ORIGINAL PAPER

SURFACES WITH VANISHING ABNORMALITY OF NORMAL DIRECTION IN MINKOWSKI SPACE

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Abstract. *This paper is investigated geometry of vector fields along spacelike curve with timelike normal vector by using anholonomic coordinates. Derivative formulas of Frenet Serret frame of the curve are stated which includes eight parameters. Surfaces with vanishing abnormality of normal direction in Minkowski space are examined. Intrinsic geometric properties of these spacelike surfaces are investigated. Finally, the relations between spacelike surfaces with vanishing abnormality of normal direction and NLS, Heisenberg spin equation are investigated as applications.*

Keywords: *anholonomic coordinates; abnormality of normal direction; spacelike surfaces.*

1. INTRODUCTION

The importance of non-Euclidean geometries such as Euclidean geometries is emphasized in many studies. Lorentzian geometry is the best known of the non-Euclidean geometries. The fact that there is a lot of interdisciplinary interactions make this geometry important. Lorentzian geometry has the most well known applications [1-5]. This geometry is a popular differential geometry research topic with physical issues involving integrable systems, soliton theory, fluid dynamics, field theories, etc. [6-9]. General relativity has been based on the mathematical theory of Lorentzian geometry ever since Albert Einstein extended Minkowski space-time to a curved space-time to explain nonzero gravitational fields. This circumstance served as a significant impetus for the advancement of novel methods in the investigation of cosmological models that are increasingly tailored to the physical world. It is crucial to use vector analysis to examine the geometrical properties of curves. Consider that $s, n,$ and b are the distances along the space curve $\alpha = \alpha(s, n, b)$ in tangential, normal and binormal directions, respectively in three-dimensional Euclidean space. The primary focus of [7] is the system which is obtained by directional derivatives of Frenet Serret frame.

The purpose of this study is to exhibit normal congruence surfaces in Lorentzian space containing spacelike curve flow with timelike normal. We examine generating surfaces with normal congruence that have spacelike curve flow with timelike normal. The metric for three dimensional Lorentzian space

$$\langle \vec{x}, \vec{y} \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

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where $\vec{x}, \vec{y} \in \mathbb{R}^3$. Through the use of anholomonic coordinates, section two aims to examine the three-dimensional vector field as well as geometric elements. We describe Frenet Serret frame by means of anholonomic coordinates. It is proved that there are no components in the direction of the principal normal vector field in the $curl$ of the tangent vector field. This demonstrates that a surface with both $\vec{T}$ and $\vec{B}$ directions exists. As a result, the last section describes a normal congruence of surface. Important results are provided together with the intrinsic geometric features of this congruence.

2. EXTENDED SERRET FRENET FORMULAS OF SPACELIKE CURVE WITH TIMELIKE NORMAL

In this section, the fundamental formulas are given by using anholomonic coordinates. Assume that $\alpha = \alpha(s, n, b)$ is a given spacelike curve with timelike normal. Then, s denotes the distance along the curve in tangential direction and unit spacelike tangent vector is is defined by

$$\vec{T} = \frac{\partial \alpha}{\partial s}.$$

The distance along the curve in normal direction is denoted by n and unit timelike normal vector of $\vec{N}$ is defined by

$$\vec{N} = \frac{\partial \alpha}{\partial n}.$$

Moreover, b denotes the distance along the curve in binormal direction and the unit spacelike binormal vector is is defined by

$$\vec{B} = \frac{\partial \alpha}{\partial b}.$$

We need to know curvature and torsion which are two independent parameters to explain the intrinsic differential geometric structure of a spacelike curve in three-dimensional Minkowski space,

Theorem 2.1. Assume that $\alpha = \alpha(s, n, b)$ is a unit speed spacelike curve with timelike normal. Derivatives of $\{\vec{T}, \vec{N}, \vec{B}\}$ are stated as follows:

i)
$$\frac{\partial}{\partial s}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ \kappa & 0 & \tau\\ 0 & \tau & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} \tag{1}$$

ii)
$$\frac{\partial}{\partial n}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & -\beta_{ns} & \mu_b - \tau\\ -\beta_{ns} & 0 & -div\vec{B}\\ \tau - \mu_b & -div\vec{B} & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} \tag{2}$$

iii)
$$\frac{\partial}{\partial b}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \tau - \mu_n & \beta_{bs}\\ \tau - \mu_n & 0 & div\vec{N} - \kappa\\ -\beta_{bs} & div\vec{N} - \kappa & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} \tag{3}$$

such that

$$\beta_{ns} = \left\langle \vec{N}, \frac{\partial \vec{T}}{\partial n} \right\rangle_L, \quad \beta_{bs} = \left\langle \vec{B}, \frac{\partial \vec{T}}{\partial b} \right\rangle_L$$

where

$$\mu_n = \langle curl\vec{N}, \vec{N} \rangle_L, \quad \mu_b = \langle curl\vec{B}, \vec{B} \rangle_L$$

are abnormalities of $\vec{N}$ and $\vec{B}$, respectively.

Proof: Equation 1 is clear. Thus, we will only give the proof of ii) and iii). There exists smooth functions; λ_i and ψ_i such that

$$\frac{\partial}{\partial n}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \lambda_1 & \lambda_2\\ \lambda_1 & 0 & \lambda_3\\ -\lambda_2 & \lambda_3 & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix},$$

$$\frac{\partial}{\partial b}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \psi_1 & \psi_2\\ \psi_1 & 0 & \psi_3\\ -\psi_2 & \psi_3 & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

for $i = 1,2,3$. For the proof of the theorem, we need to consider these functions. First of all, we get

$$\lambda_1 = \left\langle \frac{\partial \vec{T}}{\partial n}, \vec{N} \right\rangle_L = -\beta_{ns}, \qquad \psi_2 = \left\langle \frac{\partial \vec{T}}{\partial b}, \vec{B} \right\rangle_L = \beta_{bs}.$$

Divergences of Frenet Serret frame fields are obtained as follows:

$$div\vec{T} = \beta_{ns} + \psi_2,$$

$$div\vec{N} = \kappa + \psi_3$$

and

$$div\vec{B} = -\lambda_3.$$

So, we obtain

$$\psi_3 = div\vec{N} - \kappa \quad \text{and} \quad \lambda_3 = -div\vec{B}.$$

Therefore, we get

$$curl\vec{T} = (\psi_1 - \lambda_2)\vec{T} + \kappa\vec{B},$$

$$curl\vec{N} = \left(div\vec{B}\right)\vec{T} + (\psi_1 - \tau)\vec{N} + \beta_{ns}\vec{B}$$

and

$$curl\vec{B} = (div\vec{N} - \kappa)\vec{T} - \beta_{bs}\vec{N} + \mu_b\vec{B}.$$

Then, we have

$$\mu_s = \left\langle (\psi_1 - \lambda_2)\vec{T} + \kappa\vec{B}, \vec{T} \right\rangle_L = \psi_1 - \lambda_2,$$

$$\mu_n = \left\langle \left(div\vec{B}\right)\vec{T} + (\psi_1 - \tau)\vec{N} + (\beta_{ns})\vec{B}, \qquad \vec{N} \right\rangle_L = \tau - \psi_1,$$

$$\mu_b = \left\langle \left(div\vec{N} - \kappa\right)\vec{T} - \psi_2\vec{N} + (\tau + \lambda_2)\vec{B}, \qquad \vec{B}\right\rangle_L = \tau + \lambda_2.$$

This implies

$$\psi_1 = \tau - \mu_n, \qquad \lambda_2 = \mu_b - \tau.$$

Finally, we obtain

$$\frac{\partial}{\partial n}\begin{bmatrix}\vec{\mathrm{T}}\\ \vec{\mathrm{N}}\\ \vec{\mathrm{B}}\end{bmatrix} = \begin{bmatrix}0 & -\beta_{ns} & \mu_b - \tau\\ \beta_{ns} & 0 & -div\vec{B}\\ \tau - \mu_b & -div\vec{B} & 0\end{bmatrix}\begin{bmatrix}\vec{\mathrm{T}}\\ \vec{\mathrm{N}}\\ \vec{\mathrm{B}}\end{bmatrix},$$

$$\frac{\partial}{\partial b}\begin{bmatrix}\vec{\mathrm{T}}\\ \vec{\mathrm{N}}\\ \vec{\mathrm{B}}\end{bmatrix} = \begin{bmatrix}0 & \tau - \mu_n & \beta_{bs}\\ \tau - \mu_n & 0 & div\vec{N} - \kappa\\ -\beta_{bs} & div\vec{N} - \kappa & 0\end{bmatrix}\begin{bmatrix}\vec{\mathrm{T}}\\ \vec{\mathrm{N}}\\ \vec{\mathrm{B}}\end{bmatrix}.$$

3. THE INTRINSIC REPRESENTATIONS OF CURL OF FRENET FRAME FIELDS

Following equality is obtained by using relationship between abnormalities

$$\mu_s - \tau = \frac{1}{2}(\mu_s - \mu_n - \mu_b).$$

This relation shows that an important result involving Dupin theorem. By the proof of above theorem, we also get

$$curl\,\vec{T} = \mu_s\vec{T} + \kappa\vec{B}. \tag{4}$$

$curl\,\vec{T}$ has no component along $\vec{N}$. Therefore, there exists a surface which contains both s and b lines.

Then, we have

$$0 = \frac{\partial g}{\partial s}curl\vec{T} + \frac{\partial g}{\partial n}curl\,\vec{N} + \frac{\partial g}{\partial b}curl\,\vec{B} + (\frac{\partial^2 g}{\partial b\partial n} - \frac{\partial^2 g}{\partial n\partial b})\vec{T}$$

$$+(-\frac{\partial^2 g}{\partial s\partial b} + \frac{\partial^2 g}{\partial b\partial s})\vec{N} + (\frac{\partial^2 g}{\partial s\partial n} - \frac{\partial^2 g}{\partial n\partial s})\vec{B}$$

by the equality $0 = curl(gradg)$. Then, we get

$$0 = (\frac{\partial^2 g}{\partial b\partial n} - \frac{\partial^2 g}{\partial n\partial b} - \frac{\partial g}{\partial s}\mu_s + \frac{\partial g}{\partial n}div\vec{B} + \frac{\partial g}{\partial b}(div\vec{N} + \kappa))\vec{T}$$

$$+(\frac{\partial^2 g}{\partial s\partial b} - \frac{\partial^2 g}{\partial b\partial s} + \frac{\partial g}{\partial n}\mu_n - \frac{\partial g}{\partial b}\beta_{bs})\vec{N}$$

$$+(\frac{\partial^2 g}{\partial s \partial n}-\frac{\partial^2 g}{\partial n \partial s}+\frac{\partial g}{\partial s}\kappa-\frac{\partial g}{\partial n}\beta_{ns}+\frac{\partial g}{\partial b}\mu_b)\vec{B}.$$

Thus, we get the followings

$$\frac{\partial^2 g}{\partial b \partial n}-\frac{\partial^2 g}{\partial n \partial b}=-\frac{\partial g}{\partial s}\mu_s-\frac{\partial g}{\partial n}div\vec{B}-\frac{\partial g}{\partial b}(div\vec{N}-\kappa), \tag{5}$$

$$-\frac{\partial^2 g}{\partial s \partial b}+\frac{\partial^2 g}{\partial b \partial s}=\frac{\partial g}{\partial n}\mu_n+\frac{\partial g}{\partial b}\beta_{bs}, \tag{6}$$

$$\frac{\partial^2 g}{\partial s \partial n}-\frac{\partial^2 g}{\partial n \partial s}=-\frac{\partial g}{\partial s}\kappa-\frac{\partial g}{\partial n}\beta_{ns}-\frac{\partial g}{\partial b}\mu_b. \tag{7}$$

Theorem 3.1. The following conditions on κ, τ, μ_s, μ_b, $div\vec{N}$, $div\vec{B}$, β_{ns}, β_{bs} are stated by compatibility of the linear systems

$$\frac{\partial \beta_{ns}}{\partial b}+\frac{\partial(\tau-\mu_n)}{\partial n}=\kappa\mu_s+(\mu_b-\mu_n)(div\vec{N}-\kappa)+(\beta_{bs}-\beta_{ns})div\vec{B}, \tag{8}$$

$$-\frac{\partial(\tau-\mu_n)}{\partial s}+\frac{\partial \kappa}{\partial b}=-\beta_{ns}\mu_n+\beta_{bs}(2\tau-\mu_n), \tag{9}$$

$$-\frac{\partial \beta_{ns}}{\partial s}-\frac{\partial \kappa}{\partial n}=(\mu_b-\tau)\tau+\kappa^2-\beta_{ns}^2-(\mu_n-\tau)\mu_b, \tag{10}$$

$$\frac{\partial divB}{\partial s}+\frac{\partial \tau}{\partial n}=-div\vec{B}\beta_{ns}+(div\vec{N}-\kappa)\mu_b+(-\mu_b+\tau)\kappa+\tau\kappa, \tag{11}$$

$$-\frac{\partial(divN-\kappa)}{\partial s}+\frac{\partial \tau}{\partial b}=\left(div\vec{N}-2\kappa\right)\beta_{bs}-div\vec{B}\mu_n, \tag{12}$$

$$\frac{\partial(\mu_b-\tau)}{\partial b}-\frac{\partial \beta_{bs}}{\partial n}=(\beta_{ns}-\beta_{bs})(div\vec{N}+\kappa)-(\mu_b-\mu_n)div\vec{B}, \tag{13}$$

$$\frac{\partial \beta_{bs}}{\partial s}=\kappa(div\vec{N}-\kappa)+(\mu_n-\tau)\tau-\mu_n(\mu_b-\tau)-\beta_{bs}^2, \tag{14}$$

$$\frac{\partial(\mu_b-\tau)}{\partial s}=-\kappa div\vec{B}+(-\mu_b+2\tau)\beta_{ns}-\beta_{bs}\mu_b, \tag{15}$$

$$\begin{aligned}\frac{\partial div\vec{B}}{\partial b}+\frac{\partial\left(div\vec{N}-\kappa\right)}{\partial n}&=-\beta_{bs}\beta_{ns}+(\mu_n-\tau)(\mu_b-\tau)\\&-(div\vec{B})^2+(div\vec{N}-\kappa)^2+\mu_s\tau.\end{aligned} \tag{16}$$

Proof: By using Equation 5, following equality can be obtained

$$\frac{\partial^2 \vec{T}}{\partial b \partial n}-\frac{\partial^2 \vec{T}}{\partial n \partial b}=\frac{\partial \vec{T}}{\partial s}\mu_s-\frac{\partial \vec{T}}{\partial n}div\vec{B}-\frac{\partial \vec{T}}{\partial b}(div\vec{N}+\kappa).$$

By compatibility condition, we get

$$\frac{\partial^2\vec{T}}{\partial b\partial n}-\frac{\partial^2\vec{T}}{\partial n\partial b}=-\frac{\partial\beta_{ns}}{\partial b}\vec{N}-\beta_{ns}((\tau-\mu_n)\vec{T}+(div\vec{N}-\kappa)\vec{B})$$

$$+\frac{\partial(\mu_b-\tau)}{\partial b}\vec{B}+(\mu_b-\tau)(-\beta_{bs}\vec{T}+(div\vec{N}-\kappa)\vec{N})$$

$$-\frac{\partial}{\partial n}(\tau-\mu_n)\vec{N}-(\tau-\mu_n)(-\beta_{ns}\vec{T}-\left(div\vec{B}\right)\vec{B})$$

$$-\frac{\partial\beta_{bs}}{\partial n}\vec{B}-\beta_{bs}((-\mu_b+\tau)\vec{T}-\left(div\vec{B}\right)\vec{N})$$

$$=(-\beta_{ns}(\tau-\mu_n)-\beta_{bs}(\mu_b-\tau)+\beta_{ns}(\tau-\mu_n)-\beta_{bs}(-\mu_b+\tau))\vec{T}$$

$$+(-\frac{\partial\beta_{ns}}{\partial b}+(\mu_b-\tau)(div\vec{N}-\kappa)-\frac{\partial}{\partial n}(\tau-\mu_n)+\beta_{bs}div\vec{B})\vec{N}$$

$$+(-\beta_{ns}(div\vec{N}-\kappa)+\frac{\partial(\mu_b-\tau)}{\partial b}+(\tau-\mu_n)div\vec{B}-\frac{\partial\beta_{bs}}{\partial n})\vec{B}$$

$$\frac{\partial^2\vec{T}}{\partial b\partial n}-\frac{\partial^2\vec{T}}{\partial n\partial b}=(\frac{\partial\beta_{ns}}{\partial b}+(\mu_b+\tau)(div\vec{N}+\kappa)+\frac{\partial(\mu_n-\tau)}{\partial n}-\beta_{bs}div\vec{B})\vec{N}$$

$$+(\beta_{ns}(div\vec{N}+\kappa)-\frac{\partial(\mu_b+\tau)}{\partial b}+(\tau-\mu_n)div\vec{B}-\frac{\partial\beta_{bs}}{\partial n})\vec{B}.$$

Then, we also have

$$-\frac{\partial\vec{T}}{\partial s}\mu_s-\frac{\partial\vec{T}}{\partial n}div\vec{B}-\frac{\partial\vec{T}}{\partial b}(div\vec{N}-\kappa)=(-\kappa\mu_s+\beta_{ns}\left(div\vec{B}\right)-(\tau-\mu_n)(div\vec{N}-\kappa))\vec{N}$$

$$-((\mu_b-\tau)div\vec{B}-\beta_{bs}(div\vec{N}-\kappa))\vec{B}.$$

Therefore,

$$\frac{\partial\beta_{ns}}{\partial b}+\frac{\partial(\tau-\mu_n)}{\partial n}=\kappa\mu_s+(\mu_b-\mu_n)(div\vec{N}-\kappa)+(\beta_{bs}-\beta_{ns})div\vec{B}.$$

Again by using Equation 6, we may also write

$$\frac{\partial^2\vec{T}}{\partial s\partial b}-\frac{\partial^2\vec{T}}{\partial b\partial s}=-\frac{\partial\vec{T}}{\partial n}\mu_n+\frac{\partial\vec{T}}{\partial b}\beta_{bs}.$$

By compatibility condition, we get

$$\frac{\partial^2\vec{T}}{\partial b\partial s}-\frac{\partial^2\vec{T}}{\partial s\partial b}=(-\frac{\partial(\tau-\mu_n)}{\partial s}-\tau\beta_{bs}+\frac{\partial\kappa}{\partial b})\vec{N}$$

$$-((\tau-\mu_n)\tau-\frac{\partial\beta_{bs}}{\partial s}+\kappa(div\vec{N}-\kappa))\vec{B}$$

Moreover, we have

$$\frac{\partial T}{\partial n}\mu_n+\frac{\partial T}{\partial b}\beta_{bs}=(-\beta_{ns}\vec{N}+(\mu_b-\tau)\vec{B})\mu_n+((\tau-\mu_n)\vec{N}+\beta_{bs}\vec{B})\beta_{bs}$$

$$=(-\beta_{ns}\mu_n+\beta_{bs}(\tau-\mu_n))\vec{N}+((\mu_b-\tau)\mu_n+\beta_{bs}^2)\vec{B}.$$

This gives the following equation

$$-\frac{\partial(\tau-\mu_n)}{\partial s}+\frac{\partial\kappa}{\partial b}=-\beta_{ns}\mu_n+\beta_{bs}(2\tau-\mu_n).$$

Again using Equation 7, we may write

$$\frac{\partial^2\vec{T}}{\partial s\partial n}-\frac{\partial^2\vec{T}}{\partial n\partial s}=-\frac{\partial\vec{T}}{\partial s}\kappa+\frac{\partial\vec{T}}{\partial n}\beta_{ns}-\frac{\partial\vec{T}}{\partial b}\mu_b.$$

By compatibility condition, we have

$$\frac{\partial^2\vec{T}}{\partial s\partial n}-\frac{\partial^2\vec{T}}{\partial n\partial s}=-\frac{\partial\beta_{ns}}{\partial s}\vec{N}-\beta_{ns}(\kappa\vec{T}+\tau\vec{B})+\frac{\partial(\mu_b-\tau)}{\partial s}\vec{B}$$

$$+(\mu_b-\tau)\tau\vec{N}-\frac{\partial\kappa}{\partial n}\vec{N}-\kappa(-\beta_{ns}\vec{T}-\left(div\vec{B}\right)\vec{B})$$

$$=(-\frac{\partial\beta_{ns}}{\partial s}+(\mu_b-\tau)\tau-\frac{\partial\kappa}{\partial n})\vec{N}+(-\beta_{ns}\tau+\frac{\partial(\mu_b-\tau)}{\partial s}+\kappa div\vec{B})\vec{B}.$$

Then, we get

$$-\frac{\partial T}{\partial s}\kappa+\frac{\partial T}{\partial n}\beta_{ns}-\frac{\partial T}{\partial b}\mu_b=-(\kappa\vec{N})\kappa-(-\beta_{ns}\vec{N}+(\mu_b-\tau)\vec{B})\beta_{ns}$$

$$-((\tau-\mu_n)\vec{N}+\beta_{bs}\vec{B})\mu_b$$

$$=(-\kappa^2+\beta_{ns}^2-(\tau-\mu_n)\mu_b)\vec{N}+(-(\mu_b-\tau)\beta_{ns}-\beta_{bs}\mu_b)\vec{B}.$$

Thus, we obtain

$$-\frac{\partial\beta_{ns}}{\partial s}-\frac{\partial\kappa}{\partial n}=-(\mu_b+\tau)\tau-\kappa^2-\beta_{ns}^2-(\tau-\mu_n)\mu_b,$$

$$\frac{\partial(\mu_b-\tau)}{\partial s}=\tau\beta_{ns}-\kappa div\vec{B}-(\mu_b-\tau)\beta_{ns}-\beta_{bs}\mu_b.$$

Again using Equation 7, we may write

$$\frac{\partial^2\vec{B}}{\partial s\partial n}-\frac{\partial^2\vec{B}}{\partial n\partial s}=-\frac{\partial\vec{B}}{\partial s}\kappa+\frac{\partial\vec{B}}{\partial n}\beta_{ns}-\frac{\partial\vec{B}}{\partial b}\mu_b.$$

By compatibility condition, we get

$$\frac{\partial^2\vec{B}}{\partial s\partial n}-\frac{\partial^2\vec{B}}{\partial n\partial s}=(\frac{\partial(-\mu_b+\tau)}{\partial s}-\kappa div\vec{B}+\tau\beta_{ns})\vec{T}+((-\mu_b+\tau)\kappa-\frac{\partial div\vec{B}}{\partial s}-\frac{\partial\tau}{\partial n})\vec{N}.$$

Similarly, we also have

$$-\frac{\partial\vec{B}}{\partial s}\kappa+\frac{\partial\vec{B}}{\partial n}\beta_{ns}-\frac{\partial\vec{B}}{\partial b}\mu_b=-(\tau\vec{N})\kappa-((-\mu_b+\tau)\vec{T}-\left(div\vec{B}\right)\vec{N})\beta_{ns}$$

$$-(-\beta_{bs}\vec{T}-(div\vec{N}-\kappa)\vec{N})\mu_b$$

$$=((-\mu_b+\tau)\beta_{ns}+\beta_{bs}\mu_b)\vec{T}+\left(-\kappa\tau+\left(div\vec{B}\right)\beta_{ns}-\left(div\vec{N}-\kappa\right)\mu_b\right)\vec{N}.$$

Therefore, we obtain

$$-\frac{\partial div\vec{B}}{\partial s}-\frac{\partial\tau}{\partial n}=\left(div\vec{B}\right)\beta_{ns}-\left(div\vec{N}-\kappa\right)\mu_b-(-\mu_b+\tau)\kappa-\tau\kappa.$$

In similar manner, we have

$$\frac{\partial}{\partial b}\left(\frac{\partial\vec{B}}{\partial s}\right)-\frac{\partial}{\partial s}\left(\frac{\partial\vec{B}}{\partial b}\right)=(\frac{\partial(\beta_{bs})}{\partial s}-\kappa(div\vec{N}-\kappa)+\tau(\tau-\mu_n))\vec{T}$$

$$+(-\frac{\partial(div\vec{N}-\kappa)}{\partial s}+\frac{\partial\tau}{\partial b}+\kappa\beta_{bs})\vec{N}.$$

Therefore, we get

$$\frac{\partial\vec{B}}{\partial n}\mu_n+\frac{\partial\vec{B}}{\partial b}\beta_{bs}=(-\beta_{bs}^2+\mu_n(\tau-\mu_b))\vec{T}+(\beta_{bs}(div\vec{N}-\kappa)-\mu_n div\vec{B})\vec{N}.$$

Thus, we have

$$\frac{\partial\tau}{\partial b}-\frac{\partial(div\vec{N}-\kappa)}{\partial s}=-\beta_{bs}(-div\vec{N}+2\kappa)-\mu_n div\vec{B}.$$

We can obtain the last two equations similarly to the others.

4. SPACELIKE SURFACES WITH VANISHING ABNORMALITY OF NORMAL DIRECTION

There exists a normal congruence of surface $\Upsilon=\Upsilon(s,b)$ which contains s and b lines if and only if

$$\mu_n = 0. \tag{17}$$

Theorem 4.1. Gauss Mainardi-Codazzi equations are given by

$$\frac{\partial \kappa}{\partial b} - \frac{\partial \tau}{\partial s} = 2\tau\beta_{bs},$$

$$\frac{\partial \beta_{bs}}{\partial s} = -\beta_{bs}^2 - \tau^2 + \kappa(div\vec{N} - \kappa),$$

$$\frac{\partial \tau}{\partial b} - \frac{\partial(div\vec{N} - \kappa)}{\partial s} = \beta_{bs}(div\vec{N} - 2\kappa).$$

Corollary 4.1. In the case $\mu_n = 0$, $\vec{N}$ is perpendicular to the surface $\Upsilon = \Upsilon(s, b)$.

Proof: By definitions of $\vec{T}$ and $\vec{B}$, we know that

$$\frac{\partial \Upsilon}{\partial s} = \vec{T} \quad \text{and} \quad \frac{\partial \Upsilon}{\partial b} = \vec{B}.$$

Then, we obtain

$$\frac{\partial \Upsilon}{\partial s} \times_L \frac{\partial \Upsilon}{\partial b} = \vec{T} \times_L \vec{B} = -\vec{N}.$$

and so

$$n = \frac{\frac{\partial \Upsilon}{\partial s} \times_L \frac{\partial \Upsilon}{\partial b}}{\left\| \frac{\partial \Upsilon}{\partial s} \times_L \frac{\partial \Upsilon}{\partial b} \right\|} = -\vec{N}.$$

Remark 4.1. Since $\vec{N}$ is a timelike vector field, $\Upsilon = \Upsilon(s, b)$ is a spacelike surface.

Theorem 4.2. The geodesic curvature of b and s parameter curves of $\Upsilon = \Upsilon(s, b)$ are stated as $k_{g_b} = -\beta_{bs}$, $k_{g_s} = 0$, respectively.

Proof: It is known that

$$\frac{\partial^2 \Upsilon}{\partial b^2} = -\beta_{bs}\vec{T} + (div\vec{N} - \kappa)\vec{N}$$

by Equation 3. Then geodesic curvature of b parameter curve is obtained as

$$k_{g_b} = \left\langle -\beta_{bs}\vec{T} + (div\vec{N} - \kappa)\vec{N}, \vec{N} \times_L \vec{B} \right\rangle_L$$

$$= \left\langle -\beta_{bs}\vec{T} + \left(div\vec{N} - \kappa\right)\vec{N}, \quad \vec{T} \right\rangle_L = -\beta_{bs}.$$

Then, we also have

$$\frac{\partial^2 \Upsilon}{\partial s^2} = \kappa \vec{N}$$

by Equation 1. Therefore, the geodesic curvature of s parameter curve is obtained as

$$k_{g_s} = \left\langle \frac{\partial^2 \Upsilon}{\partial s^2}, \vec{N} \times_L \vec{T} \right\rangle_L = \langle \kappa \vec{N}, \vec{N} \times_L \vec{T} \rangle = \langle \kappa \vec{N}, \vec{B} \rangle = 0.$$

Corollary 4.2. All s parameter curves are the geodesics of $\Upsilon = \Upsilon(\mathrm{s}, \mathrm{b})$.

Theorem 4.3. The normal curvatures of b and s parameter curves of $\Upsilon = \Upsilon(\mathrm{s}, \mathrm{b})$ are stated as

$$k_{n_b} = (div\vec{N} - \kappa), \qquad k_{n_s} = \kappa$$

respectively.

Proof: By using the equation

$$\frac{\partial^2 \Upsilon}{\partial b^2} = -\beta_{bs}\vec{T} + (div\vec{N} - \kappa)\vec{N},$$

the normal curvature of b parameter curve is given by

$$k_{n_b} = \langle -\beta_{bs}\vec{T} + (div\vec{N} - \kappa)\vec{N}, \vec{N} \rangle_L = div\vec{N} - \kappa.$$

The normal curvature of s parameter curve is obtained as

$$k_{n_s} = \langle \kappa \vec{N}, -\vec{N} \rangle_L = \kappa.$$

Theorem 4.4. The geodesic torsion of b and s parameter curves of $\Upsilon = \Upsilon(\mathrm{s}, \mathrm{b})$ are stated as

$$\tau_{\mathrm{g_b}} = \tau, \qquad \tau_{\mathrm{g_s}} = \tau$$

respectively.

Proof: We find the geodesic torsion of b parameter curve as follows

$$\tau_{g_b} = -\langle -\tau\vec{T} - \left(div\vec{N} - \kappa\right)\vec{B}, \qquad \vec{T} \rangle_L = \tau$$

by Equation 3. The geodesic torsion of s parameter curve is found as follows

$$\tau_{g_s} = -\langle -\kappa\vec{T} - \tau\vec{B}, \vec{B} \rangle_L = \tau$$

by Equation 1.

Theorem 4.5. Gaussian and mean curvatures of $\Upsilon = \Upsilon(\mathrm{s}, \mathrm{b})$ are stated as

$$K = \kappa(div\vec{N} + \kappa), \quad H = \frac{div\vec{N} + 2\kappa}{2}$$

respectively.

Proof: The first fundamental form of $\Upsilon = \Upsilon(s, b)$ is given by

$$I = \langle \vec{T}ds + \vec{B}db, \vec{T}ds + \vec{B}db \rangle_L = ds^2 + db^2.$$

The second fundamental form of $\Upsilon = \Upsilon(s, b)$ is given by

$$II = \left\langle \vec{T}ds + \vec{B}db, (\kappa\vec{T} + \tau\vec{B})ds + (\tau\vec{T} + (div\vec{N} + \kappa\vec{B})db \right\rangle_L$$
$$= -\kappa ds^2 - (div\vec{N} - \kappa)db^2.$$

Gaussian and mean curvatures are obtained as follows:

$$K = \frac{-\kappa(-div\vec{N} - \kappa) - \tau^2}{1} = \kappa(div\vec{N} - \kappa) - \tau^2,$$

$$H = \frac{-(div\vec{N} - \kappa) - \kappa}{2} = -\frac{div\vec{N}}{2}$$

respectively.

Corollary 4.3. If the following equality is satisfied

$$\kappa(div\vec{N} - \kappa) = \tau^2,$$

then the surface $\Upsilon = \Upsilon(s, b)$ is developable.

Corollary 4.4. The surface $\Upsilon = \Upsilon(\mathrm{s}, \mathrm{b})$ is minimal surface if and only if $div\vec{N} = 0$.

5. APPLICATION

In this section, the relations between spacelike surfaces with vanishing abnormality of normal direction and NLS, Heisenberg spin equation are investigated. Consider one-parameter family of surfaces $\Upsilon = \Upsilon(s, b)$ is given. The equality

$$\frac{\partial\Upsilon}{\partial s} \times_L \frac{\partial^2\Upsilon}{\partial s^2} = \frac{\partial\Upsilon}{\partial b}$$

is satisfied if and only if $\kappa = 1$. This means that the surface $\Upsilon = \Upsilon(s, b)$ is a NLS surface if and only if $\kappa = 1$. As an example, let

$$\Upsilon(s, b) = (\cosh s, \sinh s, b)$$

be given. We obtain

$$\vec{T}(s, b) = (\sinh s, \cosh s, 0),$$

$$\vec{N}(s, b) = (\cosh s, \sinh s, 0),$$

$$\vec{B}(s, b) = (0,0,1)$$

and $\kappa = 1, \tau = 0$. Since $\kappa = 1$, surface of $\Upsilon(s, b)$ satisfies NLS equation. Moreover, one-parameter family of surfaces $\Upsilon = \Upsilon(s, b)$ is given with the equality $\beta_{bs} = 0$. Then the Heisenberg spin equation

$$\frac{\partial \vec{T}}{\partial b} = \vec{T} \times_L \frac{\partial^2 \vec{T}}{\partial s^2}$$

is satisfied for the unit tangent vector $\vec{T}$ if and only if $\kappa = -1$.

CONCLUSION

In this research, we investigate spacelike curve flow with timelike normal vector field by means of anholomonic coordinates in Minkowski space. We describe Frenet Serret frame $\{\vec{T}, \vec{N}, \vec{B}\}$ by means of anholonomic coordinates. Then, Frenet Serret formulas of a given spacelike curve are described which include eight parameters related to three partial differential equations. It is proved that the curl of tangent vector field has no component in the direction of principal normal vector field. This means that a surface with both $\vec{T}$ and $\vec{B}$ directions exists. Moreover, important results are provided together with the intrinsic geometric features of this congruence.

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