ORIGINAL PAPER

ASYMPTOTIC NORMALITY OF TRIMMED L-MOMENTS ESTIMATOR FOR ARCHIMEDEAN COPULAS

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Abstract. *In order to present a new estimation approach for multi-parameter distributions without a mean or for heavy tailed distributions, in which the L-moments method proposed by Gumbel, (1960), is invalid due to the absence of theoretical L-moments, Trimmed L-moments were first introduced by Elamir and Seheult (2003). In this paper, a new estimation method based on multi-parameter copulas' Trimmed L-moments is proposed with a simulation study. The consistency and the asymptotic normality of the new estimator also established.*
Keywords: *Archimedien copula; Trimmed L-moments; asymptotic normality.*

1. INTRODUCTION

Let $X_1, \ldots, X_d$ be a d-dimensional vector with joint distribution function $H(x_1, \ldots, x_d)$ and margins $F_j(x_j), j = 1, \ldots, d$. From Sklar's theorem (1959) we can joint H and F_j by a function C called copula, which is defined from $[0, 1]^d$ to $[0, 1]$ by:

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

The copula is a joint distribution function with uniform margins $U = (U_1, \ldots, U_d)$ with $U_j = F_j(x_j)$:

$$C(u) = F_1^{-1}\left((u_1), \ldots, F_d^{-1}(u_d)\right), u \in [0, 1]^d$$

where $F_j^{-1}(s) = inf\{x: F_j(x) \geq s\}$, is the generalized inverse function pertaining to F_j.

This describes and models the dependence structure of a multivariate data set. It characterizes many properties as the symmetry and the invariance transform. The importance of these two properties appears in measuring of association such as Kendall's tau and Spearman's rho defined in terms of the copula C, in the bivariate case by copula C, in the bivariate case by

$$\tau = 4 \int_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1 \tag{1}$$

and $\rho = 12 \int_{[0,1]^2} u_1 \, u_2 dC(u_1, u_2) - 3$

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Many parametric copula families have been introduced and applied in different fields such as the insurance, medical science, hydrology and survival analysis see [11, 8, 15]. Among these families, we cite the archimedean copula class which is introductly by [22], they have found many successful applications like the Actuarial and survey actuarial applications, in finance [6, 24, 7]. This class of copulas has nice properties, as: the ease with which it can be constructed; the great variety of families of copulas which it contains [23]. An archimedean copula is defined by

$$C(u) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + + \varphi(u_d)), \tag{2}$$

where φ is a positive continuous and strictly decreasing function on $[0, 1]$ called the generator with pseudo inverse φ^{-1}. Many families of archimedean copulas are cited in [23], [such as gumbel copula, defined by

$$C_\beta(u) = exp\left(-\left[(-\ln(u_1))^\beta + \cdots + (-\ln(u_d))^\beta\right]^{\frac{1}{\beta}}\right), \beta \geq 1$$

with generator $\varphi\,(s) = (-\,ln\,(s))^\beta$ and $\varphi^{-1}(s) = exp\,(-s)^{1/\beta}$.

Suppose that the parametric copula C belongs to a class $\mathcal{C}$ where $\mathcal{C} \coloneqq C_\theta{:}\, \theta \in \mathcal{O}$ and $\mathcal{O}$ is an open subset of $\mathbb{R}^r$ for $r \geq 1$. The problem of estimating θ under this assumption has already been the object of much work, beginning with classical methods: fully maximum likelihood (ML), Pseudo maximum likelihood (PML) and Inference function of margins (IFM) (see [12, 20]). (τ, ρ) inversion methods [24]. Minimum distance (MD) (see [27, 2]) which is based on: the empirical copula process, Kendall's dependence function which is proposed by Savu and Trede [14] and Rosenblatt's probability integral transform proposed by [25]. Many comparative studies between these methods were discussed in the literature such as in [21, 16].

Semi parametric estimation methods for multi-parametric copulas were also discussed by [4], [1] and [5] based on moments (CM) and copula L-moments (CLM). They noted that these methods are quick and does not use the density function and therefore no boundary problems arise. In a comparative sim- ulation study, they concluded that the PML and the CM based estimation perform better than the (τ, ρ)-inversion method and the main feature of CM and CLM methods is that they provide estimators with explicit forms.

The purpose of this paper is to estimate multi-parameter archimedean copulas using Trimmed L-moments. This method is analogous to L-moment method where the largest value is removed from the conceptual sample to study its influence on bias and root of the mean squared error (RMSE). Particularly, we derive TL-moments (0,1) for Gumbel copula and the results are presented in section 4.

The remainder of this article is structured as follows. Section 2 presents a brief introduction of L-moments, Trimmed L-moment and population Trimmed L- moments in terms of copula as well as the parameter estimation procedures. Section 3 discusses the consistency and asymptotic normality. An illustrative example with simulation procedure is outlined in section 4. The proofs are postponed to the Appendix.

2. TRIMMED L-MOMENTS

In this section, we first define the L-moments, TL-moments and the relation between them. Then, we explain how we can write TL-moments in terms of copula by using copula properties and we present sample TL-moments. L-moments play an important role in describing the characteristics of a probability distribution because they measure the location, scale and shape and they are related to expected values of order statistics. Let $Y_{1:r}, \dots, Y_{r:r}$ be the order statistics of a random sample $Y_1, \dots, Y_r$ of size $r \geq 1$. [18] defined the *rth* L-moment λ_r as follows:

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(Y_{r-k:r})$$

where

$$E(Y_{r-k:r}) = \frac{r!}{(r-k-1)!k!} \int_{\mathbb{R}} y\big(F(y)\big)^{r-k-1} \big(1-F(y)\big)^k dF(y).$$

Or we can write L-moments in terms of shifted Legendre polynomials as:

$$\lambda_r = \int_{\mathbb{R}} y P_{r-1} F(y) dF(y)$$

where

$$P_r(u) = \sum_{k=0}^{r} p_{r,k} u^k, \text{ with } p_{r,k} = (-1)^{r+k}(r+k)!/[(k^2)!\,(r-k)!].$$

L-moments cannot be defined for distributions for which the means do not exist or distributions with heavy tailed such as the Cauchy distribution. In that case the estimation method based on L-moments is not valid. Trimmed L-moments is an alternative measure to L-moments proposed by [10] where they replace the expected value $E(Y_{r-k:r})$ by $E\big(Y_{r+t_1-k:r+t_1+t_2}\big)$. Thus TL-moments noted $\lambda_{\mathrm{r}}^{(t_1,t_2)}$ are given as follows:

$$\lambda_{\mathrm{r}}^{(t_1,t_2)} = \frac{1}{\mathrm{r}} \sum_{\mathrm{k}=0}^{\mathrm{r}-1} (-1)^{\mathrm{k}} \frac{(\mathrm{r}-1)!}{\mathrm{k}!\,(\mathrm{r}-\mathrm{k}-1)!} E\big(Y_{r+t_1-k:r+t_1+t_2}\big), \mathrm{r} = 1,2, \dots \tag{3}$$

where t_1 and t_2 are positive integers. The case $t_1 = t_2 = 0$ yields the original L-moments. A new expression of TL-moments was given using shifted Jacobi polynomials, so (3) may be written as:

$$\lambda_{\mathrm{r}}^{(t_1,t_2)} = \frac{1}{\mathrm{r}} \sum_{\mathrm{k}=0}^{\mathrm{r}-1} (-1)^{\mathrm{k}} \frac{(\mathrm{r}-1)!}{\mathrm{k}!\,(\mathrm{r}-\mathrm{k}-1)!} \int_{\mathbb{R}} y \mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)} F(y) dF(y), \tag{4}$$

where $\mathrm{P}_{\mathrm{r}}^{*(t_1,t_2)} = \sum_{\mathrm{k}=0}^{\mathrm{r}} (-1)^{\mathrm{r}-\mathrm{k}} \binom{\mathrm{r}+t_1}{\mathrm{k}} \binom{\mathrm{r}+t_2}{\mathrm{r}-\mathrm{k}} \mathrm{u}^{\mathrm{k}} (1-\mathrm{u})^{\mathrm{r}-\mathrm{k}}$.

Shifted Jacobi polynomials satisfy the recurrence properties (see [19]) which implies that there is a relation between different degrees of trimming and between L-moments and TL-moments as:

$$(2\mathrm{r} + t_1 + t_2 - 1)\lambda_{\mathrm{r}}^{(t_1,t_2)} = (\mathrm{r} + t_1 + t_2 - 1)\lambda_{\mathrm{r}}^{(t_1,t_2-1)} - \frac{1}{\mathrm{r}}(\mathrm{r}+1)(\mathrm{r}+t_1)\lambda_{\mathrm{r}+1}^{(t_1,t_2-1)}$$

and for $t_1 = 0$, case $t_2 = 1$, we obtain:

$$\lambda_{\mathrm{r}}^{(0,1)} = \frac{\mathrm{r}+1}{2\mathrm{r}}(\lambda_r - \lambda_{r+1}), \tag{5}$$

The relation (5) is valid when all TL-moments exist. Many results concerning sample TL-moments, the existence and the unicity are presented by ([10], [19]).

2.1. COPULA TL-MOMENTS

In the following, the population TL-moments is given in details. By (3) can be written as:

$$\lambda_{\mathrm{r}}^{(t_1,t_2)} = E\left(\mathrm{Y}\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\big(F(y)\big)\right)$$
$$= \int_{\mathbb{R}} y\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\, F(y)dF(y),$$

where $\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}(\mathrm{u}) = \sum_{\mathrm{k}=0}^{\mathrm{r}-1}(-1)^{\mathrm{k}}\frac{(\mathrm{r}-1)!(\mathrm{r}+t_1+t_2)!}{\mathrm{r}\mathrm{k}!(\mathrm{r}-\mathrm{k}-1)!(\mathrm{r}+t_1-\mathrm{k}-1)!(\mathrm{k}+t_2)!}\mathrm{u}^{\mathrm{r}+t_1-\mathrm{k}-1}\{1-\mathrm{u}\}^{\mathrm{k}+t_2}$

For $\mathrm{r} = 1{,}2$ and $(t_1 = 0, t_2 = 1)$ we get

$$\mathrm{P}_0^{*(0,1)}(\mathrm{u}) = 2(1-\mathrm{u}), \mathrm{P}_1^{*(0,1)}(\mathrm{u}) = \frac{3}{2}(4\mathrm{u} - 3\mathrm{u}^2 - 1)$$

by substitution, we put $y = C(u)$ which is considered as a random variable with distribution $K_C(s) = P(C(U) \le s), s \in [0,1]$, so the *rth* TL-moments of the rv $C(U)$ may be written as:

$$\lambda_{\mathrm{r}}^{(t_1,t_2)} = \int_{[0,1]^{\mathrm{d}}} C(u)\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\, K_C(C(u))dK_C(C(u)), \mathrm{r} = 1{,}2. \tag{6}$$

Suppose that the copula C belongs to a parametric family with unknown parameter θ, we put

$$C = C_\theta, K_C = K_\theta \textit{ and } \lambda_{\mathrm{r}}^{(t_1,t_2)}(\mathrm{C}) = \lambda_{\mathrm{r}}^{(t_1,t_2)}(\theta)$$

Let $C(u) = s$, then (6) is rewritten into

$$\lambda_{\mathrm{r}}^{(t_1,t_2)}(\theta) = \int_0^1 s\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\, K_\theta(s)dK_\theta(s), \mathrm{r} = 1{,}2, \ldots$$

According to theorem 4.3.4 in [23] the density function of rv $C(u)$ may be represented in terms of the derivatives of the generator $\varphi_\theta(s)$. We have for any $s \in [0,1], k_\theta(s) = s - \varphi_\theta(s)/\varphi'_\theta(s)$. So the corresponding density is

$$k'_\theta(s) = \varphi''_\theta(s)\varphi_\theta(s)/(\varphi'_\theta(s))^2$$

and (6) becomes

$$\lambda_{\mathrm{r}}^{(t_1,t_2)}(\theta) = \int_0^1 s\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\left(s - \frac{\varphi_\theta(s)}{\varphi'_\theta(s)}\right)\left(\frac{\varphi''_\theta(s)\varphi_\theta(s)}{(\varphi'_\theta(s))^2}\right)\mathrm{ds}, \mathrm{r} = 1{,}2, \ldots \tag{7}$$

The first TL-moments for $(t_1, t_2) = (0,1)$ are

$$\lambda_1^{(0,1)}(\theta) = 2\int_0^1 s\left(1 - s + \frac{\varphi_\theta(s)}{\varphi'_\theta(s)}\right)\left(\frac{\varphi''_\theta(s)\varphi_\theta(s)}{(\varphi''_\theta(s))^2}\right) \mathrm{ds} \tag{8}$$

$$\lambda_2^{(0,1)}(\theta) = \frac{3}{2}\int_0^1 \left(4s^2 - s - 3s^3 + 6s^2\frac{\varphi_\theta(s)}{\varphi'_\theta(s)} - 3s\frac{\varphi_\theta^2(s)}{\varphi_\theta^2{}'(s)} - 4s\frac{\varphi_\theta(s)}{\varphi'_\theta(s)}\right)\left(\frac{\varphi''_\theta(s)\varphi_\theta(s)}{(\varphi'_\theta(s))^2}\right) \mathrm{ds}$$

We can also compute copula TL-moments by using relation (5) where copula L-moments are presented in [1] and we get the same formulas of $\lambda_r^{(0,1)}$.

2.2. *SEMI-PARAMETRIC TL-MOMENTS* ESTIMATION

To give an expression of sample TL-moments, which will be denoted by $\lambda_{\mathrm{r}}^{(t_1,t_2)}$, we suppose that the copula C belongs to a parametric family C_θ, with $\theta = (\theta_1, \dots, \theta_r)$, and satisfies the concordance ordering condition of copulas (see, [23]). For every

$$\theta_1, \theta_2 \in \mathcal{O}: \theta_1 \neq \theta_2 \Rightarrow C_{\theta_1} (> or <) C_{\theta_2} \tag{9}$$

The identifiability condition:

$$\text{for every } \theta_1, \theta_2 \in \mathcal{O}: \theta_1 \neq \theta_2 \Rightarrow C_{\theta_1} \neq C_{\theta_2}$$

The identifiability condition is a natural and even a necessary condition, since if the parameter is not identifiable then consistent estimator cannot exist [28] and [1]. Now, we give the procedure of sample TL-moments. Let $X_1, \dots, X_n$ a random sample from $X = (X_1, \dots, X_n)$, with empirical marginal distribution functions:

$$F_{jn}(x_j) = n^{-1}\sum_{i=1}^{n} 1_{\{X_{ji} \le x_j\}}, j = 1, \dots, d$$

and joint empirical density function C_n which is defined by [9] as:

$$C_n(u) = n^{-1}\sum_{i=1}^{n}\prod\nolimits_{\mathrm{j}=1}^{\mathrm{d}} 1_{\{\hat{U}_{ji} \le u_j\}}, \quad \mathrm{u} \in [0,1]^{\mathrm{d}}$$

where $\hat{U}_{ji} = F_{jn}(X_{ji})$ and $\hat{U}_i = (\hat{U}_{1i}, \dots, \hat{U}_{di}), \mathrm{i} = 1, \dots, \mathrm{n}$. The empirical df of the random variable $C(U)$ is given by:

$$K_n(t) = n^{-1}\sum_{i=1}^{n} 1_{\{C_n(\hat{U}_i) \le \mathrm{t}\}}, \mathrm{t} \in [0,1]$$

by replacing K_C by K_n in (6), we get

$$\hat{\lambda}_{\mathrm{r}}^{(t_1,t_2)} = \lambda_{\mathrm{r}}(C_n) = \int_0^1 s\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)} K_n(s) dK_n(s), \mathrm{r} = 1,2 \tag{10}$$

that is the sample TL-moments are given as follows

$$\hat{\lambda}_{\mathrm{r}}^{(t_1,t_2)} = n^{-1}\sum_{i=1}^{n} C_n(\hat{U}_i)\mathrm{P}_{\mathrm{r}-1}^{*(t_1,t_2)}\, K_n\left(C_n(\hat{U}_i)\right), \mathrm{r} = 1,2$$

and for $(t_1, t_2) = (0,1)$, we obtain

$$\hat{\lambda}_{\mathrm{r}}^{(0,1)} = n^{-1}\sum_{i=1}^{n} C_n(\hat{U}_i)\mathrm{P}_{\mathrm{r}-1}^{*(0,1)}\, K_n\left(C_n(\hat{U}_i)\right).$$

It is clear that we can't find explicit formulas of $\hat{\lambda}_{\mathrm{r}}^{(0,1)}$ and we get the following system

$$\begin{cases} \lambda_1^{(0,1)}(\theta_1, \dots, \theta_l) = \hat{\lambda}_1^{(0,1)} \\ \lambda_2^{(0,1)}(\theta_1, \dots, \theta_l) = \hat{\lambda}_2^{(0,1)} \\ \quad\quad\quad\quad\cdot \\ \quad\quad\quad\quad\cdot \\ \quad\quad\quad\quad\cdot \\ \lambda_l^{(0,1)}(\theta_1, \dots, \theta_l) = \hat{\lambda}_l^{(0,1)} \end{cases}, \tag{11}$$

which is solved by numerical methods to estimate the copula parameters.

3. CONSISTENCY AND ASYMPTOTIC NORMALITY

To study the asymptotic normality of the TL-moments estimator noted $\hat{\theta}^{CT\,L}$, we put

$$\mathcal{K}_l(u,\theta) = C_\theta(u)\mathrm{P}_{\mathrm{l}-1}^{*(0,1)}(K_\theta(C_\theta)) - \lambda_{\mathrm{l}}^{(0,1)}, l = 1, \dots, r \tag{12}$$

and

$$\mathcal{K}(u,\theta) = \left(\mathcal{K}_1(u,\theta), \dots, \mathcal{K}_r(u,\theta)\right)$$

Let θ_0 be the true value of θ and assume that the following assumptions $[\mathcal{A}1] - [\mathcal{A}3]$ hold.

- $[\mathcal{A}1]$ $\theta_0 \in \mathcal{O} \epsilon \mathbb{R}^r$ is the unique zero of the mapping $\theta \to \int_{[0,1]^d} \mathcal{K}(u,\theta)\, dC_{\theta_0}(u)$ which is defined from $\mathcal{O}$ to $\mathbb{R}^r$.
- $[\mathcal{A}2]$ $\mathcal{K}(.,\theta)$ is differentiable with respect to θ such that the Jacobian matrix denoted by $\dot{\mathcal{K}}(u,\theta) = [\partial\mathcal{K}_r(u,\theta)/\partial\theta_{\mathrm{k}}]_{l\times l}$ and $\dot{\mathcal{K}}(u,\theta)$ is continous both in u and θ, and the Euclidean norm $\left|\dot{\mathcal{K}}(u,\theta)\right|$ is dominated by a dC_θ-integrable function.
- $[\mathcal{A}3]$ The $r \times r$ matrix $B_0 := \int_{[0,1]^{\mathrm{d}}} \dot{\mathcal{K}}(u,\theta)\, dC_{\theta_0}(u)$ is nonsingular.

Theorem 3.1. *Assume that the concordance ordering condition (9) and assumptions* $[\mathcal{A}1] - [\mathcal{A}3]$ hold. Then, there exists a solution $\hat{\theta}^{CT\,L}$ to the system (11) which converges in probability to θ_0. Moreover

$$\sqrt{n}\left(\hat{\theta}^{CT\,L} - \theta_0\right) \xrightarrow{\mathrm{D}} \mathrm{N}(0, B_0^{-1}\mathrm{D}_0(B_0^{-1})^{\mathrm{T}}), \text{as n} \to \infty$$

where

$$\mathrm{D}_0 \coloneqq \mathrm{var}\{\mathcal{K}(\vartheta,\theta_0)+\nu(\vartheta,\theta_0)\} \text{ and } \nu(\vartheta,\theta_0)=\left(\nu_1(\vartheta,\theta_0),\dots,\nu_r(\vartheta,\theta_0)\right)$$

with

$$\nu_l(\vartheta,\theta_0)=\sum_{\mathrm{j}=1}^{\mathrm{d}}\int_{[0,1]^d}\frac{\partial\left(C_\theta(u)\mathrm{P}_{\mathrm{l}-1}^{*(0,1)}(K_\theta(C_\theta))\right)}{\partial u_j}\left(1_{\{\vartheta_{\mathrm{j}}\le u_j\}}-u_j\right)\,dC_{\theta_0}(u), l=1,\dots r$$

where ϑ is a $(0,1)$ $-$*uniform rv.*

Remark 3.1. Following [27] in the case of PML estimator and Z-estimator, one may consistently estimate the asymptotic variance $\mathrm{B}_0^{-1}\mathrm{D}_0(\mathrm{B}_0^{-1})^{\mathrm{T}}$ by the sample variance of the sequence of rv's

$$\hat{B}_{\mathrm{i}}^{-1}\mathrm{D}_{\mathrm{i}}(\hat{B}_{\mathrm{i}}^{-1})^{\mathrm{T}}, i=1,\dots,n$$

where

$$\hat{B}_i \coloneqq \int_{[0,1]^{\mathrm{d}}}\dot{\mathcal{K}}\left(u,\hat{\theta}^{CT\,L}\right)\,dC_{\hat{\theta}^{CT\,L}}(u) \text{ and } \hat{\mathrm{D}}_{\mathrm{i}}=\dot{\mathcal{K}}\left(\hat{U}_i,\hat{\theta}^{CT\,L}\right)+\nu\left(\hat{U}_i,\hat{\theta}^{CT\,L}\right)$$

4. ILLUSTRATIVE EXAMPLE AND SIMULATION

Gumbel family of copulas was first discussed by [17]. Gumbel copula also belongs to extreme value copulas see [13]. To obtain a Gumbel copula with two parameters we apply a transformation on the generator of (2). The transformed copula is defined by

$$C_{\Gamma}(u)=\Gamma^{-1}\left(\mathrm{C}\left(\Gamma(u_1),\dots,\Gamma(u_d)\right)\right)$$

where $\Gamma\colon[0,1]\to[0,1]$, is continuous, concave and strictly increasing with $\Gamma(0)=0$ and $\Gamma\Gamma(1)=1$. Suppose that $\Gamma=\Gamma_\alpha$, $\Gamma_\alpha=\exp(1-\mathrm{s}^{-\alpha}), \beta>0$ with copula C_β we obtain the transformed copula

$$C_{\alpha,\beta}(\mathrm{u})=\left(\left(\sum_{\mathrm{j}=1}^{\mathrm{d}}\left(\mathrm{u_j}^{-\alpha}-1\right)^{\beta}\right)^{1/\beta}+1\right)^{-1/\alpha}\quad for\ u=(\mathrm{u}_1,\dots,\mathrm{u}_{\mathrm{d}})\in[0,1]^{\mathrm{d}} \tag{13}$$

with generator

$$\varphi_{\alpha,\beta}(\mathrm{t})=(\mathrm{s}^{-\alpha}-1)^{\beta} \tag{14}$$

by replacing (13) in (8), we get the first TL-moments as follows

$$\lambda_1^{(0,1)}=\mathrm{A}_1-\mathrm{A}_2 \text{ and } \lambda_2^{(0,1)}=\tfrac{3}{2}(\mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3+\mathrm{B}_4)$$

where

$$\mathrm{A}_1=\frac{\alpha\beta+2\beta-1}{2\beta(\alpha+2)}, \mathrm{A}_2=\frac{2\alpha^3\beta^2+13\alpha^2\beta^2-2\alpha^2\beta+27\alpha\beta^2-\alpha\beta-4\alpha+18\beta^2+3\beta-8}{6\beta^2(2\alpha^3+13\alpha^2+27\alpha18)}$$

and

$$\mathrm{B}_1 = \frac{\alpha^3\beta+9\alpha^2\beta+26\alpha\beta+\alpha^2-15\alpha+24\beta-22}{12\beta(\alpha^3+9\alpha^2+26\alpha+24)}, \mathrm{B}_2 = -\frac{3\alpha\beta + 6\beta - 3}{2\beta^2\,(\alpha^2 + 6\alpha + 8)}$$

$$\mathrm{B}_3 = \frac{8\alpha\beta + 12\beta - 4}{3\beta^2\,(2\alpha^2 + 9\alpha + 9)}, \mathrm{B}_4 = -\frac{9\alpha\beta + 12\beta - 3}{4\beta^3\,(3\alpha^3 + 22\alpha^2 + 48\alpha + 32)}.$$

In our simulation study we select many different sample sizes with $n = 30;\ 50;\ 100;\ 200$ and 500 to assess their influence on the bias and RMSE of the estimators, and we choose different values of parameters α and β, according the degree of dependence calculated by Kendall's tau (1), that is consider three cases: $\tau = 0{,}1;\ 0{,}5$ and $0{,}8$ corresponding to (weak, moderate and strong dependence). For each choice, we make $N = 1000$ repetitions and we compute the estimation bias and RMSE:

$$Bias = \frac{1}{N}\sum_{\mathrm{i}=1}^{\mathrm{N}}\left(\hat{\theta}_{\mathrm{i}} - \theta\right), \mathrm{RMSE} = \left(\frac{1}{N}\sum_{\mathrm{i}=1}^{\mathrm{N}}\left(\hat{\theta}_{\mathrm{i}} - \theta\right)^2\right)^{1/2}$$

We can summarize the procedure of simulation as follows:

(1) Determine the value of the parameters $\alpha, \beta,$ sample sizes n and the number of simulated Samples N.

(2) Simulate a sample $(u_1, \dots, u_n)$ of size n from the copula $C_{\alpha,\beta}$ defined in (13)

(3) Compute the parameter estimates by solving the system (11).

(4) Compare the parameter estimates $\hat{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i$ and $\hat{\beta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i$ with the true parameters (presented in 1) by computing the biases and RMSE.

Table 1. True parameters of transformed Gumbel copula used for the simulation study.

τ	A	B
0.1	0.1	1.059
0.5	0.5	1.6
0.8	0.9	3.45

Table 2. Bias and RMSE of the TLM estimator of two- parameter transformed Gumbel copula for weak dependence.

$\tau = 0.1$				
n $\alpha = 0.1$ $\beta = 1.059$				
	Bias	RMSE	Bias	RMSE
30	0.331	0.153	0.322	0.146
50	0.06	0.05	0.181	0.101
100	0.04	0.04	0.163	0.09
200	0.03	0.03	0.145	0.08
500	0.009	0.02	0.09	0.07

Table 3. Bias and RMSE of the TLM estimator of two- parameter transformed Gumbel copula for moderate dependence.

$\tau = 0.5$				
n $\alpha = 0.5$ $\beta = 1.6$				
	Bias	RMSE	Bias	RMSE
30	0.787	0.359	0.493	0.326
50	0.149	0.310	-0.202	0.320
100	0.131	0.229	-0.110	0.285
200	0.116	0.186	-0.06	0.164
500	0.100	0.176	-0.04	0.131

Table 4. Bias and RMSE of the TLM estimator of two- parameter transformed Gumbel copula for strong dependence.

$\tau = 0.8$				
n $\alpha = 0.9$ $\beta = 3.45$				
	Bias	RMSE	Bias	RMSE
30	0.570	0.600	-1.005	0.648
50	0.418	0.543	0.496	0.568
100	0.400	0.500	-0.426	0.549
200	-0.310	0.448	-0.09	0.179
500	0.248	0.369	0.08	0.176

Table 5. Bias and RMSE of the CTL and CLM estimators of two-parameter transformed Gumbel copula for weak, moderate and strong dependence.

$\alpha = 0.1$ $\beta = 1.059$					
N	estimators	Bias	RMSE	Bias	RMSE
30	CTL	**0.09**	**0.110**	**0.205**	**0.106**
	CLM	0.224	0.307	-0.217	0.413
100	CTL	**0.06**	**0.07**	**0.06**	**0.104**
	CLM	0.104	0.142	-0.07	0.296
500	CTL	**0.04**	**0.06**	**0.07**	**0.110**
	CLM	0.190	0.245	-0.08	0.321
weak dependence					
30	CTL	**-0.339**	**0.276**	**-0.167**	**0.108**
	CLM	-0.428	0.482	0.222	0.290
100	CTL	**-0.120**	**0.269**	**0.04**	**0.201**
	CLM	-0.320	0.369	-0.05	0.204
500	CTL	**-0.106**	**0.236**	**0.003**	**0.126**
	CLM	-0.287	0.338	0.09	0.299
moderate dependence					
30	CTL	**-0.264**	**0.339**	**0.07**	**0.292**
	CLM	-0.696	0.795	0.100	0.327
100	CTL	**-0.201**	**0.103**	**0.05**	**0.131**
	CLM	-0.319	0.367	0.111	0.297
500	CTL	**-0.182**	**0.103**	**0.04**	**0.110**
	CLM	-0.198	0.258	0.116	0.235
strong dependence					

5. CONCLUSION

The results of the simulation study are presented in Tables 2-4. Each table corresponds to the bias and RMSE for five sample sizes and three true paramater values. We see that the bias and RMSE decrease when the sample size increase but the results in weak case appear to be better than the moderate and strong dependence. On the other hand, the time in seconds needed for computing the parameter estimates, as intended, increases with increasing sample size. The results obtained in Table 5 present a comparative study between our method and Copula L-moments method, we choose three samples for n = 30, 100 and 500 and with the same choice of parameter values we ob-serve that the TL-moments give better estimate than the L-moments for small or large samples where the biais and RMSE are lower.

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APPENDIX

Proof: For the proof of Theorem 1 we use the results of [28], [3] and [27]. To establish the consistancy and the asymptotic normality of our estimator we apply theorem 1 of [27], also the existence of a sequence of consistent roots $\hat{\theta}^{CT\,L}$ may be verified by using similar arguments as the proof of Theorem 1 in [27] and [1].

By considering CTL estimator is a RAZ-estimator according [28]. First we set

$$\Phi(\theta) = \int_{[0,1]^d} \mathcal{K}(u,\theta)\ dC_{\theta_0}(u)$$

For $\hat{U}_i = (\hat{U}_{1i}, \dots, \hat{U}_{di}), i = 1, \dots, n.$ Where $\hat{U}_{ji} = F_{jn}(X_{ji})$ with $(X_{j1}, \dots, X_{jn})$ is a random sample from the rv X_j we get the empirical version of Φ as follows

$$\Phi_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \mathcal{K}(\hat{U}_i, \theta)$$

Under the assumption $[\mathcal{A}2]$ the derivatives of Φ and Φ_n exist and are respectively given by

$$\dot{\Phi}(\theta) = \int_{[0,1]^d} \dot{\mathcal{K}}(u,\theta)\ dC_{\theta_0}(u) \text{ and } \dot{\Phi}_n(\theta) = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} \dot{\mathcal{K}}(\hat{U}_i, \theta)$$

To verify the following convergence

$$sup\{|\dot{\Phi}_n(\theta) - \dot{\Phi}(\theta)| : |\theta - \theta_0| < \varepsilon_n\} \xrightarrow{P} 0, as\ n \to \infty \tag{15}$$

for any real sequence ε_n we set

$$sup\{|\dot{\mathcal{K}}(\hat{U}_i, \theta) - \dot{\mathcal{K}}(\hat{U}_i, \theta_0)| : |\theta - \theta_0| < \varepsilon_n\} = o_P(1), i = 1, \dots, n$$

because $\dot{\mathcal{K}}(\hat{U}_i, \theta)$ is continuous in θ and according to the following inequality

$$|\dot{\Phi}_n(\theta) - \dot{\Phi}(\theta)| \le \frac{1}{n}\sum_{i=1}^{n} |\dot{\mathcal{K}}(\hat{U}_i, \theta) - \dot{\mathcal{K}}(\hat{U}_i, \theta_0)|$$

we conclude that

$$sup\{|\dot{\Phi}_n(\theta) - \dot{\Phi}(\theta)| : |\theta - \theta_0| < \varepsilon_n\} \xrightarrow{P} 0, as\ n \to \infty. \tag{16}$$

By the law of large numbers, we have

$$\frac{1}{n}\sum_{i=1}^{n} |\dot{\mathcal{K}}(\hat{U}_i, \theta_0)| \xrightarrow{P} \dot{\Phi}(\theta_0)\ as\ n \to \infty$$

In addition, the function $\dot{\mathcal{K}}(\widehat{U}_i, \theta)$ is continuous in u and by Glivenko-Cantelli theorem, we get

$$\frac{1}{n}\sum_{i=1}^{n}\left|\dot{\mathcal{K}}(\widehat{U}_i, \theta) - \dot{\mathcal{K}}(\widehat{U}_i, \theta_0)\right| \overset{P}{\rightarrow} 0$$

It follows that $\left|\dot{\Phi}_n(\theta) - \dot{\Phi}(\theta)\right| \overset{P}{\rightarrow} 0$, which together with 15, implies 16 . Conditions (MG0) and (MG3) in Theorem A.10.2 in [3] are trivially satisfied by our assumptions $[\mathcal{A}1] - [\mathcal{A}3]$.

In view of the general theorem for Z-estimators (see [28, Theorem 3.3.1]), it remains to prove that $\sqrt{n}\left|\dot{\Phi}_n(\theta) - \dot{\Phi}(\theta)\right|(\theta_0)$ converges in law to the appropriate limit. But this follows from Proposition 3 in [27], which achieves the proof of Theorem 1.