

$$^{c}\mathbf{D}_{a^+}^{\alpha_4;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\phi}\Upsilon(t) - \mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}\left(\frac{u(t)}{t^{\lambda}}\right) + c_1\frac{(\phi(t)-\phi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + c_2\frac{(\phi(t)-\phi(a))^{\alpha_3-\alpha_4}}{\Gamma(\alpha_3-\alpha_4+1)},$$

and

$$\mathbf{I}_{a^+}^{\alpha_5;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\phi}\Upsilon(t) - \mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}\left(\frac{u(t)}{t^{\lambda}}\right) + c_1\frac{(\phi(t)-\phi(a))^{\alpha_2+\alpha_3+\alpha_5}}{\Gamma(\alpha_2+\alpha_3+\alpha_5+1)} + c_2\frac{(\phi(t)-\phi(a))^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)},$$

by conditions $\phi(b)-\phi(a)=M$, $^{c}\mathbf{D}_{a^+}^{\alpha_4;\phi}u(b)=0$, and $\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(b)=\sum_{i=1}^{n}\rho_i u(\zeta_i)$, we give

$$c_1\frac{M^{\alpha_2+\alpha_3-\alpha_4}}{\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + c_2\frac{M^{\alpha_3-\alpha_4}}{\Gamma(\alpha_3-\alpha_4+1)} = \mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}\left(\frac{u(b)}{b^{\lambda}}\right) - \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\phi}\Upsilon(b)$$

and

$$c_1\frac{M^{\alpha_2+\alpha_3+\alpha_5}}{\Gamma(\alpha_2+\alpha_3+\alpha_5+1)} + c_2\frac{M^{\alpha_3+\alpha_5}}{\Gamma(\alpha_3+\alpha_5+1)} = \mathbf{I}_{a^+}^{\alpha_5;\phi}u(b) + \mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}\left(\frac{u(b)}{b^{\lambda}}\right) - \mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\phi}\Upsilon(b).$$

So,

$$\begin{aligned} c_1 = &-\frac{M^{-\alpha_5}}{\Delta\Gamma(\alpha_3-\alpha_4+1)}\sum_{i=1}^{n}\rho_i u(\zeta_i) \\ &+\frac{M^{-\alpha_5}}{\Delta\Gamma(\alpha_3-\alpha_4+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\phi}\Upsilon(b) - \mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}\left(\frac{u(b)}{b^{\lambda}}\right)\right) \\ &-\frac{M^{\alpha_4}}{\Delta\Gamma(\alpha_3+\alpha_5+1)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\phi}\Upsilon(b) - \mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}\left(\frac{u(b)}{b^{\lambda}}\right)\right) \end{aligned}$$

and

$$c_2=\frac{M^{\alpha_2-\alpha_5}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\sum_{i=1}^{n}\rho_i u\left(\zeta_i\right)$$
$$+\frac{M^{\alpha_2+\alpha_4}}{\Delta\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\phi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\phi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$-\frac{M^{\alpha_2-\alpha_5}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\phi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\phi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$

where

$$\Delta=\frac{M^{\alpha_2+\alpha_3}}{\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}-\frac{M^{\alpha_2+\alpha_3}}{\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}$$

Substituting the values of c_1, c_2 into (2.4), we find the solution.
By derivation we get:

$${}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(t)$$
$$=\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon(t)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(t)}{t^\lambda}\right)$$
$$+\left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}-\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\right)\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(b)$$
$$+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$-\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$+\frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$-\frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon(b)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right).$$

3. MAIN RESULTS

Now, let us introduce the Banach space:

$$X=\left\{u\ :\ \left(u,{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u\right)\in\left(C\left(J,\mathrm{R}\right)\right)^2\right\}$$

and the norm:

$$\|u\|_X=\max\left\{\|u\|_\infty,\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u\right\|_\infty\right\}.$$

where

$$\|u\|_\infty=\sup_{t\in[a,b]}|u(t)|,\ \text{ and }\ \left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\phi}u\right\|_\infty=\sup_{t\in[a,b]}\left|{}^c\mathbf{D}_{a^+}^{\alpha_4;\phi}u(t)\right|.$$

Then, we define the nonlinear operator $H : X \to X$ by

$$
\begin{aligned}
&\left(Hu\right)(t) \\
&= \mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\varphi}\Upsilon_u(t) - \mu\mathtt{I}_{a^+}^{\alpha_3;\varphi}\left(\frac{u(t)}{t^\lambda}\right) \\
&+\left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Delta\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)} - \frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\right)\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(b) \\
&+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&-\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Delta\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&+\frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3}}{\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&-\frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{\Delta\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^cD^{\alpha_4;\varphi}\left(Hu\right)(t) \\
&= \mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(t) - \mu\mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(t)}{t^\lambda}\right) \\
&+\left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)} - \frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\right)\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(b) \\
&+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&-\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&+\frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right) \\
&-\frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b) - \mu\mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)
\end{aligned}
$$

where if

$$e \in \left\{\alpha_3, \alpha_3-\alpha_4, \alpha_3+\alpha_5, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3-\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_5\right\}, c \in \left\{t, b\right\},$$

Then,

$$\mathtt{I}_{a^+}^{e;\varphi}\left(\frac{u(c)}{c^{\lambda}}\right)=\frac{1}{\Gamma(e)}\int_a^c \varphi'(s)(\varphi(c)-\varphi(s))^{e-1}\frac{u(s)}{s^{\lambda}}ds,$$

and

$$\begin{aligned}\mathtt{I}_{a^+}^{e;\varphi}\Upsilon_u(y)=&\frac{d_1}{\Gamma(e)}\int_a^y \varphi'(s)(\varphi(y)-\varphi(s))^{e-1}g(s,u(s),{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u(s))ds\\&+\frac{d_2}{\Gamma(e)}\int_a^y \varphi'(s)(\varphi(y)-\varphi(s))^{e-1}f(s,u(s),{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s))ds\\&+\frac{1}{\Gamma(e)}\int_a^y \varphi'(s)(\varphi(y)-\varphi(s))^{e-1}\left(h(s,u(s))+x(s)\right)ds.\end{aligned}$$

Now, we are ready to study the above problem by means of the fixed point theory.

A Unique Solution. For the sake of convenience, we introduce the following quantities:

$$\beta=\frac{M^{\alpha_5}}{\Gamma\left(1+\alpha_5\right)}.$$

$$\begin{aligned}\mathbf{N}_1=&\frac{\mu a^{-\lambda}M^{\alpha_3}}{\Gamma\left(\alpha_3+1\right)}+\frac{2\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\\&+\frac{n\left(\max\limits_{i=\overline{1,n}}\rho_i\right)M^{\alpha_2+\alpha_3-\alpha_5}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}+\frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\\&+\frac{n\left(\max\limits_{i=\overline{1,n}}\rho_i\right)M^{\alpha_2+\alpha_3-\alpha_5}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}+\frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\end{aligned}$$

$$\begin{aligned}\mathbf{N}_2=&\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1\right)}\\&+\frac{2M^{\alpha_1+\alpha_2+\alpha_3}}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3+1\right)}+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1\right)}\\&+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}\\&+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}\end{aligned}$$

$$\mathbf{N}_3 = \frac{\mu a^{-\lambda} M^{\alpha_3-\alpha_4}}{\Gamma(\alpha_3-\alpha_4+1)} + \frac{2n\left(\max\limits_{i=1,n} \rho_i\right) M^{\alpha_2+\alpha_3-\alpha_4-\alpha_5}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}$$
$$+\frac{\mu a^{-\lambda} M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3+\alpha_5+1)}$$
$$+\frac{\mu a^{-\lambda} M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)\Gamma(\alpha_3-\alpha_4+1)}$$
$$+\frac{2\mu a^{-\lambda} M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_3+\alpha_5+1)},$$

and

$$\mathbf{N}_4 = \frac{2M^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}}{\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)}$$
$$+\frac{2a^{-\lambda} M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1)}$$
$$+\frac{2a^{-\lambda} M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)}$$
$$+\frac{2a^{-\lambda} M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)}.$$

Also, we consider the following hypotheses:

A_1) $f, g : [a,b]\times R^2 \to [0,+\infty[$ are continuous functions.

A_2) There exists a constant $\delta_1, \delta_2, \delta_3 > 0$ such that

$$|g(t,u,v) - g(t,w,z)| \le \delta_1 \left(|u-w| + |v-z|\right),$$

$$|f(t,u,v) - f(t,w,z)| \le \delta_2 \left(|u-w| + |v-z|\right),$$

and

$$|h(t,u) - h(t,v)| \le \delta_3 |u-v|,$$

for all $t \in [a,b]$, $(u,v,w,z) \in R^4$.

A_3) 1 - There exist constants $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \in]0,+\infty[$, such that, for any $t \in [a,b]$, $(u,v) \in R^2$ we have

$$|g(t,u,v)| \le \varphi_1(t) + \rho_1 |u(t)| + \rho_2 |v(t)|,$$

$$|f(t,u,v)| \le \varphi_2(t) + \rho_3 |u(t)| + \rho_4 |v(t)|,$$

$$|h(t,u)| \le \varphi_3(t) + \rho_5 |u(t)|,$$

and

$$|\varphi_1(t)| \le \|\varphi_1\|_\infty, \quad |\varphi_2(t)| \le \|\varphi_2\|_\infty, \quad |\varphi_3(t)| \le \|\varphi_3\|_\infty.$$

and the function x satisfies: $\|x\|_\infty = \theta$.

Proposition 1. *Let* $q>0$ *, and* $\phi, u, v \in C^n([a,b])$*, we have*

1) $(\varphi(t)-\varphi(a))^q \le M^q$,

2) $\left|\mathbf{I}_{a^+}^{q;\varphi} \dfrac{u(t)}{t^\lambda}\right| \le \dfrac{a^{-\lambda} M^q \|u\|_X}{\Gamma(q+1)}, (a \succ 0)$

3) $\left|g(s,u(s),{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(s))\right| \le \|\phi_1\|_\infty + (\rho_1+\rho_2)\|u\|_X$,

4) $\left|f(s,u(s),{}^c\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(s))\right| \le \|\phi_2\|_\infty + (\rho_3+\rho_4\beta)\|u\|_X$,

5) $|h(s,u(s))| \le \|\phi_3\|_\infty + \rho_5\|u\|_X$,

6) $\left|g(s,u(s),{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(s)) - g(s,v(s),{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}v(s))\right| \le 2\delta_1\|u-v\|_X$.

7) $\left|f(s,u(s),{}^c\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(s)) - f(s,v(s),{}^c\mathbf{I}_{a^+}^{\alpha_5;\varphi}v(s))\right| \le \delta_2(1+\beta)\|u-v\|_X$

8) $|h(s,u(s))-h(s,v(s))| \le \delta_3\|u-v\|_X$.

9) $\left|\mathbf{I}_{a^+}^{q;\varphi}\dfrac{u(t)}{t^\lambda} - \mathbf{I}_{a^+}^{q;\varphi}\dfrac{v(t)}{t^\lambda}\right| \le \dfrac{a^{-\lambda}M^q\|u-v\|_X}{\Gamma(q+1)}, (a\succ 0)$

and

10) $|\Upsilon_u(s)| \le \left(d_1\|\phi_1\|_\infty + d_2\|\phi_2\|_\infty + \|\phi_3\|_\infty + \theta\right) + \left(d_1\rho_1 + d_2\rho_2 + d_2\rho_3 + d_2\beta\rho_4 + \rho_5\right)\|u\|_X$.

Proof :

1) We have $\phi'(t)>0, \quad \forall t\in[a,b]$, then

$$(\phi(t)-\phi(a))^q \le (\phi(b)-\phi(a))^q = M^q,$$

2)

$$\begin{aligned}\left|\mathbf{I}_{a^+}^{q;\varphi}\frac{u(t)}{t^\lambda}\right| &\le \frac{1}{\Gamma(q)}\int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{q-1}\left|\frac{u(s)}{s^\lambda}\right|ds \\ &\le \frac{\sup_{s\in[a,b]}|u(s)|}{a^\lambda\Gamma(q)}\int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{q-1}ds \\ &\le \frac{a^{-\lambda}M^q\|u\|_\infty}{\Gamma(q+1)} \le \frac{a^{-\lambda}M^q\|u\|_X}{\Gamma(q+1)}.\end{aligned}$$

3) By (A_3), we have

$$\begin{aligned}\left|g(s,u(s),{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(s))\right| &\le \phi_1(t) + \rho_1\sup_{s\in[a,b]}|u(s)| + \rho_2\sup_{s\in[a,b]}\left|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u(s)\right| \\ &\le \|\phi_1\|_\infty + \rho_1\|u\|_\infty + \rho_2\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}u\right\|_\infty \\ &\le \|\phi_1\|_\infty + (\rho_1+\rho_2)\|u\|_X .\end{aligned}$$

4) By (A_3), we have

$$\begin{aligned}\left|f(s,u(s),\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s))\right| &\le \phi_2(t)+\rho_3\sup_{s\in[a,b]}|u(s)|+\rho_4\sup_{s\in[a,b]}\left|\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s)\right| \\ &\le \phi_2(t)+\rho_3\left(\sup_{s\in[a,b]}|u(s)|\right)+\beta\rho_4\left(\sup_{s\in[a,b]}|u(s)|\right) \\ &\le \|\phi_2\|_\infty+(\rho_3+\beta\rho_4)\|u\|_X .\end{aligned}$$

6) By (A_2), we have

$$\begin{aligned}&\left|g(s,u(s),{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u(s))-g(s,v(s),{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}v(s))\right| \\ &\le \delta_1\left(\sup_{s\in[a,b]}|u(s)-v(s)|+\sup_{s\in[a,b]}\left|{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u(s)-{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}v(s)\right|\right) \\ &\le 2\delta_1\|u-v\|_X .\end{aligned}$$

7)

$$\begin{aligned}\left|f(s,u(s),{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s))-f(s,v(s),{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}v(s))\right| &\le \delta_2\left(|u(s)-v(s)|+\left|{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s)-{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}v(s)\right|\right) \\ &\le \delta_2\left(\sup_{s\in[a,b]}|u(s)-v(s)|+\sup_{s\in[a,b]}\left|{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}\left(u(s)-v(s)\right)\right|\right) \\ &\le \delta_2(1+\beta)\sup_{s\in[a,b]}\left|(u(s)-v(s))\right|\le \delta_2(1+\beta)\|u-v\|_X .\end{aligned}$$

10) By (A_3), (3) and (4) we have

$$\begin{aligned}&\left|\Upsilon_u(s)\right| \\ &\le \left|d_1g(s,u(s),{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u(s))\right|+\left|d_2f(s,u(s),{}^c\mathtt{I}_{a^+}^{\alpha_5;\varphi}u(s))\right|+|h(s,u(s))|+|x(s)| \\ &\le d_1\left(\|\phi_1\|_\infty+\rho_1\|u\|_\infty+\rho_2\left\|{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u\right\|_\infty\right)+d_2\left(\|\phi_2\|_\infty+(\rho_3+\beta\rho_4)\|u\|_\infty\right) \\ &\quad+\|\phi_3\|_\infty+\rho_5\|u\|_\infty+\|x\|_\infty \\ &\le \left(d_1\|\phi_1\|_\infty+d_2\|\phi_2\|_\infty+\|\phi_3\|_\infty+\theta\right)+\left(d_1\rho_1+d_2\rho_3+d_2\beta\rho_4+\rho_5\right)\|u\|_\infty+d_2\rho_2\left\|{}^c\mathtt{D}_{a^+}^{\alpha_4;\varphi}u\right\|_\infty \\ &\le \left(d_1\|\phi_1\|_\infty+d_2\|\phi_2\|_\infty+\|\phi_3\|_\infty+\theta\right)+\left(d_1\rho_1+d_2\rho_2+d_2\rho_3+d_2\beta\rho_4+\rho_5\right)\|u\|_X .\end{aligned}$$

Theorem 1. *Assume that hypotheses (A_1) and (A_3), are satisfied. Then, the problem (1.1) has a unique solution provided that*

$$\vartheta=\max\left\{\mathbf{N}_1+\mathbf{N}_2\left((1+\beta)d_2\delta_2+2d_1\delta_1+\delta_3\right),\mathbf{N}_3+\mathbf{N}_4\left((1+\beta)d_2\delta_2+2d_1\delta_1+\delta_3\right)\right\}<1.$$

Proof: **First step:** We consider a ball $B_r=\{u\in X,\|u\|_X\le r,r>0\}$ where:

$$\frac{k\max\{\mathbf{N}_2,\mathbf{N}_4\}}{\left[1+\omega-\max\{\mathbf{N}_1,\mathbf{N}_3\}\right]}\le r.$$

and

$$d_1\|\varphi_1\|_\infty+d_2\|\varphi_2\|_\infty+\|\varphi_3\|_\infty+\theta=k,\quad d_1\rho_1+d_2\rho_2+d_2\rho_3+d_2\beta\rho_4+\rho_5=\omega.$$

For $u\in B_r$ and $t\in J$, we have:

$$
\begin{aligned}
&\|Hu\|_{\infty}\\
&\leq \sup\left\{ \mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3;\varphi}\left|\Upsilon_u(t)\right| + \mu \mathtt{I}_{a^+}^{\alpha_3;\varphi}\left|\frac{u(t)}{t^{\lambda}}\right| \right.\\
&\quad + \left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + \frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)} \right)\mathtt{I}_{a^+}^{\alpha_5;\varphi}|u(b)|\\
&\quad + \frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right| + \mu \mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^{\lambda}}\right| \right)\\
&\quad + \frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right| + \mu \mathtt{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^{\lambda}}\right| \right)\\
&\quad + \frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right| + \mu \mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^{\lambda}}\right| \right)\\
&\quad \left. + \frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}\left(\mathtt{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right| + \mu \mathtt{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^{\lambda}}\right| \right)\right\},
\end{aligned}
$$

by Lemma 4 , and Proposition 1, we obtain

$$
\begin{aligned}
&\|Hu\|_{\infty}\\
&\leq \|u\|_X \left\{ \frac{\mu a^{-\lambda}M^{\alpha_3}}{\Gamma(\alpha_3+1)} + \frac{2\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_3-\alpha_4+1)} \right.\\
&\quad + \frac{n\left(\max\limits_{i=\overline{1,n}}\rho_i\right)M^{\alpha_2+\alpha_3-\alpha_5}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)} + \frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_4+1)\Gamma(\alpha_3+\alpha_5+1)}\\
&\quad \left. + \frac{n\left(\max\limits_{i=\overline{1,n}}\rho_i\right)M^{\alpha_2+\alpha_3-\alpha_5}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + \frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_3-\alpha_4+1)} \right\}\\
&\quad + \left(k+\omega\|u\|_X\right)\left\{ \frac{2M^{\alpha_2+\alpha_3+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+\alpha_{54}+1)} \right.\\
&\quad + \frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1)}\\
&\quad + \frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1)}\\
&\quad + \frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)}\\
&\quad \left. + \frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1)} \right\}
\end{aligned}
$$

$$\leq N_1\|u\|_X+N_2\left(k+\omega\|u\|_X\right)$$
$$\leq\left(N_1+\omega\right)\|u\|_X+kN_2.$$

Also,

$$\left\|{}^cD^{\alpha_4;\varphi}\left(Hu\right)(t)\right\|_\infty$$
$$\leq \sup\left\{\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(t)-\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(t)}{t^\lambda}\right)\right.$$
$$+\left(\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{\Delta\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\right)\mathbf{I}_{a^+}^{\alpha_5;\varphi}u(b)$$
$$+\frac{M^{-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b)+\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$

$$+\frac{M^{\alpha_2-\alpha_5}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\Upsilon_u(b)+\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$
$$+\frac{M^{\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_2+\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b)+\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)\Bigg\}$$
$$+\frac{M^{\alpha_2+\alpha_4}(\varphi(t)-\varphi(a))^{\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\Upsilon_u(b)+\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left(\frac{u(b)}{b^\lambda}\right)\right)$$

by Lemma 4 , and Proposition 1 , we obtain:

$$\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu\right\|_\infty\leq\|u\|_X\left\{\frac{\mu a^{-\lambda}M^{\alpha_3-\alpha_4}}{\Gamma\left(\alpha_3-\alpha_4+1\right)}+\frac{2n\left(\max\limits_{i=1,n}\rho_i\right)M^{\alpha_2+\alpha_3-\alpha_4-\alpha_5}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\right.$$
$$+\frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}$$
$$+\frac{\mu a^{-\lambda}M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}$$
$$\left.+\frac{2\mu a^{-\lambda}M^{\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\right\}+\left(k+\omega\|u\|_X\right)\left\{\frac{2M^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}\right.$$
$$+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3+\alpha_5+1\right)}$$
$$+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}$$
$$\left.+\frac{2a^{-\lambda}M^{\alpha_1+2\alpha_2+2\alpha_3-\alpha_4}}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}\right\}$$
$$\leq\mathbf{N}_3\|u\|_X+\mathbf{N}_4\left(k+\omega\|u\|_X\right)\leq\left(\mathbf{N}_3+\omega\right)r+k\mathbf{N}_4.$$

Hence,

$$\|Hu\|_X = \max\left\{\|Hu\|_\infty, \left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi} Hu\right\|_\infty\right\} \le \max\left\{(\mathbf{N}_1+\omega)r + k\mathbf{N}_2, (\mathbf{N}_3+\omega)r + k\mathbf{N}_4\right\}$$
$$\le \left(\max\left\{\mathbf{N}_1, \mathbf{N}_3\right\} + \omega\right) r + k\max\left\{\mathbf{N}_2, \mathbf{N}_4\right\} < r.$$

Second step: We proceed to prove that H is a contraction mapping.

For any $u,v \in X$, $t \in J$ and (A_2), Proposition 1 we give:

$$\begin{aligned}
\|Hu-Hv\|_\infty \le \sup\Bigg\{ & I_{a^+}^{\alpha_1+\alpha_3+\alpha_2;\varphi}\left(\Upsilon_u(t)-\Upsilon_v(t)\right) + \mu I_{a^+}^{\alpha_3;\varphi}\left|\frac{u(t)}{t^\lambda}-\frac{v(t)}{t^\lambda}\right| \\
& + \left(\frac{M^{\alpha_2-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + \frac{M^{-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}}{|\Delta|\Gamma(\alpha_3+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}\right) I_{a^+}^{\alpha_5;\varphi}\left|\frac{u(t)}{t^\lambda}-\frac{v(t)}{t^\lambda}\right| \\
& + \frac{M^{-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_u(b)-\Upsilon_v(b)\right)\right) + \mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}-\frac{v(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)} \\
& + \frac{M^{\alpha_2-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_u(b)-\Upsilon_v(b)\right)\right) + \mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}-\frac{v(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} \\
& + \frac{M^{\alpha_4}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_u(b)-\Upsilon_v(b)\right)\right) + \mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}-\frac{v(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)} \\
& + \frac{M^{\alpha_2+\alpha_4}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_u(b)-\Upsilon_v(b)\right)\right) + \mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}-\frac{v(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}\Bigg\}
\end{aligned}$$

by Lemma 4, and Proposition 1, we obtain:

$$\|Hu-Hv\|_\infty \le \left[\mathbf{N}_1 + \mathbf{N}_2\left((1+\beta)d_2\delta_2 + 2d_1\delta_1 + \delta_3\right)\right]\|u-v\|_X$$

also, using the same method and by Lemma , and Proposition , we obtain:

$$\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu - {}^c\,\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hv\right\|_\infty \le \left[\mathbf{N}_3 + \mathbf{N}_4\left((1+\beta)d_2\delta_2 + 2d_1\delta_1 + \delta_3\right)\right]\|u-v\|_X .$$

Then, we obtain

$$\|Hu-Hu\|_X \le \vartheta\|u-v\|_X .$$

At Least One Solution.

Theorem 2. *Assume that hypotheses (A_1) and (A_3) are fulfilled. Then, the problem (1.1) has at least one solution over* J .

Proof:

First step: First of all, we show that the operator H is completely continuous on X. We have:

$$\|Hu_n - Hu\|_\infty \le \sup\Bigg\{ I_{a^+}^{\alpha_1+\alpha_3+\alpha_2;\varphi}\left(\Upsilon_{u_n}(t)-\Upsilon_u(t)\right) + \mu I_{a^+}^{\alpha_3;\varphi}\left|\frac{u_n(t)}{t^\lambda}-\frac{u(t)}{t^\lambda}\right|$$
$$+\left(\frac{M^{\alpha_2-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}+\frac{M^{-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}}{|\Delta|\Gamma(\alpha_3+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}\right) I_{a^+}^{\alpha_5;\varphi}\left|\frac{u_n(t)}{t^\lambda}-\frac{u(t)}{t^\lambda}\right|$$
$$+\frac{M^{-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_{u_n}(t)-\Upsilon_u(t)\right)\right)+\mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u_n(b)}{b^\lambda}-\frac{u(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3-\alpha_4+1)}$$
$$+\frac{M^{\alpha_2-\alpha_5}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_{u_n}(t)-\Upsilon_u(t)\right)\right)+\mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u_n(b)}{b^\lambda}-\frac{u(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)}$$
$$+\frac{M^{\alpha_4}\left(\varphi(t)-\varphi(a)\right)^{\alpha_2+\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_{u_n}(t)-\Upsilon_u(t)\right)\right)+\mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u_n(b)}{b^\lambda}-\frac{u(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_3+\alpha_5+1)}$$
$$+\frac{M^{\alpha_2+\alpha_4}\left(\varphi(t)-\varphi(a)\right)^{\alpha_3}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_{u_n}(t)-\Upsilon_u(t)\right)\right)+\mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u_n(b)}{b^\lambda}-\frac{u(b)}{b^\lambda}\right|}{|\Delta|\Gamma(\alpha_3+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}\Bigg\}$$

by Lemma 4, and Proposition 1, we obtain:

$$\|Hu_n - Hu\|_\infty \le \left[\mathbf{N}_1 + \mathbf{N}_2\left((1+\beta)d_2\delta_2 + 2d_1\delta_1 + \delta_3\right)\right]\|u_n - u\|_X .$$

Thanks to hypothesis (A_1), so

$$\|Hu_n - Hu\|_\infty \to 0, \text{ as } n \to \infty.$$

Similarly, it can be shown that:

$$\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu_n - {}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu\right\|_\infty \le \left[\mathbf{N}_3 + \mathbf{N}_4\left((1+\beta)d_2\delta_2 + 2d_1\delta_1 + \delta_3\right)\right]\|u_n - u\|_X .$$

Therefore, it yields that:

$$\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu_n - {}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hu\right\|_\infty \to 0, \text{ as } n \to \infty.$$

We conclude that

$$\|Hu_n - Hu\|_X \to 0, \text{ as } n \to \infty$$

Consequently, H is continuous on X.

Second step: H maps bounded sets into bounded sets in X. Indeed, it is enough to show that for any $r>0$, there exists a positive constant l such that for each

$$u \in B_r = \{u \in X, \|u\|_X \leq r,\} \text{ one has } \|u\|_X \leq l \ .$$

Let $v \in B_r$.

We put:

$$l = \max\left\{\left(\mathbf{N}_1+\omega\right)r + k\mathbf{N}_2, \left(\mathbf{N}_3+\omega\right)r + k\mathbf{N}_4\right\}$$

By (A_1) and (A_3), for all $t \in J$, and by Lemma 4, and Proposition 1, we obtain,

$$\|Hv\|_\infty \leq \left(\mathbf{N}_1+\omega\right)r + k\mathbf{N}_2$$

Also

$$\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi} Hu\right\|_\infty \leq \left(\mathbf{N}_3+\omega\right)r + k\mathbf{N}_4.$$

Hence,

$$\|Hv\|_X \leq \max\left\{\left(\mathbf{N}_1+\omega\right)r + k\mathbf{N}_2, \left(\mathbf{N}_3+\omega\right)r + k\mathbf{N}_4\right\} \leq l.$$

Consequently, H is uniformly bounded on B_r.

Third step: H maps bounded sets into equicontinuous sets of X. The functions φ, u, g are continuous, hence the operator H is continuous. For any $u \in B_r$ and $t_1, t_2 \in [a,b]$ such that $t_1 < t_2$, by Proposition 1, and Lemma 5 we have

$$\begin{aligned}
\left\|\left(Hu\right)\left(t_2\right)-\left(Hu\right)\left(t_1\right)\right\|_\infty \leq \sup\Big\{ & I_{a^+}^{\alpha_1+\alpha_3+\alpha_3;\varphi}\left(\Upsilon_u(t_2)-\Upsilon_u(t_1)\right)+\mu I_{a^+}^{\alpha_3;\varphi}\left|\frac{u(t_2)}{t_2^\lambda}-\frac{u(t_1)}{t_1^\lambda}\right| \\
&+\left(\frac{M^{\alpha_2-\alpha_5}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}+\frac{M^{-\alpha_5}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_2+\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\right) I_{a^+}^{\alpha_5;\varphi}|u(b)| \\
&+\frac{M^{-\alpha_5}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_2+\alpha_3}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_u(b)\right)+\mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right) \\
&+\frac{M^{\alpha_2-\alpha_5}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left(\Upsilon_u(b)\right)+\mu I_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right) \\
&+\frac{M^{\alpha_4}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_2+\alpha_3}}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_u(b)\right)+\mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right) \\
&+\frac{M^{\alpha_2+\alpha_4}\left(\varphi(t_2)-\varphi(t_1)\right)^{\alpha_3}}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\left(I_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left(\Upsilon_u(b)\right)+\mu I_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)
\end{aligned}$$

So,

$$\begin{aligned}&\left|(Hu)(t_2)-(Hu)(t_1)\right|\\&\leq \frac{\left(k+\omega\|u\|_X\right)}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3+1\right)}(\varphi(t_2)-\varphi(t_1))^{\alpha_1+\alpha_2+\alpha_3}+\mathbf{A}_1(\varphi(t_2)-\varphi(t_1))^{\alpha_2+\alpha_3}+\mathbf{A}_2(\varphi(t_2)-\varphi(t_1))^{\alpha_3}\end{aligned}$$

where,

$$\begin{aligned}\mathbf{A}_1=\sup\Bigg\{&\frac{M^{\alpha_2-\alpha_5}\mathbf{I}_{a^+}^{\alpha_5;\varphi}\left|u(b)\right|}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\\&+\frac{M^{\alpha_2-\alpha_5}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3-\alpha_4+1\right)}\\&+\frac{M^{\alpha_4}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{|\Delta|\Gamma\left(\alpha_2+\alpha_3+1\right)\Gamma\left(\alpha_3+\alpha_5+1\right)}\Bigg\},\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}_2=\sup\Bigg\{&\frac{M^{-\alpha_5}\mathbf{I}_{a^+}^{\alpha_5;\varphi}\left|u(b)\right|}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}+\frac{\mu a^{-\lambda}\|u\|_\infty}{\Gamma\left(\alpha_3+1\right)}\\&+\frac{M^{-\alpha_5}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\\&+\frac{M^{\alpha_2+\alpha_4}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{|\Delta|\Gamma\left(\alpha_3+1\right)\Gamma\left(\alpha_2+\alpha_3+\alpha_5+1\right)}\Bigg\};\end{aligned}$$

Also,

$$\begin{aligned}&\left|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}\left(Hu\right)(t_2)-{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}\left(Hu\right)(t_1)\right|\\&\leq\frac{\left(k+\omega\|u\|_X\right)}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3-\alpha_4+1\right)}(\varphi(t_2)-\varphi(t_1))^{\alpha_1+\alpha_2+\alpha_3-\alpha_4}\\&+\mathbf{B}_1(\varphi(t_2)-\varphi(t_1))^{\alpha_2+\alpha_3-\alpha_4}+\mathbf{B}_2(\varphi(t_2)-\varphi(t_1))^{\alpha_3-\alpha_4}\end{aligned}$$

where

$$\begin{aligned}\mathbf{B}_1=\sup\Bigg\{&\frac{M^{\alpha_4}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{|\Delta|\Gamma\left(\alpha_3+\alpha_5+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\\&+\frac{M^{-\alpha_5}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right|+\mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right|+\mathbf{I}_{a^+}^{\alpha_5;\varphi}\left|u(b)\right|\right)}{|\Delta|\Gamma\left(\alpha_3-\alpha_4+1\right)\Gamma\left(\alpha_2+\alpha_3-\alpha_4+1\right)}\Bigg\},\end{aligned}$$

and

$$\mathbf{B}_2 = \sup\left\{\frac{\mu a^{-\lambda}\left\|u\right\|_\infty}{\Gamma(\alpha_3-\alpha_4+1)} + \frac{M^{\alpha_2-\alpha_5}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3+\alpha_5;\varphi}\left|\Upsilon_u(b)\right| + \mu\mathbf{I}_{a^+}^{\alpha_3+\alpha_5;\varphi}\left|\frac{u(b)}{b^\lambda}\right| + \mathbf{I}_{a^+}^{\alpha_5;\varphi}\left|u(b)\right|\right)}{\left|\Delta\right|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3-\alpha_4+1)} + \frac{M^{\alpha_2+\alpha_4}\left(\mathbf{I}_{a^+}^{\alpha_1+\alpha_2+\alpha_3-\alpha_4;\varphi}\left|\Upsilon_u(b)\right| + \mu\mathbf{I}_{a^+}^{\alpha_3-\alpha_4;\varphi}\left|\frac{u(b)}{b^\lambda}\right|\right)}{\left|\Delta\right|\Gamma(\alpha_3-\alpha_4+1)\Gamma(\alpha_2+\alpha_3+\alpha_5+1)}\right\}.$$

The right hand sides of Equation (1.1) tend to zero independently of u as $t_2 \to t_1$. As a consequence of first step second step and third step together with the Arzela-Ascoli theorem, we conclude that H is completely continuous.

Forth step: The set $\Lambda = \{u \in X : u = \sigma Hu, 0 < \sigma < 1\}$ is bounded: let $w \in \Lambda$ then $w = \sigma Hw,$ hence, we can write

$$(a) \qquad \left\|w\right\|_\infty = \left\|\sigma Hw\right\|_\infty = \sigma\left\|Hw\right\|_\infty \le \sigma\left((\mathbf{N}_1+\omega)r + k\mathbf{N}_2\right) \le \sigma r.$$

We have also,

$$(b) \qquad \left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}w\right\|_\infty = \left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}(\sigma Hw)\right\|_\infty = \sigma\left\|{}^c\mathbf{D}_{a^+}^{\alpha_4;\varphi}Hw\right\|_\infty \le \sigma\left((\mathbf{N}_3+\omega)r + k\mathbf{N}_4\right) \le \sigma r.$$

By (a) and (b) we state that $\left\|w\right\|_X < +\infty$. The set is thus bounded. As a consequence of Schauder fixed point theorem, we deduce that H has a fixed point which is a solution of the problem (1.1).

Example: *Consider the following nonlinear Lin-Emden equation of fractional orders*

$$\begin{cases} {}^c\mathbf{D}_{a^+}^{0,5;\varphi}\left({}^c\mathbf{D}_{a^+}^{0,25;\varphi}\left({}^c\mathbf{D}_{a^+}^{0,5;\varphi} + \frac{\pi}{t^{0,05}}\right)\right)u(t) = g(t,u(t),{}^c\mathbf{D}_{a^+}^{0,33;\varphi}u(t)) + 2f(t,u(t),\mathbf{I}_{a^+}^{0,2;\varphi}u(t)) \\ \qquad +h(t,u(t)) + x(t), t \in \left[\frac{1}{2},1\right], \\ \varphi(t) = t^2, x(t) = \frac{t}{t^2+1}, h(t,u(t)) = \frac{\sqrt{t}}{11} + \frac{1}{9}u(t) \\ g(t,u(t),{}^c\mathbf{D}_{a^+}^{\frac{1}{3};\varphi}u(t)) = \frac{t}{4} + \left(\sin\frac{\pi t}{2} - \frac{1}{8}\right)^2 u(t) + \frac{t^2}{5}\left({}^c\mathbf{D}_{a^+}^{\frac{1}{3};\varphi}\right)u(t) \\ f(t,u(t),\mathbf{I}_{a^+}^{\frac{1}{5};\varphi}u(t)) = \frac{t^2+1}{9} + \frac{1}{7}\left(\frac{t}{5} - \cos\pi t\right)^2 u(t) + \frac{t}{3}\left(\mathbf{I}_{a^+}^{\frac{1}{5};\varphi}\right)u(t) \\ \phi_1(t) = \frac{t}{4}, \quad \phi_2(t) = \frac{t^2+1}{9}, \quad \phi_3(t) = \frac{\sqrt{t}}{11} \end{cases}$$

Observe that the function f,g , is continuous, also

$$|h(t,u)| \le \frac{1}{11} + \frac{|u|}{9}, \text{ and } |h(t,u) - h(t,w)| \le \frac{|u-w|}{9},$$

$$|g(t,u,v)| \le \frac{1}{4} + \frac{49|u|}{64} + \frac{|v|}{5}, \text{ and } |g(t,u,v) - g(t,w,z)| \le \frac{49|u-w|}{64} + \frac{|v-z|}{5},$$

$$|f(t,u,v)| \le \frac{2}{9} + \frac{|u|}{175} + \frac{|v|}{3}, \text{ and } |f(t,u,v) - f(t,w,z)| \le \frac{|u-w|}{175} + \frac{|v-z|}{3},$$

$$\|\phi_1\|_\infty = \frac{1}{4}, \quad \|\phi_2\|_\infty = \frac{2}{9}, \quad \|\phi_3\|_\infty = \frac{1}{11}, \text{ and } \|x\|_\infty = \frac{1}{2};$$

Thus, the assumptions (A_1) and (A_2) are satisfied and Theorem 1 implies that the problem has a solution.

5. CONCLUSION

In this paper, we study a Lin-Emden integral differential problem involving three derivatives of phi-Caputo, and we prove the existence of the results by means of Shauder's theorem. After that, we investigated the uniqueness of the solution using the Banach contraction principle, and finally we also supported this with an illustrative example to demonstrate the applicability of our results.

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