ORIGINAL PAPER

NEW VARIETY TYPES OF SOLUTION TO THE FUJIMOTO-WATANABLE EQUATION WITH THE CORRESPONDING NUMERICAL SOLUTIONS

EMAD H.M. ZAHRAN[1], AHMET BEKİR[2]

Manuscript received: 22.07.2022; Accepted paper: 16.11.2022;
Published online: 30.12.2022.

Abstract. *In this article, new variety types of exact solution to the Fujimoto-Watanable- equation (FWE) that equivalent to the modified Korteweg- de Vries- equation have been derived. These new types of solutions which weren't realized before by any other technique have been established in the framework of the Ricatti-Bernolli Sub-ODE method (RBSODM). Also, the identical numerical solutions whose initial conditions are emerged from the achieved exact solutions have been constructed by using the famous numerical variational iteration method (VIM).*

Keywords: *Fujimoto-Watanable-equation; Ricatti-Bernolli Sub-ODE method; variational iteration method (VIM); exact solutions; numerical solution.*

1. INTRODUCTION

Throughout this research, we will investigate the exact solutions of the FWE [1-4] to detect the soliton behaviors arising from it. The suggested model has been discovered when classifying the 3th -order uniform rank evolution equations with non-constant separates and derived the 8th and 3th -order differential equations that describe symmetries (non-trivial Lie–Bäcklund algebras). Furthermore, Sakovich [5] explained the connection of these differential equations with the famous KdV- equation and Qiao equation. The suggested equation amounted to the modified Korteweg- de Vries (KDV) equation that represents a mathematical model of surface shallow water wave. The RBSODM [6] is the only one of the ansatze methods that doesn't depend on the balance rule has been used effectively to obtain impressive description to the exact solutions to this equation. The homogeneous balance fails of this equation, hence all the ansatze approaches methods [7-22] that depend on this rule will be fail to realize any solution for this equation. In addition, we will apply the VIM [23-24] to demonstrate the identical numerical solutions for all achieved solutions of this equation. Some unique form solutions, such as soliton solutions, may only be reliant on a single combination of the variables among the possible solutions to nonlinear evolution equations [25-32].

Few studies of this equation have been admitted sea for example, Shi and Wen [1-3] and [4] studied the bifurcation and dynamics of solitary wave solutions and the dynamical behaviors of solitary wave solutions to the FWE in two individually different published articles.

[1] BenhaUniversity, Departments of Mathematical and Physical Engineering, Shubra, Egypt.
E-mail: e_h_zahran@hotmail.com.
[2] Neighbourhood of Akcaglan, 28/4 Imarli Street, 26030 Eskisehir, Turkey.
E-mail: bekirahmet@gmail.com

The goal of this work is to introduces variety impressive solitary wave and numerical solutions to the Fujimoto-Watanable using the RBSODM and VIM respectively.

2. THE RBSODM ANALYSIS

According to [1] the solution is,

$$\mathrm{H}' = AH^{2-\beta} + BH + CH^{\beta}, \tag{1}$$

When $AC \neq 0$ and $\beta = 0$ Eq. (1) tends to Riccati equation, if $A \neq 0,\ \mathrm{C} = 0,$ and $\beta \neq 1$ it represents Bernoulli equation while A, B, C and β are unknown and must be defined later. When we substitute for u and its derivatives at Eq. (1), one can locate the following system of equations in these unknowns with the aid of a suitable choice of β and setting equivalence for the various exponential powers of H to zero. Consequently, the constructed method admits six different cases of solutions according to the transformation $\zeta = x - ct$ and the values of these constants namely,

$$H(\zeta) = C_1 e^{(A+B+C)\zeta},\ \text{for}\ \beta = 1 \tag{2}$$

$$H(\zeta) = \left(A(\beta-1)(\zeta + C_1)\right)^{1/(1-\beta)},\ \text{for}\ \beta \neq 1,\ \mathrm{B} = 0\ \text{and}\ C = 0 \tag{3}$$

$$H(\xi) = \left(-\frac{A}{B} + C_1 e^{B(\beta-1)\zeta}\right)^{1/(\beta-1)},\ \text{for}\ \beta \neq 1,\ \mathrm{B} \neq 0\ \text{and}\ C = 0 \tag{4}$$

$$\begin{gathered} H(\zeta) = \left(\frac{-B}{2A} + \frac{\sqrt{4AC - B^2}}{2A}\tan\frac{(1-\beta)\sqrt{4AC - B^2}}{2}(\zeta + C_1)\right)^{\frac{1}{(1-\beta)}}, \\ H(\zeta) = \left(\frac{-B}{2A} - \frac{\sqrt{4AC - B^2}}{2A}\cot\frac{(1-\beta)\sqrt{4AC - B^2}}{2}(\zeta + C_1)\right)^{\frac{1}{(1-\beta)}}. \end{gathered} \tag{5}$$

For $\beta \neq 1, A \neq 0$ and $B^2 - 4AC < 0$.

$$\begin{gathered} H(\zeta) = \left(\frac{-B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A}\tanh\frac{(1-\beta)\sqrt{B^2 - 4AC}}{2}(\zeta + C_1)\right)^{\frac{1}{(1-\beta)}}, \\ H(\zeta) = \left(\frac{-B}{2A} - \frac{\sqrt{B^2 - 4AC}}{2A}\coth\frac{(1-\beta)\sqrt{B^2 - 4AC}}{2}(\zeta + C_1)\right)^{\frac{1}{(1-\beta)}}. \end{gathered} \tag{6}$$

For $\beta \neq 1, A \neq 0$ and $B^2 - 4AC > 0$.

$$H(\zeta)=\left(\frac{1}{A(\beta-1)(\zeta+C_1)}-\frac{B}{2A}\right)^{1/(1-\beta)}. \tag{7}$$

For $\beta \neq 1, A \neq 0$ and $B^2-4AC=0$, where C_1 an arbitrary constant.

Now, we will extract the solitary wave solution FEW [1] which is

$$w_t = w^3 w_{xxx} + \frac{3}{2} w^2 w_x w_{xx} + \alpha w^2 w_x = 0 \tag{8}$$

In the framework of the RBSODM as follow

When Eq. (8) surrenders to the transformation $w(x,t)=H(\zeta), \zeta = x-ct$ it will be converted to

$$-cH' = H^3 H''' + \frac{3}{2} H^2 H' H'' + \alpha H^2 H', \tag{9}$$

Divide by H^2, integrating once with respect to H, we get,

$$H^2 H'' + \frac{1}{4} H H'^2 + \alpha H^2 + gH - c = 0, \tag{10}$$

Now from Eq. (1) by differentiate we get,

$$\begin{aligned} H'' &= AB(3-\beta)H^{2-\beta} + A^2(2-\beta)H^{3-2\beta} \\ &\quad + \beta C^2 H^{2\beta-1} + BC(\beta+1)H^{\beta} + (2AC+B^2)H, \end{aligned} \tag{11}$$

According to the proposed method, inserting H'^2, H'' into the suggested equation Eq. (10), with suitable choice of β and equating the coefficients of different powers of H to zero, we get system of equations from which the following results achieved

$$B=-i\sqrt{2}\sqrt{AC},\; C=0,\; \alpha=\frac{-3AB}{2}, g=\frac{-A^2}{4}, \tag{12}$$

The obtained results lead only to the solution form (2) of the proposed method which is $H(\zeta)=\left(A(\beta-1)(\zeta+C_1)\right)^{1/(1-\beta)}$, for $\beta \neq 1, B=0$ *and* $C=0$

$$H(\zeta)=\frac{\pm 2\sqrt{g}\; i}{(\zeta+C_1)}, \tag{13}$$

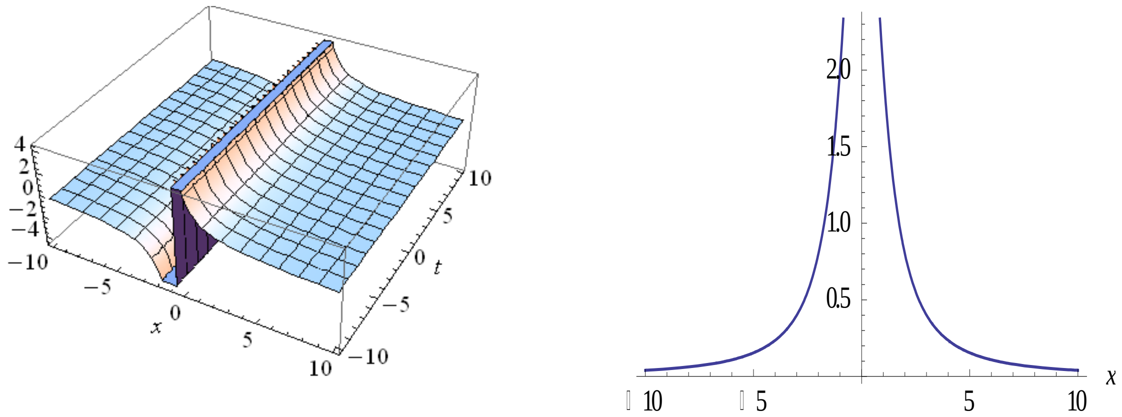

Figure 1. a) The solution behavior of Eq.(13) in 3D and 2D with values:
$A = 2\sqrt{g}\, i, B = 0, C = 0, C_1 = 1, \beta = 2, g = 4$.

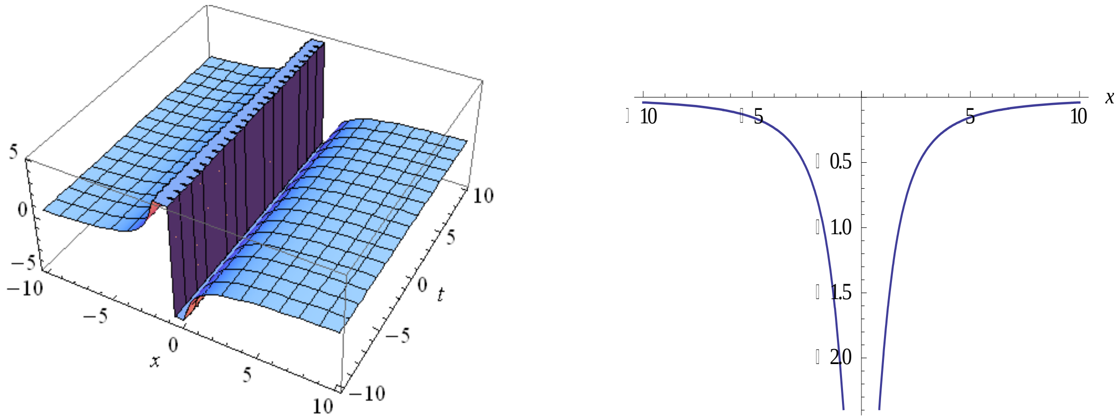

Figure 1. b) The solution behavior of Eq.(13) in 3D and 2D with values:
$A = -2\sqrt{g}\, i, B = 0, C = 0, C_1 = 1, \beta = 2$.

3. THE VARIATIONAL ITERATION METHOD

To investigate this algorithm, let us suppose

$$L\psi + N\psi = h(\xi) \tag{14}$$

Whose correction functional is

$$H_{M+1}(\xi) = H_M(\xi) + \int_0^{\xi} l(t)(LH_M(t) + N\,\tilde{H}_M(t) - g(t))dt. \tag{15}$$

where l appearing in this equation is Lagrange multiplier.
The general iteration formula is

$$H_{M+1}(\xi) = H_M(\xi) + \tfrac{(-1)^M}{(M-1)!}\int_0^{\xi} (t-x)^{M-1}\left(H^{(M)} + f(H', H'', H''' ..., H^{(M-1)}) - g(t)\right)dt,$$
$$\lambda = \tfrac{(-1)^M}{(M-1)!}(t-x)^{M-1} \tag{16}$$

where M related to the rank of differential equation, and the zeros approximation $H_0(\zeta)$ for the ODE of order M is

$$H_0(\xi) = H_0(0) + H'(0)\xi + \tfrac{1}{2!}H''(0)\xi^2 + \tfrac{1}{3!}H'''(0)\xi^3 .... + \tfrac{1}{(m-1)!}H^{M-1}(0)\xi^{M-1}. \quad (17)$$

4. THE NUMERICAL SOLUTIONS TO THE FWE

For the FWE mentioned above which is

$$HH'' + \frac{1}{4}HH'^2 + \alpha H^2 + gH - C = 0$$

The initial conditions are $H(0) = \pm 4,\ H'(0) = \mp 4,$ the first and second iterations according to the VIM are

$$H_0(\zeta) = H(0) + \zeta H'(0), H_0(\zeta) = 4 - 4\zeta,$$

$$H_1(\zeta) = H_0(\zeta) - \int_0^{\zeta} \Big(H_0(t)H_0''(t) + 0.25 H_0(t)H_0'^2(t) + \alpha H_0^2(t) + gH_0(t) - C \Big) dt, \quad (18)$$

$$H_1 = 4 - 4\zeta - 32\int_0^{\zeta}(1-t)dt = 4 - 36\zeta + 16\zeta^2,$$

$$H_2(\zeta) = H_1(\zeta) - \int_0^{\zeta} \Big(H_1(t)H_1''(t) + 0.25 H_1(t)H_1'^2(t) + \alpha H_1^2(t) + gH_1(t) - C \Big) dt$$

$$H_2(\zeta) = 4 - 36\zeta + 16\zeta^2 - \int_0^{\zeta} \begin{pmatrix} -32(4-36t+16t^2)+ \\ 0.25(4-36t+16t^2)(-36-32t)^2 + 4(4-36t+16t^2) \end{pmatrix} dt, \quad (19)$$

$$H_2(\zeta) = 819\zeta^5 - 4992\zeta^3 - 4160\zeta^2 + 1148\zeta + 4$$

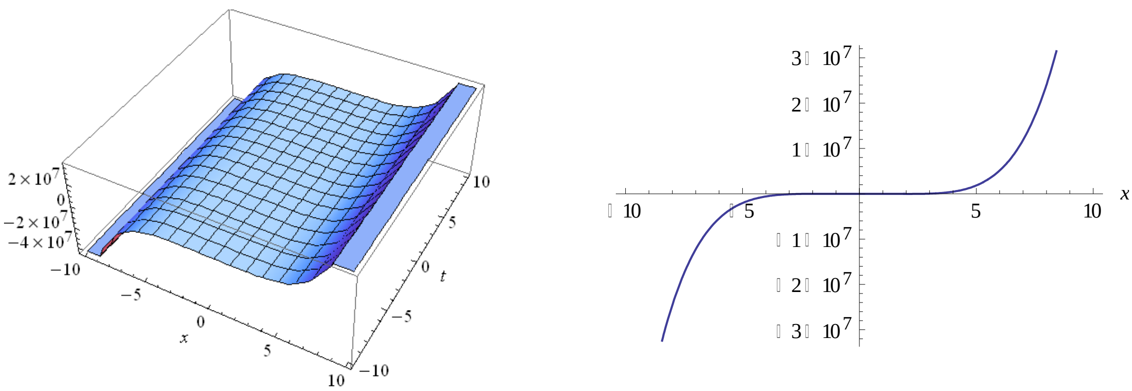

Figure 2. a) The solution behavior of Eq.(19) in 3D and 2D with values:
$A = 2\sqrt{g}\, i, B = 0, C = 0, C_1 = 1, \beta = 2, g = 4$.

Similarly, we will repeat these calculations for the second case in which,

$$H_0(\zeta)=H(0)+\zeta H'(0),\, H_0(\zeta)=-4+4\zeta,$$
$$H_1(\zeta)=H_0(\zeta)-\int_0^{\zeta}\Big(H_0(t)H_0''(t)+0.25H_0(t)H_0'^2(t)+\alpha H_0^2(t)+gH_0(t)-C\Big)dt, \tag{20}$$
$$H_1=-4+4\zeta-32\int_0^{\zeta}(1-t)dt=36\zeta-16\zeta^2-4,$$

$$H_2(\zeta)=H_1(\zeta)-\int_0^{\zeta}\Big(H_1(t)H_1''(t)+0.25H_1(t)H_1'^2(t)+\alpha H_1^2(t)+gH_1(t)-C\Big)dt,$$
$$H_2(\zeta)=36\zeta-16\zeta^2-4-\int_0^{\zeta}\begin{pmatrix}-32(36\zeta-16\zeta^2-4)+\\ 0.25(36\zeta-16\zeta^2-4)(-36+16t)^2+4(36\zeta-16\zeta^2-4)\end{pmatrix}dt, \tag{21}$$
$$H_2(\zeta)=3277\zeta^5-11520\zeta^4+29568\zeta^3-6496\zeta^2+1200\zeta-4.$$

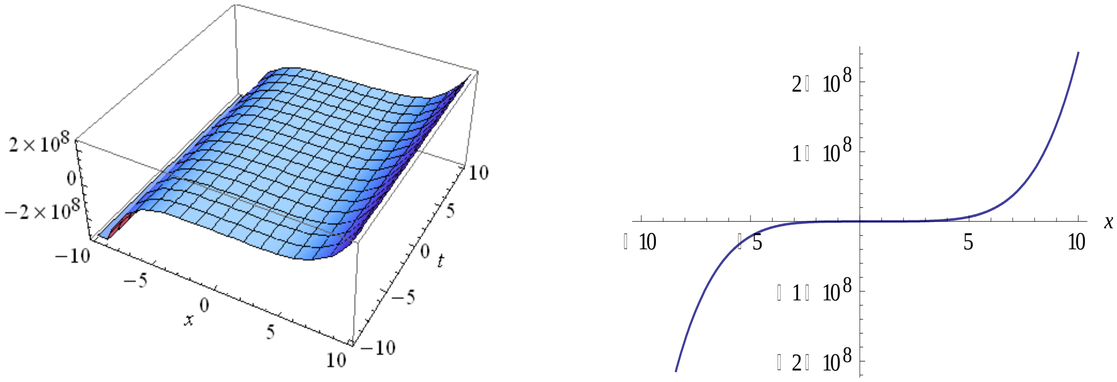

Figure 2. b) The solution behavior of Eq.(21) in 3D and 2D with values:
$A=2\sqrt{g}\,i, B=0, C=0, C_1=1, \beta=2, g=4$.

And for these two cases we can get the other iterations as follows,

$$H_3(\zeta)=H_2(\zeta)-\int_0^{\zeta}\Big(H_2(t)H_2''(t)+0.25H_2(t)H_2'^2(t)+\alpha H_2^2(t)+gH_2(t)-C\Big)dt.$$

....

$$H_{M+1}(\zeta)=H_M(\zeta)-\int_0^{\xi}\Big(H_M(t)H_M''(t)+0.25H_M(t)H_M'^2(t)+\alpha H_M^2(t)+gH_M(t)-C\Big)dt.$$

Remark, the exact solution is $H(\zeta)=\lim_{\zeta\to\infty}H_M(\zeta)$.

5. CONCLUSIONS

The RBSODM which is famous ansatze methods that doesn't rely on the balance rule and reduce large volume of calculation has been used for the first time to finding new exact solutions to the FWE. While, all the ansatze approaches methods which depends on the balance rule fails to realize any solution. Furthermore, the VIM has been also applied to show the numerical solutions that are identical for all realized exact solutions to this model. These new established solutions are very important to discuss the interaction of optical fiber phenomonos as well as other branches of applied mathematics and physical field sciences.It is clear that there is consistent between the exact and numerical solutions in the interval (0≤ x ≤10).

REFERENCES

[1] Shi, L.J., Wen, Z.S., *Communications in Theoretical Physics*, **69**, 631, 2018.
[2] Shi, L.J., Wen, Z.S., *Chinese Physics B*, **27**, 090201, 2018.
[3] Shi, L.J., Wen, Z.S., *Chinese Physics B*, **28**, 040201, 2019.
[4] Dong, H., Fang, Y., Guo, B., Liu, Y., *Journal Quaestiones Mathematicae*, **43**, 1349, 2020.
[5] Sakovich, S.Y., Journal *of Physics A: Mathematical and Theoretical*, **24**, 519, 1991.
[6] Shehata, M.S.M., Rezazadeh, H., Zahran, E.H.M., Tala-Tebue, E., Bekir, A., Communications in Theoretical Physics, **71**, 1275, 2019.
[7] Khater, M.M., Zahran, E.H.M., Shehata, M.S., *Journal of Egyptian Mathematical Society*, **25**, 8, 2017.
[8] Bekir, A., Uygun, F., *Arab Journal of Mathematical Sciences*, **18**, 73, 2012.
[9] Bekir, A., Zahran, E.H.M., *Optical and Quantum Electronics*, **232**, 166539, 2021.
[10] Bekir, A., Zahran, E.M.H.., *Optik-International Journal for Light and Electron Optics*, **228**, 166157, 2021.
[11] Bekir, A., Zahran, E.H.M., *Physica Scripta*, **96,** 055212, 2021.
[12] Bekir, A., Zahran, E.M.H.., *Pramana – Journal of Physics.* **95**, 158, 2021.
[13] Wazwaz A.M., *Chaos Solitons Fractals*, **12**, 1549, 2001.
[14] Yang, X.F., Deng, Z.C., Wei, Y., *Advances in Difference Equations*, **1**, 117, 2015.
[15] Khater, M.M., Lu, D., Zahran, E.H., *Applied Mathematics and Information Sciences*, **11**, 1, 2017.
[16] Shehata, M.S.M., Rezazadeh, H., Zahran, E.H.M., Eslami, M., Bekir, A., *Journal of Science and Arts*, **21**, 91, 2021.
[17] Bekir, A., Zahran, E.H.M., Shehata, M.S.M., *Journal of Science and Arts*, **20**, 251, 2020.
[18] Zahran, E.H.M., Ahmed, H., Saeed, T., Botmart, T., *Results in Physics*, **35**, 105320, 2022.
[19] Zahran, E.H.M., Bekir, A., Alotaibi, M.F., Omri, M., Ahmed, H., *Results in Physics*, **29**, 104730, 2021.
[20] Zahran, E.H.M, Bekir, A., Ahmed, H., *Modern Physics Letters B*, **35**, 2150509, 2021.
[21] Biswas, A, Ekici, M, Sonmezoglu A, Alqahtani, R.T., *Optik*, **165**, 102, 2018.
[22] Eslami, M., Rezazadeh, H., *Calcolo*, **53**, 475, 2016.
[23] Zhou, Q., Ekici, M., Sonmezoglu, A., Mirzazadeh, M., Eslami, M., *Nonlinear Dynamics*, **84**, 1883, 2016.
[24] Wazwaz, A.M., *Applied Mathematics and Computation*, **212**, 120, 2009.
[25] Zafar, A., Shakeel, M., Ali, A., Rezazadeh, H., Bekir, A., *Journal of Nonlinear Optical Physics & Materials*, **32**, 2350010, 2023.
[26] Zafar, A., Mushtaq, T., Malik, A., Bekir, A., *Journal of Ocean Engineering and Science*, **7**, 170, 2022.
[27] Zafar, A., *Nonlinear Engineering*, **8**, 350, 2019.

[28] Zafar, A., Rezazadeh, H., Razzaq, W., Bekir, A., *Modern Physics Letters B*, **35,** 2150132, 2021.
[29] Akbulut, A., Tascan, F., *Chaos, Solitons and Fractals*, **100**, 1, 2017.
[30] Kaplan, M., Akbulut, A. *Opt Quant Electron* **53**, 498, 2021.
[31] K Hosseini, Akbulut, A., Baleanu, D., Salahshour, S., *Commun. Theor. Phys.* **74,** 025001, 2022.
[32] Mirzazadeh, M., Akbulut, A., Tascan, F., Akinyemi, L., *Optik-International Journal for Light and Electron Optics*, **252**, 168529, 2022