ORIGINAL PAPER

ASYMPTOTIC BEHAVIOR AND L^p PROPERTIES OF NON-OSCILLATORY SOLUTIONS TO THE THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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Abstract. *This article deals with the asymptotic behavior of non oscillatory solutions of third order differentiel equation.*

$$x''' + x'' + a(t)f(x)x' + b(t)g(x) = 0.$$

The behavior of non-oscillatory solutions are shown to be bounded and $L^p[0,\infty)$ under the specified conditions, the derivative are shown to be in $L^2[0,\infty)$ and the solutions as well as their derivatives shown to approach 0 as $t \longrightarrow \infty$ implying stability.
We obtain results which extend and complement those known in [1]. Finally, several examples illustrating the usefulness of the procedure are given.

Keywords: *integrable; square-integrable; nonlinear; bounded; L^p solutions, non-oscillatory.*

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1. INTRODUCTION

Linear third order differential equations appear as the more basic mathematical models in several areas of science and engineering. And also the study of boundary value problem for ordinary differential equations arise in variety of different areas of applied mathematics, physics and many applications of engineering and sciences. For example, the deformations of an elastic beam are described by a differential equation, often referred to as the beam equation, and spectral problems for differential equations arise in many different physical applications arise in astro- physics, i.e., the narrow convecting layers bounded by stable layers which are believed to surround stars may be modeled by boundary value problems, also this problems arise in hydrodynamic and magnetohydro dynamic stability theory, and by derived a model for beams and pipes that the resulting differential equation after separation of variables leads to a differential equation. The study of asymptotic behavior for linear ordinary differential equations has been achieved their breakthroughs in the recent research, for more research and reading. See the following references [2-11].

In 2013, Kroopnick [12] discussed the existence of non-oscillatory solutions to the second order differential equations $x'' + a(t)f(x)x' + b(t)g(x) = 0$. The problem of obtaining sufficient conditions to ensure that all solutions of second order nonlinear differential equations are non oscillatory has been studied by a number of authors.

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In this paper we prove the well, posedness and we study the asymptotic behavior. Of non-oscillatory L^p solutions for a third order non linear differential equation

$$x''' + x'' + a(t)f(x)x' + b(t)g(x) = 0 \tag{1}$$

approach 0 as $t \longrightarrow \infty$. The same will be true of their derivatives, too, the solutions are asymptotic stable. The following conditions will be used. Assume that, $a(t)$ and $b(t)$ are elements of $C^1[0,\infty)$ are positive on $[0,\infty)$, and both possess derivatives are non positive and there exists a positive continuous function $f(x)$ on $C(-\infty,\infty)$ and $g(x)$, too, must be continuous on $C(-\infty,\infty)$ such that for any $x \neq 0$, $xg(x) > 0$ and, finally

$$G(x) = \int^x g(u)du \longrightarrow \infty \text{ as } |x| \longrightarrow \infty.$$

Moreover, it is important to we shall show that these solutions is said to in L^psolutions when $a(t) > a_0, b(t) > b_0, f(x) > f_0$ for $a_0 \geq 0$, $b_0 \geq 0$, $f_0 \geq 0$ and $xg(x) \geq kx^p$ for any $k \geq 0$.

By an L^psolution, which is a solution to (1) such that $\int^\infty |x(t)|^p dt < \infty$ where $p \geq 1$. Moreover, we give some examples.

2. PRELIMINARIES

For definiteness, we consider real-valued functions. Analogous results apply to complex-valued functions.

Definition 2.1. Let $(X;\ A;\ \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p(X)$consists of equivalence classes of measurable functions $f : X \longrightarrow \mathbb{R}$ such that

$$\int |\boldsymbol{f}|^{\boldsymbol{p}}\, \boldsymbol{d}\mu < \infty,$$

where two measurable functions are equivalent if they are equal μ-a.e. The L^p-norm of $f \in L^p(X)$ is defined by

$$\|\boldsymbol{f}\|_{L^p} = \left(\int |f|^p\, d\mu\right)^{\frac{1}{p}}.$$

The notation $L^p(X)$ assumes that the measure μon X is understood. We say that $f_n \longrightarrow f$ in L^p if $\|f_n - f\|_{L^p} \longrightarrow 0$. The reason to regard functions that are equal a.e. as equivalent is so that $\|f\|_{L^p} = 0$ implies that $f = 0$.

The space $L^\infty(X)$ is defined in a slightly different way. First, we introduce the notion of essential supremum.

Definition 2.2. Let$f : X \longrightarrow \mathbb{R}$ be a measurable function on a measure space $(X;\ A;\ \mu)$. The essential supremum of f on X is

$$\boldsymbol{ess}\sup_{\boldsymbol{X}} \boldsymbol{f} = \inf\left\{\sup_X g :\ \ g = f \text{ pointwise a.e.}\right\}.$$

Thus, the essential supremum of a function depends only on its μ-a.e. equivalence class. We say that f is essentially bounded on X if

$$\underset{X}{ess\,sup}|f| < \infty.$$

Definition 2.3. Let $(X;\ A;\ \mu)$ be a measure space. The space $L^{\infty}(X)$ consistsof pointwise a.e.-equivalence classes of essentially bounded measurable functions $f : X \longrightarrow \mathbb{R}$ with norm

$$\|f\|_{L^{\infty}} = \underset{X}{ess\,sup}\, f$$

We rarely want to use the supremum instead of the essential supremum when the two have different values, so this notation should not lead to any confusion.

2.1. MINKOWSKI AND HÖLDER INEQUALITIES

We state without proof two fundamental inequalities.

Theorem 2.1. (Minkowski inequality). *If* $f;\ g \in L^p(X)$ *where* $1 \le p \le \infty$, *then* $f + g \in L^p(X)$ *and*

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}\ .$$

This inequality means, as stated previously, that $\|f\|_{L^p}$ is a norm on $L^p(X)$ for $1 \le p \le \infty$. If $0 < p < 1$, then the reverse inequality holds

$$\|f\|_{L^p} + \|g\|_{L^p} \le \|f + g\|_{L^p},$$

So $\|.\|_{L^p}$ is not a norm in that case. Nevertheless, for $0 < p < 1$ we have

$$|f + g|^p \le |f|^p + |g|^p\ ,$$

so $L^p(X)$ is a linear space in that case also. To state the second inequality, we define the Hölder conjugate of an exponent

Definition 2.4. Let $1 \le p \le \infty$. The Hölder conjugate $p'$of p is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad if\ 1 < p < \infty\ .$$

Note that $1 \le p' \le \infty$., and the Hölder conjugate of p' is p.

Theorem 2.2. (Hölder's inequality) *Suppose that* $(X;\ A;\ \mu)$ *is a measure space and* $1 \le p \le \infty$. *If* $f \in L^p(X)$ *and* $g \in L^{p'}(X)$, *then* $fg \in L^1(X)$ *and*

$$\int |fg|\, d\mu \le \|f\|_{L^p} \|g\|_{L^{p'}}.$$

For $p = p' = 2$, this is the Cauchy-Schwartz inequality.

2.2. THE BEHAVIOR OF NON-OSCILLATORY SOLUTIONS

We study the behavior of non-oscillatory solutions of (1). We use an approach that leads to only three independent conditions, but we obtain sufficient conditions which guarantee that every non-oscillatory solution of (1) tends to zero as $t \longrightarrow \infty$.

Definition 2.5. A solution x of (1) is said to be non-oscillatory if it is positive or negative. Otherwise, it is said to be oscillatory.

3. MAIN RESULTS

In this section, we are in a position to give the main result of this work .

Theorem 3.3. Give the differential equation (1). Assume the following hypotheses:

(i) $a(t), b(t) \in C^1[0,\infty)$ *with non-positive derivatives and, both* $f(x), g(x) \in C(-\infty,\infty)$ *then any non-oscillatory solution is bounded as* $t \longrightarrow \infty$.

(ii) $a(t), b(t), f(x)$ *are greater than constants* $a_0 \geq 0,\ b_0 \geq 0,\ f_0 \geq 0$ *and* $xg(x) > 0$ *for* $x \neq 0$ *then any non-oscillatory solution approaches 0 as* $t \longrightarrow \infty$.

(iii) *We also assume that there exists* $p \geq 2$ *such that* $xg(x) \geq k|x|^p$. *Then the solution of* (1) *is an element of* $L^p[0,\infty)$.

Proof: Multiplying equation (1) by $x''(t)$ and then integrating from 0 to t. We obtain,

$$\frac{(x''(t))^2}{2} + \int_0^t (x''(s))^2\, ds + a(t)F(x(t)) - \int_0^t a'(s)F\big(x(s)\big)ds +$$

$$+\int_0^t b(s)g\big(x(s)\big)\, x''(s)ds = \frac{(x''(0))^2}{2} + a(0)F(x(0)) \tag{2}$$

Assume that $F(x) = \int^x f(u)u''du$ where we have the third term in (1) can be integrated by parts.

Next multiplying equation (1) by $x'(t)$ and then integrating from 0 to t Thus we obtain

$$x''(t)x'(t) - \int_0^t (x''(s))^2 ds + \frac{(x'(t))^2}{2} + \int_0^t a(s)F(x(s))(x'(s))^2\, ds + b(t)G(x(t)) -$$

$$-\int_0^t b'(s)G\big(x(s)\big)ds = x''(0)x'(0) + \frac{\big(x'(0)\big)^2}{2} + b(0)G\big(x(0)\big) \tag{3}$$

Assume that $G(x) = \int^x g(u)du$

We will now look at equation (3), it follows that both $x(t)$ and $x'(t)$ are bounded as $t \longrightarrow \infty$.

Either, the LHS of (2) would become infinite as $t \longrightarrow \infty$ which is impossible. Should $a(t)$ and $f(x)$ will remain greater than some positive constants a_0 and f_0 respectively, we can conclude that $x'(t)$ is square-integrable, and, hence in $L^2[0,\infty)$.

We next show that $x(t)$ is in $L^p[0,\infty)$ when $xg(x) \geq kx^p$. Multiply equation (1) by $x(t)$ and integrate from 0 to t thus we obtain,

$$x''(t)x(t) - \frac{\left(x'(t)\right)^2}{2} + x(t)x'(t) - \int_0^t \left(x'(s)\right)^2 ds + a(t)H\left(x(t)\right) - \int_0^t a'(s)H\left(x(s)\right) ds +$$

$$+ \int_0^t b(s)g\left(x(s)\right) x(s) ds = x''(0)x(0) - \frac{\left(x'(0)\right)^2}{2} + a(0)H\left(x(0)\right) + x(0)x'(0) \tag{4}$$

Assume that $H(x) = \int^x f(u) u du$

Where we have the first two terms of equation (1) can be integrated by parts.

Examining equation (4) shows us that as long as $b(t) > b_0 > 0$, then $x(t)$ is an element in $L^p[0,\infty)$ under the hypotheses of the theorem 3.3.

We next can prove $|x(t)|$ and $|x'(t)|$ approach 0 as $t \longrightarrow \infty$.

Without any loss of generality, assume that $x(t) > 0$ for $t \geq 0$. A similar argument works for $x(t) < 0$. We will seethat both $x(t)$ and $x'(t)$ approach 0, we first observe that $x'(t)$ must be of fixed sign. Otherwise, whenever $x'(t) = x''(t) = 0$, we have from equation (1)

$$x'''(t) = -b(t)g(x) < 0,$$

so we have an infinite number of consecutive relative maxima one after the other which is impossible because a negative second derivative means $x'(t)$ is always decreasing between consecutive zeroes of $x'(t)$.

Moreover, if we have two consecutive critical points, then $x'''(t) = x''(t) = 0$ between these two points which we have shown is not possible. Furthermore, $x'(t)$ must be negative or else $x(t)$ will increase without bound and hence $x(t)$ could not be in $L^p[0,\infty)$ should $x'(t)$ become positive. Hence, by the monotonicity of both $x(.)$ and $x'(.)$ and the L^p property of $x(.)$, we must have that $x(.)$ and $x'(.)$ approach 0 as $t \longrightarrow \infty$.

4. APPLICATIONS

The following examples illustrate applications of theoretical results presented in this paper.

4.1. EXAMPLE

Consider the non-linear differential equation

$$x''' + x'' + \frac{2}{t+2} x' + 6x^4 = 0 \tag{5}$$

For equation (5), we have non-oscillatory solution to is $x(t) = \frac{1}{t+2}$.
The solution of (5) is non oscillatory and an element of $L^5[0,\infty)$.
After a straightforward computation, we conclude by Theorem 3.3.
That $xg(x) = x^5$ moreover, its derivative $x'(t) = -(t+2)^{-2}$ can be seen easily an element of $L^2[0,\infty)$ withal $a(t) = \frac{2}{t+2}$ is not bounded away from zero.

4.2. EXAMPLE

Consider the non-linear differential equation

$$x''' + x'' + x^2x' + x^3 = 0 \tag{6}$$

A non-oscillatory solution to (6) is $x(t) = e^{-x}$. The solution of (6) is non oscillatory and an element of $L^4[0,\infty)$. After a straightforward computation, we conclude by Theorem 3.3, that $xg(x) = x^4$. Moreover, its derivative $x'(t) = -e^{-x}$ is as before easily seen to be in $L^2[0,\infty)$.

CONCLUSION

In this research ee have studied the asymptotic behavior of non-oscillating solutions of a third-order differential equation. We demonstrated all sufficient specific conditions to obtain the behavior of constrained non-volatile solutions. We also set the necessary conditions for stability solutions. We can apply these basic results to a fourth-order or higher order differential equation. With all conditions imposed to achieve our goal.

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