ORIGINAL PAPER

ON THE HASIMOTO SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. *In the present study, we work on the Hasimoto surfaces in three-dimensional Euclidean space by using q-frame. Calculating the coefficients of fundamental forms, we present Gaussian and mean curvatures of these Hasimoto surfaces. Lastly, we find some characterization of parameter curves of these Hasimoto surfaces.*
Keywords: *curvature; Hasimoto surface; q-frame; vortex filament.*

1. INTRODUCTION

As well known, the study of surfaces like ruled surfaces, canal surfaces, Hasimoto surfaces, and so on is a long-standing interest for mathematicians and having an understanding of how surfaces are constructed is very important. Hasimoto surfaces are special type of surfaces generated by evolving a regular space curve $\rho(u,v)$ formulated as

$$\rho_v = \rho_u \times \rho_{uu}.$$

The approximation of the local induction approximation to solutions of the nonlinear Schrodinger equation was introduced and the relation between vortex filament equation and non-linear Schrodinger equation was given by Hasimoto [1].

After this physical exploration, there have been studies of the relationship between moving frame curves and soliton equations in ambient spaces. In [2], geometric interpretation of Backlund transformations related to Hasimoto surfaces and regular space curves was explained; that is, using a special Hashimoto flow for doubly discrete curves, an evolution using double Backlund transformations was obtained. Based on the Gauss-Weingarten equations, the Hasimoto surfaces in Euclidean 3-space by applying numerical integration to their fundamental form coefficients were derived [3] while a description of parameter curves of these surfaces in Minkowski 3-space was given in [4]. Kelleci et al. [5] and Elzawy [6] worked on Hasimoto surfaces in detail. Using Hasimoto surfaces, both parallel surfaces and harmonic evolute surfaces were recently obtained and characterizations of these surfaces were given in Euclidean 3-space [7, 8].

In this present paper, we give the formula for the differential of the Hasimoto transformation in Euclidean 3-space, and various geometric properties such as the Gaussian and mean curvatures are calculated.

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2. GEOMETRIC BACKGROUND

In this part, we focus on some geometric background information about metric and frames; especially, the basics of q-frame can be found in detail [9-12]. Suppose that $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be two vectors in three dimensional Euclidean space endowed with the metric

$$< u, v >= u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The norm of the vector u is given $||u|| = \sqrt{< u, u >}$, and the Euclidean cross product of the vectors u and v is defined as

$$u \times v = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Let $\alpha(s)$ be a space curve with $\alpha''(s)$ is not zero. The Frenet formulas are written as

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{1}$$

where $v = \|\alpha'(s)\|$. Frenet vectors and curvature functions are

$$t = \frac{\alpha'}{\|\alpha'\|}, b = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, n = b \wedge t,$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2},$$

respectively [13]. Besides Frenet frame, we use q-frame $\{t, n_q, b_q, k\}$ where k is the projection vector. We choose the Euclidean angle between the principal normal n and q-normal n_q vectors to establish the relationship between the q-frame and Frenet frame. The relation matrix can therefore be stated as

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{2}$$

or

$$\begin{bmatrix} \mathrm{t} \\ \mathrm{n} \\ \mathrm{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathrm{t} \\ \mathrm{n}_q \\ \mathrm{b}_q \end{bmatrix}. \tag{3}$$

where q-frame vectors and q-curvatures are

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}_s = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}, \tag{4}$$

$$\begin{aligned} k_1 &= < t', n_q > \\ k_2 &= < t', b_q > \\ k_3 &= < n'_q, b_q >. \end{aligned} \tag{5}$$

respectively.

Let M be a regular surface given with the parametrization $\varphi(u, v)$ in E^3. The vectors φ_u and φ_v span the tangent space of M at every given location. Then the unit normal vector field of M is defined as

$$N = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}. \tag{6}$$

The coefficients of the both first and second fundamental forms of the surface M are given as

$$\begin{aligned} E &= \langle \varphi_u, \varphi_u \rangle, \\ F &= \langle \varphi_u, \varphi_v \rangle, \\ G &= \langle \varphi_v, \varphi_v \rangle, \end{aligned} \tag{7}$$

and

$$\begin{aligned} e &= \langle \varphi_{uu}, N \rangle, \\ f &= \langle \varphi_{uv}, N \rangle, \\ g &= \langle \varphi_{vv}, N \rangle \end{aligned} \tag{8}$$

respectively, where $\langle ., . \rangle$ be the Euclidean inner product. The Gaussian curvature and the mean curvature of M are given by

$$K = \frac{eg - f^2}{EG - F^2} \tag{9}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \tag{10}$$

respectively.

3. THE HASIMOTO SURFACES WITH Q-FRAME

In this part, we use the q-frame $\{\mathrm{t}, \mathrm{n}_q, \mathrm{b}_q\}$ to calculate the time derivatives, which are given by

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}_t = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix} \tag{11}$$

by the help of the equation (4) where the α, β and γ are smooth functions. Taking derivatives of (4) and (11) with respect to t and s, we get

$$t_{st} = (-\alpha k_1 - \beta k_2)t + ((k_1)_t - \gamma k_2)n_q + (k_1\gamma + (k_2)_t)b_q$$

and

$$t_{ts} = (-\alpha k_1 - \beta k_2)t + (\alpha_s - \beta k_3)n_q + (\alpha k_3 + \beta_s)b_q,$$

respectively. Applying the compatibility condition $t_{ts} = t_{st}$, we get

$$\begin{aligned} (k_1)_t &= \alpha_s - \beta k_3 + \gamma k_2, \\ (k_2)_t &= \beta_s + \alpha k_3 - k_1\gamma. \end{aligned}$$

Similarly,

$$(n_q)_{ts} = (-\alpha_s - k_2\gamma)t + (-\alpha k_1 - \gamma k_3)n_q + (-\alpha k_2 + \gamma_s)b_q$$

and

$$(n_q)_{st} = (-(k_1)_t - k_3\beta)t + (-k_1\alpha - k_3\gamma)n_q + (-k_1\beta + (k_3)_t)b_q.$$

Therefore, we have the time derivative of the first and the third q-curvature functions as follows:

$$\begin{aligned} (k_1)_t &= \alpha_s - \beta k_3 + \gamma k_2, \\ (k_3)_t &= \gamma_s - \alpha k_2 + k_1\beta. \end{aligned}$$

Similarly, applying the compatibility condition $(b_q)_{ts} = (b_q)_{st}$ to the equations

$$(b_q)_{ts} = (-\beta_s + \gamma k_1)t + (-\gamma_s - \beta k_1)n_q + (-\beta k_2 - \gamma k_3)b_q$$

and

$$(b_q)_{st} = (-(k_2)_t + k_3\alpha)t + (-k_2\alpha - (k_3)_t)n_q + (-k_2\beta - k_3\gamma)b_q,$$

we get the following functions

$$\begin{aligned} (k_2)_t &= \beta_s + \alpha k_3 - k_1\gamma, \\ (k_3)_t &= \gamma_s - \alpha k_2 + k_1\beta. \end{aligned}$$

Each of the three cases can be combined to have

$$\begin{aligned}(k_1)_t &= \alpha_s - \beta k_3 + \gamma k_2,\\ (k_2)_t &= \beta_s + \alpha k_3 - k_1\gamma,\\ (k_3)_t &= \gamma_s - \alpha k_2 + k_1\beta.\end{aligned} \tag{12}$$

The velocity vector can be written as

$$\rho_t = \lambda t + \mu n_q + \eta b_q.$$

The derivative of ρ_t with respect to s is given

$$(\rho_t)_s = (\lambda_s - \mu k_1 - \eta k_2)t + (\mu_s + \lambda k_1 - \mu k_3)n_q + (\eta_s + \lambda k_2 + \mu k_3)b_q.$$

Since $(\rho_t)_s = (\rho_s)_t$, we derive the folllowing equalities

$$\begin{aligned}\lambda_s - \mu k_1 - \eta k_2 &= 0,\\ \mu_s + \lambda k_1 - \mu k_3 &= \alpha,\\ \eta_s + \lambda k_2 + \mu k_3 &= \beta.\end{aligned}$$

If $\rho(s,t)$ is the position of vortex filament, then one can write

$$\rho_t = \rho_s \times \rho_{ss}.$$

The velocity vector can be rewritten

$$\rho_t = \rho_s \times \rho_{ss} = t \times \left(k_1 n_q + k_2 b_q\right) = -k_2 n_q + k_1 b_q$$

for a solution of smoke ring equation. Then, $(\lambda, \mu, \eta) = (0, -k_2, k_1)$. That is,

$$\begin{aligned}\alpha &= -(k_2)_s - k_1 k_3,\\ \beta &= (k_1)_s - k_2 k_3.\end{aligned}$$

Substituting these findings in third equation (12), and differentiating with respect to s give us

$$\gamma = \int (k_3)_t ds - \frac{k_1^2 + k_2^2}{2}.$$

Therefore, the equation (11) turns out to be

$$\begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}_t = \begin{bmatrix} 0 & -(k_2)_s - k_1 k_3 & (k_1)_s - k_2 k_3 \\ (k_2)_s + k_1 k_3 & 0 & \int (k_3)_t ds - \frac{k_1^2+k_2^2}{2} \\ k_2 k_3 - (k_1)_s & \frac{k_1^2+k_2^2}{2} - \int (k_3)_t ds & 0 \end{bmatrix} \begin{bmatrix} t \\ n_q \\ b_q \end{bmatrix}.$$

Theorem 3.1. Let $\rho(s,t)$ be a position vector for the surface $\varphi(u,v)$ in E^3. Then the Gaussian curvature and the mean curvature of $\varphi(u,v)$ are given by

$$K = \frac{1}{k_1^2+k_2^2}\left[\frac{(k_2)_t k_1 - k_2(k_1)_t}{k_1^2+k_2^2} + \int (k_3)_t ds - \frac{k_1^2+k_2^2}{2} - \left(\frac{(k_2k_1)_t}{k_1^2+k_2^2} + k_3\right)^2\right] \tag{13}$$

and

$$H = \frac{1}{2(k_1^2+k_2^2)}\left[\frac{(k_2)_t k_1 - k_2(k_1)_t}{k_1^2+k_2^2} + \int (k_3)_t ds + \frac{k_1^2+k_2^2}{2}\right], \tag{14}$$

respectively.

Proof: Basic calculations show that the coeeffients of the first and second fundamental forms of $\rho(s,t)$ are given

$$E = 1,$$

$$F = 0,$$

$$G = k_1^2 + k_2^2$$

and

$$\begin{aligned} e &= 1 \\ f &= \frac{(k_2k_1)_t}{k_1^2+k_2^2} + k_3 \\ g &= \frac{(k_2)_t k_1 - k_2(k_1)_t}{k_1^2+k_2^2} + \int (k_3)_t ds - \frac{k_1^2+k_2^2}{2}. \end{aligned}$$

Using these findings and equations (9) and (10), (13) and (14) are easily found.

Theorem 3.2. Let $\rho(s,t)$ be the Hasimoto surface in E^3. In this case, the following conditions are satisfied.
i) s- parameter curves of the surface is geodesic.
ii) t- parameter curves of the surface is geodesic only when

$$(k_1)_s k_1 = -k_2(k_2)_s$$

and

$$k_2\left\{\frac{k_1^2+k_2^2}{2} - \int (k_3)_t ds - (k_2)_t\right\} = k_1\left\{\int (k_3)_t ds - \frac{k_1^2+k_2^2}{2} + (k_1)_t\right\}$$

Proof: **i)** We know that

$$\rho_{ss} = k_1 n_q + k_2 b_q.$$

The normal vector of the surface is

$$N = -\frac{1}{k_1^2 + k_2^2}(k_1 n_q + k_2 b_q).$$

Thus, the normal vector of the surface and the s-parameter curves of the surface are linear dependent; that is, s-parameter curves of the surface is geodesic.

ii)

$$\begin{aligned}\rho_{tt} = (-k_2 n_q + k_1 b_q)_s &= -\{(k_1)_s k_1 + k_2(k_2)_s\}t \\ &= \{\frac{k_1^2 + k_2^2}{2} - \int (k_3)_t ds - (k_2)_t\}n_q \\ &= \{\int (k_3)_t ds - \frac{k_1^2 + k_2^2}{2} + (k_1)_t\}b_q.\end{aligned}$$

Using the similar argument in **i)**, we can prove theorem easily.

Theorem 3.3. Let $\rho(s,t)$ be the Hasimoto surface in E^3. In this case, the following conditions are satisfied.
i) s- parameter curves of the Hasimoto surface is asymptotic when $k_1 = 0$.
ii) t- parameter curves of the Hasimoto surface is asymptotic only when

$$k_1^2 + k_2^2 = 2\{\int (k_3)_t ds + (k_2)_t\}.$$

Proof:
i) Since $\rho_{ss} = k_1 n_q + k_2 b_q$, k_1 must be zero in order that s-parameter curves of the Hasimoto surface is asymptotic.

ii) We say the normal component of ρ_{tt} becomes zero if and only if

$$\frac{k_1^2 + k_2^2}{2} - \int (k_3)_t ds - (k_2)_t = 0.$$

Therefore, t-parameter curves of the Hasimoto surfaces are asymptotic curves if and only if

$$k_1^2 + k_2^2 = 2\{\int (k_3)_t ds + (k_2)_t\}.$$

4. CONCLUSION

In this paper, q-frame is used to study Hasimoto surfaces in three-dimensional Euclidean space. Later, some geometric properties of these surfaces are presented by finding coefficients of fundamental forms. Lastly, necessary and sufficient conditions are given for the parameter curves of these Hasimoto surfaces to be asymptotic and geodesic.

REFERENCES

[1] Hasimoto, H., *J. Fluid. Mech.*, **51**, 477, 1972.
[2] Hoffmann, T., Discrete Hashimoto surfaces and a doubly discrete smokering flow, arXiv:math/0007150.
[3] Abdel-All, N.H., Hussien, R.A., Youssef, T., *Life Science Journal*, **9**(3), 556, 2012.
[4] Erdogdu, M., Ozdemir, M., *Mathematical Physics, Analysis and Geometry*, **17** 169, 2014.
[5] Kelleci, A., Bektas, M., Ergut, M., *Adiyaman University Journal of Science*, **9**, 13, 2019.
[6] Elzawy, M., *Journal of the Egyptian Mathematical Society*, **29**(1), 5, 2021.
[7] Cakmak, A., *BEU Journal of Science*, **7**(1), 125, 2018.
[8] Eren, K., Kelleci Akbay, A., *Adiyaman University Journal of Science*, **11**(1), 87, 2021.
[9] Dede, M., Ekici, C., Görgülü, A., *International Journal of Advanced Research In Computer Science and Software Engineering*, **5**(12), 775, 2015.
[10] Ekici, C., Kaymanli, G.U., Dede, M., *Erzincan University Journal of Science and Technology*, **13**, 33, 2020.
[11] Korpinar, T., Kaymanli, G.U., *Journal of Science and Arts*, **20**(1), 55, 2020.
[12] Kaymanli, G.U., Korpinar, T., *Journal of Science and Arts*, **20**(4), 881, 2020.
[13] Gray, A., Salamon, S., Abbena, E., Curves in Space. In Abbena, E., Salamon, S., Gray, A. (Eds.), *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 3rd Ed., Chapman and Hall/CRC, 2006, p.38.