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THE NOTES ON ENERGY AND ELECTROMAGNETIC FIELD VECTORS IN THE NULL CONE Q^2

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Abstract. *In this paper, we characterize the directional derivatives in accordance with the asymptotic orthonormal frame* $\{x, \alpha, y\}$ *in* $\boldsymbol{Q}^2$. *Also, we express the extended Serret-Frenet relations by using cone Frenet formulas and we explain the geometrical understanding of energy on each asymptotic orthonormal vector fields in null cone. Furthermore, we express the bending elastic energy function for the same particle according to curve* $x(s, \xi, \eta)$ *and we finalize our results by providing energy variation sketches according to directional derivatives for different cases. Additionally, we explain a geometrical interpretation of the energy for unit vector fields and we express Maxwell's equations for the electric and magnetic field vectors in null cone 3-space.*

Keywords: *Electromagnetic field vectors; asymptotic orthonormal frame; null cone; energy.*

1. INTRODUCTION

The theory of integrable systems has an enormous impact in mathematics and in a wide variety of non-linear phenomena in physics. For example, the curves and surfaces arise naturally in numerous areas of the physical sciences and within areas of pure mathematics itself. The greatest effect in the research of curves and surfaces was the discovery of the calculus. In [1, 2], the impact of magnetic fields on the moving particle trajectories by variational approach to the magnetic flow associated with the Killing magnetic field on lightlike cone is examined and different magnetic curves are found in the 2-dimensional lightlike cone using the Killing magnetic field of these curves. Also, some characterizations for x-magnetic curve and x-magnetic surface of rotation are given using the Killing magnetic field of this curve in Q^2. In [3, 4], the energy and volume of vector fields are investigated by the authors. In [5], the similar studies are made about volume of a unit vector field defined as the volume of the submanifold in the unit tangent bundle. In [6, 7], the effects of geometric phase rotation are expressed topological features of traditional Maxwellian theory originating at the quantum level but surviving the associated principle limit into the classical level and differential geometry to broaden the preceding traditional analysis of the uniform helix and attain the comprehensive consequence for arbitrary fiber trajectory are comprehended. In [8], the pioneering connection between the solutions of the cubic nonlinear Schrodinger equation and the solutions localized induction equation are investigated by the authors. In [9], the kinematics properties of a moving particle lying in De-Sitter 3-space are given and the

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bending elastic energy function for the same particle are determined by the authors. Also, the change of the Willmore energy of curves in 3-dimensional Lorentzian space is investigate and given the variation of Frenet vector fields, the curvature and the torsion of the curve by the authors, [10]. In [11], the rotation of the plane of polarization of light spreading in a monomode optical fiber tracing a non-planar trajectory are expressed and are given the rotation in the helical optical fiber with a constant-torsion with some measurements on the fiber bent into a helix. In [12], the geometric generalization of the action belonging to curvature of the moving particle's trajectory in different spacetimes are given by the authors. In [13], a classical problem in hydrodynamics originally posed by Gilbarg has been reduced to that of solving a solitonic Heisenberg spin equation subject to a geometric constraint and some formulations are given to lead to a class of solutions of the Gilbarg problem according to travelling wave solutions.

Broading speaking, the study of the vector fields of curves represents the beginning of a major area of mathematical physics. In mathematical physics, one of the important topics is electromagnetic wave that is constrained to travel along with a space curve. In this context, the geometry of the curve provides to relate the geometric phase and the rotation of the polarization of the electromagnetic wave vectors. In the light of this information, some studies have been made. For example, a rotation of the polarization of light propagating in a monochromatic optical fiber wrapping around the conductor induced by the magnetic field according to the electric current flowing in an optical fiber current transformer are examined in [14]. The unit vector field's energy on a Riemannian manifold is described to be equal to the energy of the mapping defined as unit tangent bundle equipped with Sasaki metric [15].

2. MATERIALS AND METHODS

Let E_1^3 be the $3-$dimensional pseudo-Euclidean space with

$$\tilde{g}(X,Y) = \langle X,Y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

for all $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in E_1^3$. E_1^3 is a flat pseudo-Riemannian manifold of signature (2,1).

Let M be a submanifold of E_1^3. If the pseudo-Riemannian metric $\tilde{g}$ of E_1^3 induces a pseudo-Riemannian metric g(respectively, a Riemannian metric, a degenerate quadratic form) on M, then M is called a time-like(respectively, space-like, degenerate) submanifold of E_1^3. Let c be a fixed point in E_1^3. The pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$\mathbf{Q}_1^2(c) = \{x \in E_1^3 : g(x-c, x-c) = 0\},$$

where the point c is called the center of $\mathbf{Q}_1^2(c)$. When $c = 0$, we merely indicate $\mathbf{Q}_1^2(0)$ by $\mathbf{Q}^2$ be and call it the null cone.

Let E_1^3 be 3-dimensional Minkowski space and $\mathbf{Q}^2$ the lightlike cone in E_1^3. A vector $V \neq 0$ in E_1^3 is called space-like, time-like or light-like, if $\langle W,W\rangle > 0$, $\langle W,W\rangle < 0$ or $\langle W,W\rangle = 0$, respectively. The norm of a vector $x \in E_1^3$ is given by $\|x\| = \sqrt{\langle x,x\rangle}$, [16].

We suppose that curve $x: I \to \mathbf{Q}^2 \subset E_1^3$ is a regular curve in $\mathbf{Q}^2$ for $t \in I$. Below, we always suppose that the curve is regular. A frame field $\{x, \alpha, y\}$ on E_1^3 is called an asymptotic orthonormal frame field, if

$$\langle x,x\rangle = \langle y,y\rangle = \langle x,\alpha\rangle = \langle y,\alpha\rangle = 0, \langle x,y\rangle = \langle \alpha,\alpha\rangle = 1.$$

By using $x'(s)=\alpha(s)$ we have that $\{x(s),\alpha(s),y(s)\}$ from an asymptotic orthonormal frame throughout the curve $x(s)$ and the cone Frenet formulas of $x(s)$ are written as

$$x'(s)=\alpha(s)$$

$$\alpha'(s)=\kappa(s)x(s)-y(s) \tag{2.1}$$

$$y'(s)=-\kappa(s)\alpha(s),$$

where the function $\kappa(s)$ is called cone curvature function of the curve $x(s)$, [17].

Let $x: I \to \mathbf{Q}^2 \subset E_1^3$ be a space-like curve in $\mathbf{Q}^2$ with an arc length parameter s. Then $x=x(s)=(x_1,x_2,x_3)$ can be given as

$$x(s)=\frac{f_s^{-1}}{2}(f^2-1,2f,\ f^2+1), \tag{2.2}$$

for some non constant function $f(s)$ and $f_s=f'$, [18].

Definition 1. For two Riemannian manifolds (M,ϱ) and (N,h) the energy of a differentiable map $f:(M,\varrho)\to(N,h)$ is given as

$$energy(f)=\frac{1}{2}\int_M \sum_{a=1}^{n} h(df(e_a),df(e_a))v, \tag{2.3}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [15, 19].

Definition 2. Let $Q: T(T^1M)\to T^1M$ be the connection map. Then, the following conditions satisfy:

i) $\omega o Q=\omega o d\omega$ and $\omega o Q=\omega o \varpi$ where $\varpi: T(T^1M)\to T^1M$ is the tangent bundle projection;

ii) for $\varrho \in T_xM$ and a section $\xi: M\to T^1M;$ we have

$$Q(d\xi(\varrho))=D_\varrho \xi, \tag{2.4}$$

where D is the Levi-Civita covariant derivative [15, 19].

Definition 3. For$\varsigma_1,\varsigma_2 \in T_\xi(T^1M)$, Riemannian metric on TM is defined as

$$\varrho_S(\varsigma_1,\varsigma_2)=\varrho(d\omega(\varsigma_1),d\omega(\varsigma_2))+\varrho(Q(\varsigma_1),Q(\varsigma_2)). \tag{2.5}$$

Here, as knownϱ_S is called the Sasaki metric that also makes the projection $\omega: T^1M\to M$ a Riemannian submersion [15, 19].

3. THE REPRESENTATION OF THE EXTENDED SERRET-FRENET RELATIONS IN NULL CONE $\mathbf{Q}^2$

In this section, we describe the directional derivatives in accordance with the asymptotic orthonormal frame $\{x,\alpha,y\}$ in $\mathbf{Q}^2$. Also, we express the extended Serret-Frenet relations using cone Frenet formulas.

Let $x = x(s, \xi, \eta)$ be a space curve lying in null cone 3-space. we know that some Formulas used to characterize 3D vector space and geometry of curvatures of vector lines in anholonomic coordinates. Then, the following cases are held:

1) s is the distance along the s-lines of the curve in s −direction. Hence, tangent vector of s-lines is described by $\alpha = \alpha(s, \xi, \eta) = x'$.

2) ξ is the distance along the ξ-lines of the curve in ξ −direction. Hence, tangent vector of ξ-lines is described by $x = x(s, \xi, \eta) = \partial_\xi x$.

3) η is the distance along the η-lines of the curve in η −direction and tangent vector of η-lines is described by $y = y(s, \xi, \eta) = \partial_\eta x$.

First to find the extended Frenet relations let's think the the gradient operator ∇ given by

$$\nabla = \overleftarrow{x}\frac{\partial}{\partial s} + \overleftarrow{\alpha}\frac{\partial}{\partial \xi} + \overleftarrow{y}\frac{\partial}{\partial \eta}.$$

Thus, we obtain the other geometric quantities given by

a) For $div\overleftarrow{x}$, since $\frac{\partial \overleftarrow{x}}{\partial s} = \alpha$ we get

$$div\overleftarrow{x} = \nabla\overleftarrow{x} = \overleftarrow{\alpha}\frac{\partial \overleftarrow{x}}{\partial \xi} + \overleftarrow{y}\frac{\partial \overleftarrow{x}}{\partial \eta}.$$

b) For $div\overleftarrow{\alpha}$, since $\frac{\partial \overleftarrow{\alpha}}{\partial s} = \kappa x - y$ we get

$$div\overleftarrow{\alpha} = \nabla\overleftarrow{\alpha} = -1 + \overleftarrow{y}\frac{\partial \overleftarrow{\alpha}}{\partial \eta}.$$

c) For $div\overleftarrow{y}$, since $\frac{\partial \overleftarrow{y}}{\partial s} = -\kappa\alpha$ we get

$$div\overleftarrow{y} = \nabla\overleftarrow{y} = \overleftarrow{\alpha}\frac{\partial \overleftarrow{y}}{\partial \xi}.$$

d) For $curl\overleftarrow{x}$, since $\frac{\partial \overleftarrow{x}}{\partial s} = \alpha$ we get

$$curl\overleftarrow{x} = \left(\overleftarrow{y}\frac{\partial \overleftarrow{x}}{\partial \xi} - \overleftarrow{\alpha}\frac{\partial \overleftarrow{x}}{\partial \eta}, 0,1\right) = \left(-P_{xy}^{\xi} + M_{x\alpha}^{\eta}, 0,1\right).$$

e) For $curl\overleftarrow{\alpha}$, since $\frac{\partial \overleftarrow{\alpha}}{\partial s} = \kappa x - y$ we get

$$curl\overleftarrow{\alpha} = \left(\overleftarrow{y}\frac{\partial \overleftarrow{\alpha}}{\partial \xi}, \overleftarrow{x}\frac{\partial \overleftarrow{\alpha}}{\partial \eta} - \kappa, -\overleftarrow{x}\frac{\partial \overleftarrow{\alpha}}{\partial \xi}\right) = \left(-div\overleftarrow{y}, M_{x\alpha}^{\eta} - \kappa, -P_{x\alpha}^{\xi}\right).$$

f) For $curl\overleftarrow{y}$, since $\frac{\partial \overleftarrow{y}}{\partial s} = -\kappa\alpha$ we get

$$curl\overleftarrow{y} = \left(-\overleftarrow{\alpha}\frac{\partial \overleftarrow{y}}{\partial \eta}, \overleftarrow{x}\frac{\partial \overleftarrow{y}}{\partial \eta}, -\kappa - \overleftarrow{x}\frac{\partial \overleftarrow{y}}{\partial \xi}\right) = \left(div\overleftarrow{\alpha} + 1, M_{xy}^{\eta}, -\kappa - P_{xy}^{\xi}\right),$$

where the previous equations can be written as

$$-\overleftarrow{y}\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=\overleftarrow{\alpha}\frac{\partial\overleftarrow{y}}{\partial\xi}=div\overleftarrow{y};\overleftarrow{x}\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=-\overleftarrow{\alpha}\frac{\partial\overleftarrow{x}}{\partial\xi}=P^{\xi}_{x\alpha};\overleftarrow{x}\frac{\partial\overleftarrow{y}}{\partial\xi}=-\overleftarrow{y}\frac{\partial\overleftarrow{x}}{\partial\xi}=P^{\xi}_{xy};$$

$$-\overleftarrow{\alpha}\frac{\partial\overleftarrow{y}}{\partial\eta}=\overleftarrow{y}\frac{\partial\overleftarrow{\alpha}}{\partial\eta}=div\overleftarrow{\alpha}+1;\overleftarrow{x}\frac{\partial\overleftarrow{y}}{\partial\eta}=-\overleftarrow{y}\frac{\partial\overleftarrow{x}}{\partial\eta}=M^{\eta}_{xy};\overleftarrow{x}\frac{\partial\overleftarrow{\alpha}}{\partial\eta}=-\overleftarrow{\alpha}\frac{\partial\overleftarrow{x}}{\partial\eta}=M^{\eta}_{x\alpha};$$

$$\overleftarrow{\alpha}\frac{\partial\overleftarrow{y}}{\partial\xi}=div\overleftarrow{y};\overleftarrow{y}\frac{\partial\overleftarrow{\alpha}}{\partial\eta}=div\overleftarrow{\alpha}+1;div\overleftarrow{x}=\overleftarrow{\alpha}\frac{\partial\overleftarrow{x}}{\partial\xi}+\overleftarrow{y}\frac{\partial\overleftarrow{x}}{\partial\eta}.$$

For the derivatives of the vector fields x,α,y with respect to ξ, we can calculate as follows, respectively

a) For $\frac{\partial\overleftarrow{x}}{\partial\xi}$, we have

$$\frac{\partial\overleftarrow{x}}{\partial\xi}=a_1^1\overleftarrow{x}+a_2^1\alpha+a_3^1\overleftarrow{y}\Rightarrow a_1^1=\overleftarrow{y}\frac{\partial\overleftarrow{x}}{\partial\xi}=-P^{\xi}_{xy};a_2^1=\overleftarrow{\alpha}\frac{\partial\overleftarrow{x}}{\partial\xi}=-P^{\xi}_{x\alpha};a_3^1=0$$

$$\frac{\partial\overleftarrow{x}}{\partial\xi}=-P^{\xi}_{xy}\overleftarrow{x}-P^{\xi}_{x\alpha}\overleftarrow{\alpha}.$$

b) For $\frac{\partial\overleftarrow{\alpha}}{\partial\xi}$, we have

$$\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=a_1^2\overleftarrow{x}+a_2^2\alpha+a_3^2\overleftarrow{y}\Rightarrow a_1^2=\overleftarrow{y}\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=-div\overleftarrow{y};a_2^2=0;a_3^2=\overleftarrow{x}\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=P^{\xi}_{x\alpha}$$

$$\frac{\partial\overleftarrow{\alpha}}{\partial\xi}=-div\overleftarrow{y}\overleftarrow{x}+P^{\xi}_{x\alpha}\overleftarrow{y}.$$

c) For $\frac{\partial\overleftarrow{y}}{\partial\xi}$, we have

$$\frac{\partial\overleftarrow{y}}{\partial\xi}=a_1^3\overleftarrow{x}+a_2^3\alpha+a_3^3\overleftarrow{y}\Rightarrow a_1^3=0;a_2^3=\overleftarrow{\alpha}\frac{\partial\overleftarrow{y}}{\partial\xi}=div\overleftarrow{y};a_3^3=\overleftarrow{x}\frac{\partial\overleftarrow{y}}{\partial\xi}=P^{\xi}_{x\alpha}$$

$$\frac{\partial\overleftarrow{y}}{\partial\xi}=div\overleftarrow{y}\overleftarrow{\alpha}+P^{\xi}_{xy}\overleftarrow{y}.$$

Therefore, from the last equations, we write Serret-Frenet relations in the following forms

$$\frac{d}{d\xi}\begin{bmatrix}x\\ \alpha\\ y\end{bmatrix}=\begin{bmatrix}-P^{\xi}_{xy} & -P^{\xi}_{x\alpha} & 0\\ -div\overleftarrow{y} & 0 & P^{\xi}_{x\alpha}\\ 0 & div\overleftarrow{y} & P^{\xi}_{xy}\end{bmatrix}\begin{bmatrix}x\\ \alpha\\ y\end{bmatrix}.$$

Also, from the equations $curl\overleftarrow{x},\ curl\overleftarrow{\alpha},\ curl\overleftarrow{y}$, we can write as follows

$$curl\overleftarrow{x}=\left(-P^{\xi}_{xy}+M^{\eta}_{x\alpha}\right)\overleftarrow{x}+\overleftarrow{y};$$

$$curl\overleftarrow{\alpha}=(-div\overleftarrow{y})\overleftarrow{x}+\left(M^{\eta}_{x\alpha}-\kappa\right)\overleftarrow{\alpha}+\left(-P^{\xi}_{x\alpha}\right)\overleftarrow{y};$$

$$curl\overleftarrow{y} = (div\overleftarrow{\alpha} + 1)\overleftarrow{x} + \left(M_{xy}^{\eta}\right)\overleftarrow{\alpha} + \left(-\kappa - P_{xy}^{\xi}\right)\overleftarrow{y}$$

and by using these equations, we obtain

$$\overleftarrow{y}curl\overleftarrow{x} = -P_{xy}^{\xi} + M_{x\alpha}^{\eta};\ \overleftarrow{x}curl\overleftarrow{x} = 1;$$

$$\overleftarrow{y}curl\overleftarrow{\alpha} = -div\overleftarrow{y};\ \overleftarrow{\alpha}curl\overleftarrow{\alpha} = M_{x\alpha}^{\eta} - \kappa;\ \overleftarrow{x}curl\overleftarrow{\alpha} = -P_{x\alpha}^{\xi};$$

$$\overleftarrow{y}curl\overleftarrow{y} = div\overleftarrow{\alpha} + 1;\ \overleftarrow{\alpha}curl\overleftarrow{y} = M_{xy}^{\eta};\ \overleftarrow{x}curl\overleftarrow{y} = -\kappa - P_{xy}^{\xi}$$

or

$$div\overleftarrow{y} = -\overleftarrow{y}curl\overleftarrow{\alpha};\ div\overleftarrow{\alpha} = \overleftarrow{y}curl\overleftarrow{y} - 1;\ M_{xy}^{\eta} = \overleftarrow{\alpha}curl\overleftarrow{y};$$

$$M_{x\alpha}^{\eta} = \overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa;\ P_{x\alpha}^{\xi} = -\overleftarrow{x}curl\overleftarrow{\alpha};\ P_{xy}^{\xi} = -\kappa - \overleftarrow{x}curl\overleftarrow{y}$$

and

$$\frac{d}{d\xi}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix} = \begin{bmatrix} \kappa + \overleftarrow{x}curl\overleftarrow{y} & \overleftarrow{x}curl\overleftarrow{\alpha} & 0 \\ -div\overleftarrow{y} & 0 & -\overleftarrow{x}curl\overleftarrow{\alpha} \\ 0 & div\overleftarrow{y} & -\kappa - \overleftarrow{x}curl\overleftarrow{y} \end{bmatrix}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}.$$

For the derivatives of the vector fields x, α, y with respect to η , we can compute as follows

a) For $\frac{\partial \overleftarrow{x}}{\partial \eta}$, we get

$$\frac{\partial \overleftarrow{x}}{\partial \eta} = b_1^1\overleftarrow{x} + b_2^1\alpha + b_3^1\overleftarrow{y} \Rightarrow b_1^1 = \overleftarrow{y}\frac{\partial \overleftarrow{x}}{\partial \eta} = -M_{xy}^{\eta}; b_2^1 = \overleftarrow{\alpha}\frac{\partial \overleftarrow{x}}{\partial \eta} = -M_{x\alpha}^{\eta}; b_3^1 = 0$$

$$\frac{\partial \overleftarrow{x}}{\partial \eta} = -M_{xy}^{\eta}\overleftarrow{x} - M_{x\alpha}^{\eta}\overleftarrow{\alpha}.$$

b) For $\frac{\partial \overleftarrow{\alpha}}{\partial \eta}$, we get

$$\frac{\partial \overleftarrow{\alpha}}{\partial \eta} = b_1^2\overleftarrow{x} + b_2^2\overleftarrow{\alpha} + b_3^2\overleftarrow{y} \Rightarrow b_1^2 = \overleftarrow{y}\frac{\partial \overleftarrow{\alpha}}{\partial \eta} = div\overleftarrow{\alpha} + 1; b_2^2 = 0; b_3^2 = \overleftarrow{x}\frac{\partial \overleftarrow{\alpha}}{\partial \eta} = M_{x\alpha}^{\eta}$$

$$\frac{\partial \overleftarrow{\alpha}}{\partial \eta} = (div\overleftarrow{\alpha} + 1)\overleftarrow{x} + M_{x\alpha}^{\eta}\overleftarrow{y}.$$

c) For $\frac{\partial \overleftarrow{y}}{\partial \eta}$, we get

$$\frac{\partial \overleftarrow{y}}{\partial \eta} = b_1^3\overleftarrow{x} + b_2^3\overleftarrow{\alpha} + b_3^3\overleftarrow{y} \Rightarrow b_1^3 = 0; b_2^3 = \overleftarrow{\alpha}\frac{\partial \overleftarrow{y}}{\partial \eta} = -(div\overleftarrow{\alpha} + 1); b_3^3 = \overleftarrow{x}\frac{\partial \overleftarrow{y}}{\partial \eta} = M_{xy}^{\eta};$$

$$\frac{\partial \overleftarrow{y}}{\partial \eta} = -(div\overleftarrow{\alpha} + 1)\overleftarrow{\alpha} + M_{xy}^{\eta}\overleftarrow{y}.$$

Therefore, from the last equations, we write out Serret-Frenet relations in the following forms

$$\frac{d}{d\eta}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix} = \begin{bmatrix} -M_{xy}^{\eta} & -M_{x\alpha}^{\eta} & 0 \\ div\overleftarrow{\alpha}+1 & 0 & M_{x\alpha}^{\eta} \\ 0 & -(div\overleftarrow{\alpha}+1) & M_{xy}^{\eta} \end{bmatrix}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}.$$

Also, from the equations $curl\overleftarrow{x}$, $curl\overleftarrow{\alpha}$, $curl\overleftarrow{y}$, and by using these equations, we obtain

$$\overleftarrow{y}curl\overleftarrow{x} = -P_{xy}^{\xi} + M_{x\alpha}^{\eta};\ \overleftarrow{x}curl\overleftarrow{x} = 1;$$

$$\overleftarrow{y}curl\overleftarrow{\alpha} = -div\overleftarrow{y};\ \overleftarrow{\alpha}curl\overleftarrow{\alpha} = M_{x\alpha}^{\eta} - \kappa;\ \overleftarrow{x}curl\overleftarrow{\alpha} = -P_{x\alpha}^{\xi};$$

$$\overleftarrow{y}curl\overleftarrow{y} = div\overleftarrow{\alpha} + 1;\ \overleftarrow{\alpha}curl\overleftarrow{y} = M_{xy}^{\eta};\ \overleftarrow{x}curl\overleftarrow{y} = -\kappa - P_{xy}^{\xi}$$

or

$$div\overleftarrow{y} = -\overleftarrow{y}curl\overleftarrow{\alpha};\ div\overleftarrow{\alpha} = \overleftarrow{y}curl\overleftarrow{y} - 1;\ M_{xy}^{\eta} = \overleftarrow{\alpha}curl\overleftarrow{y};$$

$$M_{x\alpha}^{\eta} = \overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa;\ P_{x\alpha}^{\xi} = -\overleftarrow{x}curl\overleftarrow{\alpha};\ P_{xy}^{\xi} = -\kappa - \overleftarrow{x}curl\overleftarrow{y}$$

and

$$\frac{d}{d\eta}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix} = \begin{bmatrix} -\overleftarrow{\alpha}curl\overleftarrow{y} & -(\overleftarrow{\alpha}curl\overleftarrow{\alpha}+\kappa) & 0 \\ div\overleftarrow{\alpha}+1 & 0 & \overleftarrow{\alpha}curl\overleftarrow{\alpha}+\kappa \\ 0 & -(div\overleftarrow{\alpha}+1) & \overleftarrow{\alpha}curl\overleftarrow{y} \end{bmatrix}\begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}.$$

4. THE REPRESENTATION OF MAXWELL'S EQUATIONS OF ELECTROMAGNETIC WAVE VECTOR FIELDS IN NULL CONE Q^2

If we want to understand the electromagnetic theory, we have to know Maxwell's equations. So that Electromagnetic waves propagated along the optical fiber and the electromagnetic waves spread through the optical fiber in which its axis is expressed by the curve x. On account of the vectorial nature of the light electromagnetic waves are defined by using the vector fields. The orientation of the electromagnetic wave in the fiber is defined by using the asymptotic orthonormal frame of vectors $\{x, \alpha, y\}$ in null cone 3-space.

The spreading of the electromagnetic wave is in the direction of α, direction of the electric field vector E are expressed as the polarization of the electromagnetic wave. Also, the electromagnetic wave carries magnetic field vector B. Consequently, the electromagnetic vectors E and B may be considered as a physically coordinate frame, which are expressed according to asymptotic orthonormal unit vectors $\{x, \alpha, y\}$. For an electromagnetic wave of a space curve, the electric field vector E and the magnetic field vector B are expected to perform a rotation in the tangential $\alpha -$direction according to the asymptotic orthonormal unit vectors $\{x, \alpha, y\}$.

Let E and B be the vectors of the electromagnetic wave, so that E and B are perpendicular to the tangent vector field $\alpha = x'$ along $x(s, \xi, \eta)$ [20]. The change of the electric vector E with respect to $\xi -$direction along the curve x is written as

$$\frac{\partial \overleftarrow{E}}{\partial \xi} = c_1^2\overleftarrow{x} + c_2^2\overleftarrow{\alpha} + c_3^2\overleftarrow{y}.$$

Also, since E and B are perpendicular to the tangent vector field $\alpha = x'$ along $x(s, \xi, \eta)$, we have

$$\langle \alpha, E\rangle = 0, \langle E, E\rangle = cons.; \overleftarrow{\alpha}.\frac{\partial \overleftarrow{E}}{\partial \xi} = -\overleftarrow{E}.\frac{\partial \overleftarrow{\alpha}}{\partial \xi};$$

$$\langle \alpha, B\rangle = 0, \langle B, B\rangle = cons.; \overleftarrow{\alpha}.\frac{\partial \overleftarrow{B}}{\partial \xi} = -\overleftarrow{B}.\frac{\partial \overleftarrow{\alpha}}{\partial \xi}$$

and

$$\overleftarrow{E}.\frac{\partial \overleftarrow{E}}{\partial \xi} = c_1^2\overleftarrow{E}.\overleftarrow{x} + c_2^2\overleftarrow{E}.\overleftarrow{\alpha} + c_3^2\overleftarrow{E}.\overleftarrow{y} = c_1^2\overleftarrow{E}.\overleftarrow{x} + c_3^2\overleftarrow{E}.\overleftarrow{y} = 0,$$

by using previous equation $\overleftarrow{E}.\overleftarrow{x} = 0 \wedge \overleftarrow{E}.\overleftarrow{y} = 0$, hence we obtain $\overleftarrow{E} = E^1\overleftarrow{x} + E^3\overleftarrow{y}$, for the components of the electric vector field we write

$$c_1^2 = \overleftarrow{y}\frac{\partial \overleftarrow{E}}{\partial \xi} = -\overleftarrow{E}\frac{\partial \overleftarrow{y}}{\partial \xi};\ c_2^2 = \overleftarrow{\alpha}\frac{\partial \overleftarrow{E}}{\partial \xi} = -\overleftarrow{E}\frac{\partial \overleftarrow{\alpha}}{\partial \xi};\ c_3^2 = \overleftarrow{x}\frac{\partial \overleftarrow{E}}{\partial \xi} = -\overleftarrow{E}\frac{\partial \overleftarrow{x}}{\partial \xi},$$

by using the derivatives $\frac{\partial \overleftarrow{y}}{\partial \xi}, \frac{\partial \overleftarrow{\alpha}}{\partial \xi}, \frac{\partial \overleftarrow{x}}{\partial \xi}$, we have

$$\frac{\partial \overleftarrow{E}}{\partial \xi} = \left(E^3P_{xy}^{\xi}\right)\overleftarrow{x} + \left(E^1 div\overleftarrow{y} + E^3P_{x\alpha}^{\xi}\right)\overleftarrow{\alpha} + \left(E^1P_{xy}^{\xi}\right)\overleftarrow{y}.$$

The change of the electric vector field $\overleftarrow{E}$ with respect to η −direction $\frac{\partial \overleftarrow{E}}{\partial \eta}$, we can write

$$\frac{\partial \overleftarrow{E}}{\partial \eta} = d_1^2\overleftarrow{x} + d_2^2\overleftarrow{\alpha} + d_3^2\overleftarrow{y}.$$

Also, the following equations hold

$$\overleftarrow{E}.\frac{\partial \overleftarrow{E}}{\partial \eta} = d_1^2\overleftarrow{E}.\overleftarrow{x} + d_2^2\overleftarrow{E}.\overleftarrow{\alpha} + d_3^2\overleftarrow{E}.\overleftarrow{y} = d_1^2\overleftarrow{E}.\overleftarrow{x} + d_3^2\overleftarrow{E}.\overleftarrow{y} = 0;$$

$$\overleftarrow{E}.\overleftarrow{x} = 0 \wedge \overleftarrow{E}.\overleftarrow{y} = 0 \Rightarrow \overleftarrow{E} \in Sp\{\overleftarrow{x}, \overleftarrow{y}\} \text{ and } \overleftarrow{E} = E^1\overleftarrow{x} + E^3\overleftarrow{y},$$

hence, we get

$$d_1^2 = \overleftarrow{y}\frac{\partial \overleftarrow{E}}{\partial \eta} = -\overleftarrow{E}\frac{\partial \overleftarrow{y}}{\partial \eta};\ d_2^2 = \overleftarrow{\alpha}\frac{\partial \overleftarrow{E}}{\partial \eta} = -\overleftarrow{E}\frac{\partial \overleftarrow{\alpha}}{\partial \eta};\ d_3^2 = \overleftarrow{x}\frac{\partial \overleftarrow{E}}{\partial \eta} = -\overleftarrow{E}\frac{\partial \overleftarrow{x}}{\partial \eta},$$

from the derivatives of the vector fields $\frac{\partial \overleftarrow{y}}{\partial \eta}, \frac{\partial \overleftarrow{\alpha}}{\partial \eta}, \frac{\partial \overleftarrow{x}}{\partial \eta}$ we obtain

$$\frac{\partial \overleftarrow{E}}{\partial \eta} = \left(E^3M_{xy}^{\eta}\right)\overleftarrow{x} + \left(-E^1(div\overleftarrow{\alpha} + 1) + E^3M_{x\alpha}^{\eta}\right)\overleftarrow{\alpha} + \left(M_{xy}^{\eta}E^1\right)\overleftarrow{y}.$$

Similarly, for the change of the electric vector field $\overleftarrow{E}$ with respect to s −direction $\frac{\partial \overleftarrow{E}}{\partial \eta}$, we obtain

$$\frac{\partial \overleftarrow{E}}{\partial s} = d_1^3\overleftarrow{x} + d_2^3\overleftarrow{\alpha} + d_3^3\overleftarrow{y};$$

$$\overleftarrow{E}.\frac{\partial\overleftarrow{E}}{\partial s}=d_1^3\overleftarrow{E}.\overleftarrow{x}+d_2^3\overleftarrow{E}.\overleftarrow{\alpha}+d_3^3\overleftarrow{E}.\overleftarrow{y}=d_1^2\overleftarrow{E}.\overleftarrow{x}+d_3^2\overleftarrow{E}.\overleftarrow{y}=0$$

$$\overleftarrow{E}.\overleftarrow{x}=0\wedge\overleftarrow{E}.\overleftarrow{y}=0\Rightarrow\overleftarrow{E}\in Sp\{\overleftarrow{x},\overleftarrow{y}\}\text{ and }\overleftarrow{E}=E^1\overleftarrow{x}+E^3\overleftarrow{y}.$$

Therefore, the components of $\frac{\partial\overleftarrow{E}}{\partial s}$ are obtained as follows

$$d_1^3=\overleftarrow{y}\frac{\partial\overleftarrow{E}}{\partial s}=-\overleftarrow{E}\frac{\partial\overleftarrow{y}}{\partial s};\ d_2^2=\overleftarrow{\alpha}\frac{\partial\overleftarrow{E}}{\partial s}=-\overleftarrow{E}\frac{\partial\overleftarrow{\alpha}}{\partial s};\ d_3^2=\overleftarrow{x}\frac{\partial\overleftarrow{E}}{\partial s}=-\overleftarrow{E}\frac{\partial\overleftarrow{x}}{\partial s}.$$

Also, from the equations $\frac{\partial\overleftarrow{y}}{\partial s}$, $\frac{\partial\overleftarrow{\alpha}}{\partial s}$, $\frac{\partial\overleftarrow{x}}{\partial s}$ we have

$$\frac{\partial\overleftarrow{E}}{\partial s}=-\overleftarrow{E}.(-\kappa\overleftarrow{\alpha})\overleftarrow{x}-\overleftarrow{E}.(\kappa\overleftarrow{x}-\overleftarrow{y})\overleftarrow{\alpha}-\overleftarrow{E}.\overleftarrow{\alpha}\overleftarrow{y}=-(E^1\kappa+E^3)\overleftarrow{\alpha}.$$

Hence, we compute that

$$\nabla.\overleftarrow{E}=\left(\overleftarrow{x}\frac{\partial}{\partial s}+\overleftarrow{\alpha}\frac{\partial}{\partial s}+\overleftarrow{y}\frac{\partial}{\partial s}\right).\overleftarrow{E}=\left(\overleftarrow{x}\frac{\partial\overleftarrow{E}}{\partial s}+\overleftarrow{\alpha}\frac{\partial\overleftarrow{E}}{\partial\xi}+\overleftarrow{y}\frac{\partial\overleftarrow{E}}{\partial\xi}\right);$$

$$\nabla.\overleftarrow{E}=\begin{pmatrix}\overleftarrow{x}(-(E^1\kappa+E^3)\overleftarrow{\alpha})\\+\overleftarrow{\alpha}\left(\left(E^3P_{xy}^{\xi}\right)\overleftarrow{x}+\left(E^1div\overleftarrow{y}+E^3P_{x\alpha}^{\xi}\right)\overleftarrow{\alpha}+\left(E^1P_{xy}^{\xi}\right)\overleftarrow{y}\right)\\+\overleftarrow{y}\begin{pmatrix}(E^3M_{xy}^{\eta})\overleftarrow{x}+\begin{pmatrix}-E^1(div\overleftarrow{\alpha}+1)\\+E^3M_{x\alpha}^{\eta}\end{pmatrix}\overleftarrow{\alpha}\\+(M_{xy}^{\eta}E^1)\overleftarrow{y}\end{pmatrix}\end{pmatrix},$$

which implies that

$$\nabla.\overleftarrow{E}=E^1div\overleftarrow{y}+E^3\left(P_{x\alpha}^{\xi}+M_{xy}^{\eta}\right)=0.$$

The Hall effect is a classical phenomenon for uniform magnetic fields in Euclidean space and thus in a non-gravity environment. The dynamics of an electric current flow E in R^3 is expressed when exposed to a perpendicular uniform magnetic field B. The basic physical principle underlying the Hall effect Lorentz force appearing in the Lorentz force equation. Therefore, E experiences a force, the Lorentz force, acting normal to both E and B, and it moves in response to this force and the force affected by its internal electric field. The Lorentz force of a magnetic field B on (M^n,g) is defined to be the skew-symmetric operator given by $g(\Phi(X),Y)=B(X,Y)$, $\forall X,Y\in\chi(M)$ [21].

The associated magnetic trajectories are curves in M^n that satisfy the Lorentz equation. That is, α-magnetic trajectories of B are x on $\mathbf{Q}^2$ that satisfy the Lorentzian equation [21]:

$$\nabla_{x'}x'=\Phi(x').$$

A very special class of magnetic fields on a Riemannian manifold is that made up of parallel 2-forms, $\nabla B=0$ [21]. If B is a Killing vector in $\mathbf{Q}^2$, the equation for Lorentz force of F_B is $\Phi(X)=B\times X$, $\forall X\in\mathbf{Q}^2$. Therefore, the Lorentz equation is defined as

$$\nabla_{x'}x'=\Phi(x')=B\times x',$$

then every Killing vector field is defined a magnetic field which will be called a Killing magnetic field. In particular, uniform magnetic fields, $\nabla B = 0$, are obviously Killing, [21].

Hence, since $\overleftarrow{E} = (E^1, 0, E^3)$ we can suppose

$$\overleftarrow{B} = (B^1, B^2, B^3),$$

also we know that $\langle B, E \rangle = 0$, we get

$$\langle B, E \rangle = 0 \Rightarrow \langle (B^1, B^2, B^3), (E^1, 0, E^3) \rangle = B^1E^1 - B^3E^3 = 0;$$

$$B^1 = \pm E^3 \text{ and } B^3 = \pm E^1 \text{ or } \overleftarrow{B} = \pm(E^3, 0, E^1).$$

5. THE ENERGY OF THE VECTOR FIELDS ON A PARTICLE IN NULL CONE $\mathbf{Q}^2$

In this section, we investigate the bending energy formula for tangent vector of s-lines (ξ −lines, η −lines respectively) of elastic curve written by extended Serret-Frenet relations along the curve x in null cone 3-space.

*5.1. THE ENERGY OF UNIT TANGENT VECTOR OF **s** −LINES ON A MOVING PARTICLE IN NULL CONE 3-SPACE*

In the section, we calculate the energy of the unit tangent vector of ξ −-lines of the curve in $\mathbf{Q}^2$ and we also investigate the bending energy formula for an elastic curve given by extended Serret-Frenet relations along the curve $x(s, \xi, \eta)$ in $\mathbf{Q}^2$.

Let A be a moving particle in null cone 3-space such that it corresponds to a curve $x(s, \xi, \eta)$ with parameter s, which s is the distance along the s-lines of the curve in s −direction and tangent vector of s-lines is described by $\frac{\partial x}{\partial s}$. Hence, by using Sasaki metric and the equations (2.3), (2.4), (2.5), the energy on the particle in vector field $\frac{\partial x}{\partial s}$ can be written as

$$energy_{x_s} = \frac{1}{2} \int \rho_s(dx(x), dx(x)) ds$$

and

$$\rho_s(dx(x), dx(x)) = \rho_s(x, x) + \rho_s(D_x x, D_x x) = \rho(\alpha, \alpha) = 1,$$

since $D_x x = \overleftarrow{\alpha}$, we obtain

$$energy_{x_s} = \frac{1}{2} \int ds = \frac{s}{2} + c.$$

Also, the energy on the particle in vector field $\frac{\partial \alpha}{\partial s}$ is written as

$$energy_{\alpha_s} = \frac{1}{2} \int \rho_s(d\alpha(\alpha), d\alpha(\alpha)) ds,$$

$$\rho_s(d\alpha(\alpha), d\alpha(\alpha)) = \rho_s(\alpha, \alpha) + \rho_s(D_\alpha \alpha, D_\alpha \alpha) = 1 - 2\kappa,$$

since $D_\alpha \alpha = \kappa \overleftarrow{x} - \overleftarrow{y}$, we get

$$energy_{\alpha_s} = \frac{1}{2} \int (1 - 2\kappa) ds = \frac{s}{2} - \int \kappa ds.$$

Similarly, the particle in vector field $\frac{\partial y}{\partial s}$ is written as

$$energy_{y_s} = \frac{1}{2}\int \rho(dy(y), dy(y))ds$$

and

$$\rho_s(dy(y), dy(y)) = \rho_s(y, y) + \rho_s(D_y y, D_y y) = \kappa^2,$$

since $D_y y = -\kappa\overleftarrow{\alpha}$, we get

$$energy_{y_s} = \frac{1}{2}\int \kappa^2 ds.$$

Then, the following theorem can be given.

Theorem 1. Energy of asymptotic frame field of s −lines with Sasaki metric in null cone 3-space are given by

$$energy_{x_s} = \frac{s}{2} + c;$$
$$energy_{\alpha_s} = \frac{s}{2} - \int \kappa ds;$$
$$energy_{y_s} = \frac{1}{2}\int \kappa^2 ds.$$

5.2. THE ENERGY OF UNIT TANGENT VECTOR OF $\boldsymbol{\xi}$ −LINES ON A MOVING PARTICLE IN NULL CONE 3-SPACE

In the section, we calculate the energy of the unit tangent vector of ξ −lines of the curve in $\mathbf{Q}^2$ and we also investigate the bending energy formula for an elastic curve given by extended Serret-Frenet relations along the curve $x(s, \xi, \eta)$ in $\mathbf{Q}^2$.

Let A be a particle moving in a zero cone corresponding to a curve $x(s, \xi, \eta)$ with parameter ξ, which ξ is the distance along the ξ −lines of the curve in ξ −direction and the tangent vector of ξ −lines is described by $\frac{\partial x}{\partial \xi}$. Hence, the energy on the particle in vector field $\frac{\partial x}{\partial \xi}$ can be written as

$$energy_{x_\xi} = \frac{1}{2}\int \rho_\xi(dx(x), dx(x))d\xi,$$

from (2.3), (2.4), (2.5), we get

$$\rho_\xi(dx(x), dx(x)) = \rho_\xi(x, x) + \rho_\xi(D_x x, D_x x) = \rho(D_x x, D_x x) = \left(P^{\xi}_{x\alpha}\right)^2,$$

by using the extended Serret-Frenet relations according to parameter ξ, since $D_x x = -P^{\xi}_{xy}\overleftarrow{x} - P^{\xi}_{x\alpha}\overleftarrow{\alpha}$ or $D_x x = (\kappa + \overleftarrow{x} curl \overleftarrow{y})\overleftarrow{x} + (\overleftarrow{x} curl \overleftarrow{\alpha})\overleftarrow{\alpha}$, we get

$$energy_{x_\xi} = \frac{1}{2}\int \left(P^{\xi}_{x\alpha}\right)^2 d\xi = \frac{1}{2}\int (\overleftarrow{x} curl \overleftarrow{\alpha})^2 d\xi.$$

Also, the energy on the particle in vector field $\frac{\partial \alpha}{\partial \xi}$ is written as

$$energy_{\alpha_\xi} = \frac{1}{2}\int \rho_\xi(d\alpha(\alpha), d\alpha(\alpha))d\xi$$

and

$$\rho_\xi\big(d\alpha(\alpha), d\alpha(\alpha)\big) = \rho_\xi(\alpha, \alpha) + \rho_\xi(D_\alpha\alpha, D_\alpha\alpha) = 1 - 2div\breve{y}P^{\xi}_{x\alpha} = 1 + 2div\breve{y}.\breve{x}curl\breve{\alpha},$$

since

$$D_\alpha\alpha = (-div\breve{y})\breve{x} + P^{\xi}_{x\alpha}\breve{y} \text{ or } D_\alpha\alpha = (-div\breve{y})\breve{x} + (-\breve{x}curl\breve{\alpha})\breve{y},$$

we get

$$energy_{\alpha_\xi} = \frac{1}{2}\int \left(1 - 2div\breve{y}P^{\xi}_{x\alpha}\right)d\xi = \frac{1}{2}\int \big(1 + 2div\breve{y}.(\breve{x}curl\breve{\alpha})\big)d\xi.$$

And the energy on the particle in vector field $\frac{\partial y}{\partial \xi}$ is expressed as

$$energy_{y_\xi} = \frac{1}{2}\int \rho(dy(y), dy(y))d\xi$$

and

$$\rho_\xi(dy(y), dy(y)) = \rho_\xi(y, y) + \rho_\xi\big(D_y y, D_y y\big) = (div\breve{y})^2,$$

since $D_y y = (div\breve{y})\breve{\alpha} + P^{\xi}_{xy}\breve{y}$ or $D_y y = (div\breve{y})\breve{\alpha} + (-\kappa - \breve{x}curl\breve{y})\breve{y}$, we get

$$energy_{y_\xi} = \frac{1}{2}\int (div\breve{y})^2 d\xi.$$

Then, the following theorem can be given.

Theorem 2. Energy of asymptotic frame field of ξ −lines with Sasaki metric in null cone 3-space are given by

$$energy_{x_\xi} = \frac{1}{2}\int (\breve{x}curl\breve{\alpha})^2 d\xi;$$
$$energy_{\alpha_\xi} = \frac{1}{2}\int \big(1 + 2div\breve{y}.(\breve{x}curl\breve{\alpha})\big)d\xi;$$
$$energy_{y_\xi} = \frac{1}{2}\int (div\breve{y})^2 d\xi.$$

5.3. THE ENERGY OF THE TANGENT VECTOR OF η −LINES ON A MOVING PARTICLE IN NULL CONE 3-SPACE

In the section, we compute energy of the unit tangent vector of η −lines of the curve in $\mathbf{Q}^2$ and we also investigate the bending energy formula for an elastic curve given by extended Serret-Frenet relations along the curve $x(s, \xi, \eta)$ in $\mathbf{Q}^2$.

Let A be a particle moving in a zero cone corresponding to a curve $x(s, \xi, \eta)$ with parameter η, which η is the distance along the η −lines of the curve in η −direction and tangent vector of η −lines is described by $\frac{\partial x}{\partial \eta}$. Hence, by using Sasaki metric the energy on the particle in vector field $\frac{\partial x}{\partial \eta}$ can be written

$$energy_{x_\eta} = \frac{1}{2}\int \rho_\eta(dx(x), dx(x))d\eta$$

and from (2.3), (2.4), (2.5) we get

$$\rho_\eta(dx(x), dx(x)) = \rho_\eta(x,x) + \rho_\eta(D_x x, D_x x) = \left(M_{x\alpha}^{\eta}\right)^2 = (\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)^2$$

also from extended Serret-Frenet relations with respect to parameter η, since $D_x x = \left(-M_{xy}^{\eta}\right)\overleftarrow{x} - M_{x\alpha}^{\eta}\overleftarrow{\alpha}$ or $D_x x = (-\overleftarrow{\alpha}curl\overleftarrow{y})\overleftarrow{x} + \left(-(\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)\right)\overleftarrow{\alpha}$, we get

$$energy_{x_\eta} = \frac{1}{2}\int \left(M_{x\alpha}^{\eta}\right)^2 d\eta = \frac{1}{2}\int (\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)^2 d\eta.$$

Similarly, the energy on the particle in vector field $\frac{\partial\alpha}{\partial\eta}$ is written as

$$energy_{\alpha_\eta} = \frac{1}{2}\int \rho(d\alpha(\alpha), d\alpha(\alpha))d\eta;$$

$$\begin{aligned}\rho_\eta(d\alpha(\alpha), d\alpha(\alpha)) &= \rho_\eta(\alpha,\alpha) + \rho_\eta(D_\alpha\alpha, D_\alpha\alpha) = 2M_{x\alpha}^{\eta}(div\overleftarrow{\alpha} + 1)\\ &= 2(div\overleftarrow{\alpha} + 1)(\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)\end{aligned}$$

since $D_\alpha\alpha = (div\overleftarrow{\alpha} + 1)\overleftarrow{x} + M_{x\alpha}^{\eta}\overleftarrow{y}$ or $D_\alpha\alpha = (div\overleftarrow{\alpha} + 1)\overleftarrow{x} + (\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)\overleftarrow{y}$, we get

$$energy_{\alpha_\eta} = \int M_{x\alpha}^{\eta}(div\overleftarrow{\alpha} + 1)d\eta = \int (div\overleftarrow{\alpha} + 1)(\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)d\eta$$

and for the energy on the particle in vector field $\frac{\partial y}{\partial\eta}$ is also obtained as

$$energy_{y_\eta} = \frac{1}{2}\int \rho_\eta(dy(y), dy(y))d\eta$$

and

$$\rho_\eta(dy(y), dy(y)) = \rho_\eta(y,y) + \rho_\eta\left(D_y y, D_y y\right) = (div\overleftarrow{\alpha} + 1)^2,$$

since $D_y y = -(div\overleftarrow{\alpha} + 1)\overleftarrow{\alpha} + M_{xy}^{\eta}\overleftarrow{y}$ or $D_y y = -(div\overleftarrow{\alpha} + 1)\overleftarrow{\alpha} + (\overleftarrow{\alpha}curl\overleftarrow{y})\overleftarrow{y}$, we get

$$energy_{y_\eta} = \frac{1}{2}\int (div\overleftarrow{\alpha} + 1)^2 d\eta.$$

Then, the following theorem is given.

Theorem 3. Energy of asymptotic frame field of η −lines with Sasaki metric in null cone 3-space are given by

$$energy_{x_\eta} = \frac{1}{2}\int (\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)^2 d\eta;$$

$$energy_{\alpha_\eta} = \int (div\overleftarrow{\alpha} + 1)(\overleftarrow{\alpha}curl\overleftarrow{\alpha} + \kappa)d\eta;$$

$$energy_{y_\eta} = \frac{1}{2}\int (div\overleftarrow{\alpha} + 1)^2 d\eta.$$

6. CONCLUSIONS

In this paper, it was discussed that the analysis of the directional derivatives in accordance with the asymptotic orthonormal frame $\{x, \alpha, y\}$ in $\boldsymbol{Q}^2$ and the extended Serret-Frenet relations by using cone Frenet formulas. Also, the geometrical understanding of energy on each asymptotic orthonormal vector fields were given in null cone. Furthermore, we express the bending elastic energy function for the same particle according to curve $x(s, \xi, \eta)$ were expressed in null cone space in terms of geometric perspective, and the results by providing energy variation sketches according to directional derivatives were given. Additionally, a geometrical interpretation of the energy of unit vector fields are expressed and Maxwell's equations for the electric and magnetic field vectors are solved in null cone 3-space.

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