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SPINOR DESCRIPTION OF B-DARBOUX EQUATIONS

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Abstract. *In this study, spinor representations of the curves on surfaces have been expressed via B-Darboux frame in Euclidean space* $\mathbb{E}^3$*. The relation between Darboux and B-Darboux frame has been established according to their spinor formulations. Moreover, all these spinor characterizations have been interpreted in the meaning of Darboux frame (via the curvatures) in Euclidean 3-space. Finally, an application has been presented about the characterizations of the relations between the B-Darboux frame and the spinors.*
Keywords: *Spinors; Bishop-Darboux frame; Darboux frame.*

1. INTRODUCTION

In the study of curves and surfaces in $\mathbb{E}^3$, it is generally admitted that vectors are one of the most basic instruments of differential geometry. The behaviour of a differentiable curve is studied by building a triad at each point whose vectors are mutually orthogonal to each other. The changing of these vectors along the curve also defines the functions called as curvature and torsion. These invariants even excepting its position and direction in space completely characterize the curve on a surface.

The different adapted frames have been defined along the space curves and thus the characterizations of the curves have been given in detail. The most effective of these ones is Frenet frame except in the vanishing case of the second derivative of the curve at some points [1,2]. An alternative frame called as Bishop (parallel transport) frame has been constructed in Euclidean space. This frame is a well-defined one even if the second derivative of the curve vanishes at some points and this frame was used in many researches [3-6]. On the other hand, Darboux frame has a crucial role for featuring the curves on the surfaces in differential geometry [7,8].

The triad of orthonormal vectors determined with regards to a specific vector with two complex components, is said to be a spinor [9-11]. Spinors are mostly used in quantum mechanics for examining the features of the intrinsic angular momentum of the electron and other fermions [12]. Recently, spinors are widely used in physics applications. In the fields of mathematics such as differential geometry, and global analysis, spinors get the vast utilizations [13-15]. The Frenet equations characterized by a particular expression of single spinor equation which is proportionate to the three usual vector equations, is a result of the link between spinors and orthogonal Frenet basis vectors.

With the aid of these theories, spinor representations of Frenet frame along a space curve have been given in Euclidean 3-space [10,11]. The characterizations of Bishop frame have been investigated in the sense of spinors in Euclidean and Lorentzian 3-spaces [16-18]. Spinors related to a curve on a surface have been studied with Darboux frame in Euclidean

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space [19]. Recently a new frame was introduced along the curve on the surface, called as the Bishop Darboux frame, which is formed of three orthonormal vectors $\{t, b_1, b_2\}$. The B-darboux frame which is developed as the Bishop version of the Darboux frame, has a tangent parallel to the tangent of the Darboux frame, but the vector fields b_1 and b_2 are obtained by the parallel transport method unlike the Darboux frame [20,21].

In this study, the curve on a surface has been investigated via Bishop Darboux frame (abbr. B-Darboux frame) in $\mathbb{E}^3$ in the view of the spinors. These spinor formulations have been given in terms of the geodesic and the normal curvatures of the Darboux frame. The relations between Darboux and B-Darboux frame have been acquired with the aid of their spinor representations. Finally, an application has been made about the relation between curves on surfaces due to B-Darboux frame and their spinor representations.

2. PRELIMINARIES

The Euclidean 3-space is given with the standard flat metric presented by

$$\langle \ , \ \rangle = dx_1^2 + dx_2^2 + dx_3^2 \tag{1}$$

where (x_1, x_2, x_3) is a rectilinear coordinate system of $\mathbb{E}^3$. Remind that the norm of an arbitrary vector $X \in \mathbb{E}^3$ is delivered by $\|X\| = \sqrt{\langle X, X\rangle}$. Let $\alpha: I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be an arbitrary curve in the Euclidean space $\mathbb{E}^3$. The curve α is said to be a unit speed one if $\langle \frac{d\alpha}{ds}, \frac{d\alpha}{ds} \rangle = 1$. The derivative equations of the Frenet frame vector fields (Frenet-Serret formula) is endowed

$$\begin{aligned} \frac{dt}{ds} &= \kappa n, \\ \frac{dn}{ds} &= -\kappa t + \tau b \\ \frac{db}{ds} &= -\tau n, \end{aligned} \tag{2}$$

where $\{t, n, b\}$ is called as Frenet orthonormal basis of $\alpha(s)$, and the functions κ, τ are the curvature and the torsion of the curve $\alpha(s)$, respectively [1].

On the other hand, the variation of the Darboux frame $\{t, g, n\}$ is characterized with the following equations:

$$\begin{aligned} \frac{dt}{ds} &= \kappa_g g + \kappa_n n, \\ \frac{dg}{ds} &= -\kappa_g t + \tau_g n, \\ \frac{dn}{ds} &= -\kappa_n t - \tau_g n, \end{aligned} \tag{3}$$

where $\alpha(s)$ is a curve lying on a surface $M = (u, v)$, n is the normal of surface and $g = n \times t$. Here, the geodesic curvature κ_g, the normal curvature κ_n, and the relative torsion τ_g are expressed with the subsequent equations [19]:

$$\kappa_g = \langle t', g \rangle, \quad \kappa_n = \langle t', n \rangle, \quad \tau_g = \langle g', n \rangle. \tag{4}$$

One of the alternative frames of the curves on the surfaces is Bishop Darboux frame abbreviated as B-Darboux frame in this study. The B-Darboux frame of a curve-surface pair is formed of three orthonormal vectors $\{t, b_1, b_2\}$. Here, the tangent vector t is unique and two arbitrary basis vectors b_1 and b_2 lie in the normal plane of the surface. The variation equations of the B-Darboux triad $\{t, b_1, b_2\}$ on the surface are given as follows;

$$\begin{aligned} \frac{dt}{ds} &= n_1 b_1 + n_2 b_2, \\ \frac{db_1}{ds} &= -n_1 t, \\ \frac{db_2}{ds} &= -n_2 t, \end{aligned} \tag{5}$$

where n_1 and n_2 are B-Darboux curvatures which are expressed with the following equations,

$$\begin{aligned} n_1 &= \kappa_g sin\theta + \kappa_n cos\theta, \\ n_2 &= \kappa_n sin\theta - \kappa_g cos\theta. \end{aligned} \tag{6}$$

Here, the relation between curvatures of Darboux and B-Darboux frames can be written from the Eq. (6)

$$\kappa_g^2 + \kappa_n^2 = n_1^2 + n_2^2. \tag{7}$$

The case of the curve $\alpha(s)$ to be asymptotic one is,

$$\frac{n_1}{n_2} = -tan\theta, \tag{8}$$

and to be geodesic one is

$$\frac{n_1}{n_2} = cot\theta. \tag{9}$$

Additionally, the rotation matrix between B-Darboux and Darboux frames is given with the following skew-symmetric matrix:

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} t \\ g \\ n \end{bmatrix}. \tag{10}$$

Here, the angle θ between n and b_1 is as follows:

$$\theta - \theta_0 = \int \tau_g dt, \tag{11}$$

where θ_0 is an arbitrary integration constant [20].

The group of rotation around the origin announced by $SO(3)$ in $\mathbb{E}^3$ becomes homomorphic to the group of unitary complex 2x2 matrices whose determinants are one, implied by $SU(2)$. Accordingly, a two-to-one homomorphism exists between $SU(2)$ and $SO(3)$. While the elements of $SO(3)$ appear on the vectors with three real components (the points of $\mathbb{E}^3$), the elements of $SU(2)$ behave on vectors with two complex components which are called spinors [10].

A spinor is defined with the subsequent matrix

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{12}$$

by virtue of three vectors $a, b, c \in \mathbb{R}^3$ such that

$$a + ib = \Psi^t \sigma \Psi, \quad \mathrm{c} = -\widehat{\Psi}^t \sigma \Psi, \tag{13}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose cartesian components are the complex symmetric 2x2 matrices as follows:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

Let $\widehat{\Psi}$ be the mate (or conjugate) of Ψ, and $\overline{\Psi}$ be complex conjugation of Ψ [16]. Therefore

$$\widehat{\Psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\Psi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\Psi}_1 \\ \overline{\Psi}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\Psi}_2 \\ \overline{\Psi}_1 \end{pmatrix}. \tag{15}$$

Taking $a + ib = (x_1, x_2, x_3)$, from the Eq. (13) and (14) we have

$$x_1 = \Psi^t \sigma_1 \Psi = \Psi_1^2 - \Psi_2^2, \quad x_2 = \Psi^t \sigma_2 \Psi = i(\Psi_1^2 + \Psi_2^2), \quad x_3 = \Psi^t \sigma_3 \Psi = -2\Psi_1 \Psi_2,$$

where the superscript t denotes transposition, that is,

$$a + ib = (x_1, x_2, x_3) = \Psi^t \sigma \Psi = (\Psi_1^2 - \Psi_2^2, i(\Psi_1^2 + \Psi_2^2), -2\Psi_1 \Psi_2). \tag{16}$$

In the same manner, we see that

$$c = (c_1, c_2, c_3) = (\Psi_1 \overline{\Psi}_2 + \overline{\Psi}_1 \Psi_2, i(\Psi_1 \overline{\Psi}_2 - \overline{\Psi}_1 \Psi_2), |\Psi_1|^2 - |\Psi_2|^2). \tag{17}$$

After all, the vector $a + ib \in \mathbb{C}^3$ is an isotropic vector, one straightforwardly finds that a, b and c mutually orthogonal vectors. Furthermore $|a| = |b| = |c| = \overline{\Psi}^t \Psi$, and

$$\langle a \wedge b, c \rangle = \det(a, b, c) > 0.$$

Conversely, if the vectors a, b and c are reciprocally orthogonal vectors with the equal magnitude such that $\det(a, b, c) > 0$, thus as an appropriate to sign, the definition of a spinor satisfies the Eq. (13).

Based upon these cases, the two arbitrary spinors ϕ and Ψ hold the coming results:

$$\begin{gathered} \overline{\phi^t \sigma \Psi} = -\hat{\phi}^t \sigma \widehat{\Psi}, \\ \widehat{a\phi + b\Psi} = \bar{a}\hat{\phi} + \bar{b}\widehat{\Psi}, \\ \widehat{\widehat{\Psi}} = -\Psi, \end{gathered} \tag{18}$$

where a and b are complex numbers [9,10].

The relation between the spinors and the orthogonal bases (given by the Eq. (13)) is two-to-one because of the spinors Ψ and $-\Psi$ resembled to the same ordered orthogonal bases $\{a, b, c\}$ with $|a| = |b| = |c|$ and $\langle a \wedge b, c\rangle > 0$.

Additionally, the ordered triads $\{a, b, c\}, \{b, c, a\}, \{c, a, b\}$ serve to different spinors. Since the matrices σ (given by the Eq. (18)) are symmetric, any couple of spinors ϕ and Ψ satisfy

$$\phi^t \sigma \Psi = \Psi^t \sigma \phi. \tag{19}$$

Also, the non-vanishing of $\Psi \neq 0$ ensures that the set $\{\widehat{\Psi}, \Psi\}$ becomes linearly independent [9,10].

Conforming to the consequences dealt with the spinor, the spinor Ψ satisfies the following equality for Darboux frame from the Eqs. (12) and (13)

$$g + in = \Psi^t \sigma \Psi, \ t = -\widehat{\Psi}^t \sigma \Psi, \tag{20}$$

with $\widehat{\Psi}^t \Psi = 1$. So, the spinor Ψ represents the set $\{g, n, t\}$ and the changes of this set along the curve α on surface M have to associate to the explanation for $\frac{d\Psi}{ds}$. Particularly, the Darboux derivative equations become identical to the unique spinor equation

$$\frac{d\Psi}{ds} = \left(-i\frac{\tau_g}{2}\right)\Psi\left(\frac{-i\kappa_n + \kappa_g}{2}\right)\widehat{\Psi}, \tag{21}$$

where κ_n is normal curvature, κ_g is geodesic curvature and τ_g is geodesic torsion of the curve $\alpha(s)$. The Eq. (21) is said to be spinor Darboux equation in $\mathbb{E}^3$ [10].

3. SPINOR B-DARBOUX EQUATIONS IN $\mathbb{E}^3$

The B-Darboux frame of the curve $\alpha(s)$ on the surface M depends on the notion that while the tangent vector $t(s)$ is uniquely determined for a given surface curve model, the set of other vector fields b_1 and b_2 can be chosen for any proper basis. So, the set $\{b_1(s), b_2(s)\}$ lies in the normal plane perpendicular to $t(s)$ at each point. Thus, the B-Darboux frame is expressed in terms of a single spinor equation with the following theorem:

Theorem 1. Let the spinor ϕ be a representation of the B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed curve lying on a surface $M = M(u, v)$. Then, a single spinor equation can be written as

$$\frac{d\phi}{ds} = \frac{n_1 + i\, n_2}{2}\ \hat{\phi}, \tag{22}$$

where n_1 and n_2 are B-Darboux curvatures of the curve.

Proof: The following equations are satisfied for B-Darboux frame from the Eq. (20)

$$b_1 + ib_2 = \phi^t \sigma \phi, \quad t = -\hat{\phi}^t \sigma \phi, \tag{23}$$

where $\bar{\phi}^t\phi = 1$ and the spinor ϕ corresponds to the triad $\{b_1, b_2, t\}$. Additionally, the set $\{\phi, \hat{\phi}\}$ express a basis for the spinors ($\phi \neq 0$) with two components. So, since two arbitrary functions f and g are likely complex-valued functions, the variations of the set $\{b_1, b_2, t\}$ along the curve lying on a surface M can be written as follows:

$$\frac{d\phi}{ds} = f\phi + g\hat{\phi}. \tag{24}$$

Differentiating the first equation of the Eq. (23) with regard to s, it is obtained

$$\frac{db_1}{ds} + i\frac{db_2}{ds} = \frac{d}{ds}(\phi^t\sigma\phi) = (\frac{d\phi}{ds})^t\,\sigma\,\phi + \phi^t\sigma\left(\frac{d\phi}{ds}\right). \tag{25}$$

Substituting the Eqs. (5), (23), (24) into (25), and rearranging, one finds

$$-(n_1 + in_2)t = 2\,f\,(b_1 + ib_2) - 2\,g\,t, \tag{26}$$

where n_1 and n_2 are B-Darboux curvatures. From the last equation, it is obvious that

$$f = 0, \qquad g = \frac{n_1 + i\,n_2}{2}. \tag{27}$$

If the Eq. (27) is substituted into the Eq. (24), the proof is completed.

Corollary 1. The spinor representation of B-Darboux frame $\{b_1, b_2, t\}$ of a unit speed curve lying on a surface M is given in terms of Darboux curvatures with the aid of the Eqs. (6) and (22) as follows:

$$\frac{d\phi}{ds} = \frac{e^{i\theta}}{2}\left(\kappa_n - i\,\kappa_g\right)\hat{\phi}, \tag{28}$$

where θ is the angle between the vectors n and b_1.

Corollary 2. If the curve $\alpha(s)$ lying on a surface M is asymptotic, the spinor representation of B-Darboux frame in terms of Darboux curvatures is given by the following equation from the Eq. (8),

$$\frac{d\phi}{ds} = -i\frac{e^{i\theta}}{2}\,\kappa_g\,\hat{\phi}. \tag{29}$$

Corollary 3. If the curve $\alpha(s)$ lying on a surface M is geodesic, the spinor representation of B-Darboux frame in terms of Darboux curvatures is given by the following equation from the Eq. (9),

$$\frac{d\phi}{ds} = \frac{e^{i\theta}}{2}\,\kappa_n\,\hat{\phi}. \tag{30}$$

Theorem 2. The relation between the spinor formulations of B-Darboux and Darboux frames for a curve lying on a surface is given as

$$\begin{gathered}\phi^t\sigma\phi = -ie^{-i\theta}(\Psi^t\sigma\Psi),\\ t = t.\end{gathered} \tag{31}$$

where the spinors ϕ and Ψ mean the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$, respectively.

Proof. Considering the Eq. (10), it is reached

$$\begin{aligned} t &= t, \\ b_1 &= g\sin\theta + n\cos\theta, \\ b_2 &= -g\cos\theta + n\sin\theta, \end{aligned}$$

and

$$\begin{aligned} b_1 + ib_2 &= sin\theta g + cos\theta n + i(-cos\theta g + sin\theta n) \\ &= -i(g + in)e^{i\theta}. \end{aligned} \tag{32}$$

The Eq. (31) is straightforwardly obtained from the Eqs. (20), (23) and (32). On the other hand, using the Eq. (2) and the Eq. (10), the rotation matrix between B-Darboux and Frenet frames is

$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{33}$$

Then, the following equation is obtained

$$t = t, \;\; b_1 = b, \;\; b_2 = -n. \tag{34}$$

Here, since the spinors ϕ and λ represent the B-Darboux frame $\{b_1, b_2, t\}$ and the Frenet frame $\{n, b, t\}$ respectively, the following equations can be given from Theorem 2 for the relation of spinor representation between Darboux and Frenet frame

$$b_1 + ib_2 = -i(n + ib), \tag{35}$$

and

$$\begin{aligned} \phi^t \sigma \phi &= -i(\lambda^t \sigma \lambda), \\ t &= t. \end{aligned} \tag{36}$$

Corollary 4. The relation between B-Darboux and Frenet frames is expressed by means of spinor description as in the Eq. (36).

Lemma 1. Let α be a regular curve lying in a surface in Euclidean space $\mathbb{E}^3$. If the rotation angle between the B-Darboux frame $\{b_1, b_2, t\}$ and the Darboux frame $\{g, n, t\}$ equals to θ, there is a relation between ϕ and Ψ that corresponds to these frames respectively, as follows:

$$\phi = \sqrt{-i} e^{i\frac{\theta}{2}} \Psi. \tag{37}$$

Proof: Let $g + in = (1, i, 0)$ be an isotropic vector. Furthermore, from the Eq. (20), we write the following equation:

$$g + in = (a_1, a_2, a_3) = (1, i, 0) = \Psi^t \sigma \Psi = (\Psi_1^2 - \Psi_2^2, i(\Psi_1^2 + \Psi_2^2), -2\Psi_1\Psi_2),$$

and then

$$\Psi_1 = \mp\sqrt{\frac{a_1 - ia_2}{2}}, \quad \Psi_2 = \mp\sqrt{\frac{-a_1 - ia_2}{2}}.$$

Using $(a_1, a_2, a_3) = (1, i, 0)$ in the above equations, we reach $\Psi\ (\Psi_1, \Psi_2) = (\pm 1, 0)$. Now, the Darboux frame $\{g, n, t\}$ is rotated with the angle θ. Hence, from the Eq. (10), it is known that

$$\begin{aligned} b_1 &= \ \ \sin\theta g + \cos\theta n, \\ b_2 &= -\cos\theta g + \sin\theta n. \end{aligned}$$

Here, the spinor Ψ rotates to the spinor ϕ while the Darboux frame $\{g, n, t\}$ rotating to the B-Darboux frame $\{b_1, b_2, t\}$, then it can be written as

$$\begin{aligned} b_1 + ib_2 &= (g + in)\left(-ie^{-i\theta}\right) = (a_1, a_2, a_3)\left(-ie^{-i\theta}\right) \\ &= \phi^t \sigma \phi = (\phi_1^2 - \phi_2^2, i(\phi_1^2 + \phi_2^2), -2\phi_1\phi_2) \end{aligned}$$

and

$$\phi_1 = \mp\sqrt{-i}e^{i\frac{\theta}{2}}\sqrt{\frac{a_1 - ia_2}{2}}, \quad \phi_2 = \mp\sqrt{-i}e^{i\frac{\theta}{2}}\sqrt{\frac{-a_1 - ia_2}{2}}.$$

Thus, from the equality $(a_1, a_2, a_3) = (1, i, 0)$, it is seen that

$$\phi_1 = \mp\sqrt{-i}e^{i\frac{\theta}{2}}, \quad \phi_2 = 0.$$

Finally, it is reached

$$\phi = (\phi_1, \phi_2) = \left(\mp\sqrt{-i}e^{i\frac{\theta}{2}}, 0\right) = \sqrt{-i}e^{i\frac{\theta}{2}}(\mp 1, 0) = \sqrt{-i}e^{i\frac{\theta}{2}}\Psi.$$

Corollary 5. The relation between the spinors ϕ and λ which respectively represent B-Darboux frame and Frenet Frame is obtained from the Eqs. (34), (35) and (37) as follows:

$$\phi = \sqrt{-i}\lambda, \tag{38}$$

for the isotropic vector $n + ib = (1, i, 0)$.

Corollary 6. Let $\vec{\varphi}$ and $\vec{\psi}$ be the vectors with complex components corresponding to the spinor pairs ϕ and Ψ, respectively. The Eq. (37) expresses a homothetic rotation on the complex plane as follows:

$$\vec{\varphi} = he^{i\frac{\theta}{2}}\vec{\psi}, \tag{39}$$

where $h = \sqrt{-i}$ is homothetic scale, as seen in [22].

4. APPLICATION

Let the spinors ϕ and Ψ hold that $\bar{\phi}^t\phi = 1$ and $\bar{\Psi}^t\Psi = 1$, then the following equation is known from [10]

$$\overline{(\Psi \pm \phi)}^t(\Psi \pm \phi) = 2 \pm Re(\bar{\Psi}^t\phi).$$

Suppose that the spinors ϕ and Ψ which represent the B-Darboux and Darboux frames, respectively, correspond to two curves on a surface M whose curvature functions n_1 and n_2 are equal. Then by Eq. (22), it can be written by the following equation,

$$\begin{aligned}\frac{d}{ds}(\bar{\Psi}^t\phi) &= \overline{\left(\frac{1}{2}(n_1 + in_2)\hat{\Psi}\right)}^t \phi + \bar{\Psi}^t\left(\frac{1}{2}(n_1 + in_2)\hat{\phi}\right)\\ &= \frac{1}{2}n_1\overline{(\hat{\Psi})}^t\phi - \frac{i}{2}n_2\overline{(\hat{\Psi})}^t\phi + \frac{1}{2}n_1\bar{\Psi}^t\hat{\phi} + \frac{i}{2}n_2\bar{\Psi}^t\hat{\phi}\\ &= \frac{1}{2}n_1\left(\bar{\Psi}^t\hat{\phi} + \overline{(\hat{\Psi})}^t\phi\right) + \frac{i}{2}n_2\left(\bar{\Psi}^t\hat{\phi} - \overline{(\hat{\Psi})}^t\phi\right)\end{aligned}$$

which is pure imaginary. If the equations $\left(\bar{\Psi}^t\hat{\phi} + \overline{(\hat{\Psi})}^t\phi\right)$ and $\left(\bar{\Psi}^t\hat{\phi} - \overline{(\hat{\Psi})}^t\phi\right)$ are written explicitly, it seems that the first one is pure imaginary and the second one is pure real. If the last equation is given by the curvatures of Darboux frame κ_g, κ_n from the Eq. (6) and the cases of $\alpha(s)$ on M to be geodesic or asymptotic, the below equations can be obtained as,

$$\frac{d}{ds}(\bar{\Psi}^t\phi) = \frac{1}{2}\kappa_n cos\theta\left(\bar{\Psi}^t\hat{\phi} + \overline{(\hat{\Psi})}^t\phi\right) + \frac{i}{2}\kappa_n sin\theta\left(\bar{\Psi}^t\hat{\phi} - \overline{(\hat{\Psi})}^t\phi\right), \tag{40}$$

where $\alpha(s)$ is geodesic, and

$$\frac{d}{ds}(\bar{\Psi}^t\phi) = \frac{1}{2}\kappa_g sin\theta\left(\bar{\Psi}^t\hat{\phi} + \overline{(\hat{\Psi})}^t\phi\right) - \frac{i}{2}\kappa_g cos\theta\left(\bar{\Psi}^t\hat{\phi} - \overline{(\hat{\Psi})}^t\phi\right), \tag{41}$$

where $\alpha(s)$ is asymptotic.

Therefore, if the set $\{b_1, b_2, t\}$ attributes to these two curves on a surface M coinciding for some value of s by convenient translation and rotation at that point ϕ, then it pairs Ψ or with $-\Psi$, and this case is provided for all s.

5. CONCLUSION

In this research, The spinor B-Darboux equations of curves on surfaces have been handled in $\mathbb{E}^3$. In this context, the spinor representation of a curve on a surface has been examined by B-Darboux triad $\{t, b_1, b_2\}$. The relations between both B-Darboux – Darboux frames and B-Darboux – Frenet frames have been expressed via spinor modelling. Additionally, some results have been obtained for special cases of the curvatures of B-Darboux and Darboux frame. An application about spinor featuring of the curve – surface pair via B-Darboux frame has been provided, and more specifically, for the case of the curvatures to be vanishing, the application has been evaluated at the end of the research.

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