ORIGINAL PAPER

ON FIXED POINTS OF INTERPOLATIVE CONVEX AND ALMOST CONVEX-TYPE CONTRACTIVE MAPPINGS

MERVE ÖZKAN[1], SAFEER HUSSAIN KHAN[2], MURAT ÖZDEMİR[1]

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Abstract. *In this work, we introduce interpolative (α,q)-convex contractions and almost (α,q)-convex contractive mappings via simulation function in metric spaces and prove some existence results for fixed point of such mappings. Consequently, our results generalize numerous results in the literature.*

Keywords: *fixed point; interpolative (α,q)-convex contraction; almost (α,q)-convex contractive mapping; simulation function*

1. INTRODUCTION

Banach Contraction Principle [1] has been studied in different ways. For instance, Berinde [2] suggested an attractive concept named almost contraction in 2004. Recently, Khojasteh et al. [3] initiated the concept of simulation functions in 2015. Istratescu [4] initiated a novel class, namely convex contraction mappings in 1982. After that, convex contractions have been studied by many authors to get extensions for various forms of contractions. Khan et al. [5] defined generalized convex contractions of type-2. Miandaragh et al. [6] studied approximate fixed points of generalized convex contractions.

In 2010, Goebel and Sims [7] gave the notions of α-nonexpansive mapping and (α,q)-nonexpansive mapping. Later, Khan et al. [8] introduced two definitions in 2018, respectively (α,q)-contraction and (α,q)-convex contraction.

Firstly, Karapınar [9] introduced the definition of interpolative Kannan type contraction. Karapınar et al. [10] amended this definition. Later, Karapınar et al. [11] gave the definition of interpolative Hardy-Rogers type contraction and furthermore, Karapinar et al. [10] gave interpolative Reich-Rus-Ciric type contractions. Also, extended interpolative single and multivalued $F-$contractions were given by Yıldırım [12]. Many authors proposed novel concepts by combining these publications and references therein.

In this work, we introduce the concept of generalized almost (α,q)-convex contractive mappings via simulation function and interpolative (α,q)-convex contractive mappings in metric spaces using the recent contractions of Karapınar [9], Berinde [2], and Istratescu [4]. We establish some fixed results for such contractions. Our results are extensions of the latest fixed point results of Karapınar et al. [9-11], Istratescu [4], Khojasteh et al. [3], Khan et al. [8], Goebel and Pineda [13], and Berinde [14], and other various results in the literature. The new concepts lead to further investigations and applications. After analyzing the existence of a fixed point for this novel type contraction, we express some consequences. As a result, our

[1] Ataturk University, Department of Mathematics, 25050 Erzurum, Turkey. E-mail: merve.ozkan@atauni.edu.tr; mozdemir@atauni.edu.tr.
[2] Government College University, Abdus Salam School of Mathematical Sciences, 54000 Lahore, Pakistan. E-mail: safeer@sms.edu.pk.

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results extend and unify various well-known results in the existing literature.

2. MATERIALS AND METHODS

Throughout this presentation, $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ denote the set of natural numbers, positive real numbers, and real numbers, respectively. We start this section by recalling some definitions related to our work.

Let (E,ρ) be a metric space and $h: E \to E$ be a self-mapping. Given $\epsilon > 0$, $t_0 \in E$ is said to be an ϵ-fixed point of h, whenever $\rho(t_0, ht_0) < \epsilon$. Every fixed point is ϵ-fixed point but the converse need not to be true.

$$F_\epsilon h = \{t \in E: \rho(t, ht) < \epsilon\} \text{ and } Fixh = \{t \in E: \rho(t, ht) = 0\}.$$

Definition 2.1. [15] **(**If for all $\epsilon > 0$, there exists an ϵ-fixed point of h i.e., $inf_{t\in E}\rho(t, ht) = 0$ or for all ϵ, $F_\epsilon h \neq \emptyset$, then we say that h has the approximate fixed point property (AFPP).

For details about this topic, see [15-18].

Definition 2.2. Let (E,ρ) be a metric space, $h: E \to E$ a self mapping and $\{t_n\}$ be a sequence in E.

1. [19] h is called an asymptotically regular at a point $t \in E$ if $\lim_{n\to\infty}\rho(h^n t, h^{n+1} t) = 0$.
2. [20] $\{t_n\}$ is called an asymptotically h-regular, if $\lim_{n\to\infty}\rho(t_n, ht_n) = 0$.
3. [5] $\{t_n\}$ is called an asymptotically h^2-regular, if $\lim_{n\to\infty}\rho(t_n, h^2 t_n) = 0$.
4. [5] $\{t_n\}$ is called an asymptotically (h, h^2)-regular, if $\lim_{n\to\infty}\rho(t_n, ht_n) = 0$ and $\lim_{n\to\infty}\rho(t_n, h^2 t_n) = 0$.

Lemma 2.1. [21] If h is an asymptotically regular self mapping on E, that is $\rho(h^n t, h^{n+1} t) = 0$ for all $t \in E$, then h has the AFPP.

Lemma 2.2. [5] If a sequence $\{t_n\}$ in E is asymptotically (h, h^2)-regular in E, then $lim_{n\to\infty}\rho(ht_n, h^2 t_n) = 0$.

The definition of almost contraction is given below.

Definition 2.3. [13] Let (E,ρ) be a metric space. A mapping $h: E \to$ E is called an almost contraction if there exists a constant $\delta \in (0,1)$ and $L \geq 0$ such that

$$\rho(ht, hs) \leq \delta\rho(t, s) + L\rho(s, ht) \tag{1}$$

for all $t, s \in E$.

Istratescu [4] gave the following definitions.

Definition 2.4. [4] Let (E,ρ) be a metric space and $h: E \to$ E be a self-mapping. For all $t, s \in E$,

1. h is called a convex contraction of order 2 if there exist $d_1, d_2 \in (0,1)$ such that $d_1 + d_2 < 1$ and

$$\rho(h^2t, h^2s) \leq d_1\rho(ht, hs) + d_2\rho(t, s). \tag{2}$$

2. h is called two-sided convex contraction mappings if there exist $d_j \in (0,1)$ for all $j = 1,2, \ldots, 4$ such that $\sum_{j=1}^{j=4} k_j < 1$ and

$$\rho(h^2t, h^2s) \leq d_1\rho(t, ht) + d_2\rho(ht, h^2t) + d_3\rho(s, hs) + d_4\rho(hs, h^2s). \tag{3}$$

Khojasteh [3] initiated the concept of simulation function.

Definition 2.5. [3] A mapping $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if the following two conditions hold:

1. $\zeta(t, s) < s - t$ for all $t, s > 0$
2. if $\{t_n\}$, $\{s_n\}$ are sequences in (0,1) such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the family of all simulation functions by Γ. Goebel and Pineda [13] gave following definitions.

Definition 2.6. [13] Let (E, ρ) be a metric space and $h: E \to E$ be a mapping. If for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{j=1}^{n} \alpha_j = 1$, $\alpha_j \geq 0$ for all j and $\alpha_1, \alpha_n > 0$ the inequality

$$\sum_{j=1}^{n} \alpha_j \rho(h^j t, h^j s) \leq \rho(t, s). \tag{4}$$

is satisfied, then h is called an α-nonexpansive mapping.

Definition 2.7. [13] Let (E, ρ) be a metric space and $h: E \to E$ be a mapping. If for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{j=1}^{n} \alpha_j = 1$, $\alpha_j \geq 0$ for all j, and $\alpha_1, \alpha_n > 0$, and $q \in 1, \infty)$, the inequality

$$\sum_{j=1}^{n} \alpha_j \rho^q(h^j t, h^j s) \leq \rho^q(t, s) \tag{5}$$

is satisfied, then h is called an (α, q)-nonexpansive mapping.

For $n = 2$ in (5), we have

$$\alpha_1 \rho^q(ht, hs) + \alpha_2 \rho^q(h^2t, h^2s) \leq \rho^q(t, s) \tag{6}$$

and we say that h is an $((\alpha_1, \alpha_2), q)$-nonexpansive mapping.

Khan et al. [8] introduced following two definitions in 2018.

Definition 2.8. [8] Let (E, ρ) be a metric space. Then $h: E \to E$ is said to be (α, q)-contraction if for some $\alpha \in (0,1)$ and $q \geq 1$, there exists $k \in (0,1)$ satisfying the following inequality

$$\alpha\rho^q(ht, hs) + (1 - \alpha)\rho^q(h^2t, h^2s) \leq k\rho^q(t, s) \tag{7}$$

for all $t, s \in E$.

If $\alpha = \alpha_1$ and $1 - \alpha = \alpha_2$, $k = 1$ in (7), then h is an $((\alpha_1, \alpha_2), q)$-contractive mapping.

Definition 2.9. [8] Let (E, ρ) be a metric space. Then $h: E \to E$ is said to be (α, q)-convex contraction, if for some $\alpha \in (0,1)$ and $q \geq 1$, there exists $k_i \in (0,1)$ for all $i = 1,2, \ldots, 5$ such that $\sum_{i=1}^{i=5} k_i < 1$ satisfying the following inequality

$$\begin{aligned} &\alpha\rho^q(ht, hs) + (1-\alpha)\rho^q(h^2t, h^2s) \\ &\leq k_1\rho^q(t, s) + k_2\rho^q(t, ht) + k_3\rho^q(ht, h^2t) \\ &+k_4\rho^q(s, hs) + k_5\rho^q(hs, h^2s) \end{aligned} \tag{8}$$

for all $t, s \in E$.

If $\alpha = \alpha_1$ and $1 - \alpha = \alpha_2$, $k = 1$ in (8) inequality, then h is an $((\alpha_1, \alpha_2), q)$-convex contractive mapping. Karapınar [9] gave the following definition of interpolative Kannan type contraction.

Definition 2.10. [9] Let (E, ρ) be a metric space. Then the mapping $h: E \to E$ is said to be an interpolative Kannan type contraction if there exists $\lambda \in (0,1)$ and $b \in (0,1)$ such that

$$\rho(ht, hs) \leq \lambda[\rho(t, ht)]^b[\rho(s, hs)]^{1-b} \tag{9}$$

for every $t, s \in E$ with $t \neq ht$.

Karapınar [10] made a correction in [9] as follows.

Theorem 2.1. [10] Let (E, ρ) be a complete metric space and $h: E \to E$ be a self mapping. If there exist $\lambda \in (0,1)$ and $b \in (0,1)$ such that

$$\rho(ht, hs) \leq \lambda[\rho(t, ht)]^b[\rho(s, hs)]^{1-b} \tag{10}$$

for every $t, s \in E - Fixh$.

Later, Karapınar [9] gave the definition of interpolative Hardy-Rogers type contraction.

Theorem 2.2. [9] Let (E, ρ) be a complete metric space. Then a self mapping $h: E \to$ E is an interpolative Hardy-Rogers type contraction, if there exist $\lambda \in (0,1)$ and $b_i \in (0,1)$, $i = 1,2,3$ with $b_1 + b_2 + b_3 < 1$ such that

$$\begin{aligned} &\rho(ht, hs) \\ &\leq \lambda[\rho(t, s)]^{b_1}[\rho(t, ht)]^{b_2}[\rho(s, hs)]^{b_3}\left[\frac{\rho(t, ht) + \rho(s, hs)}{2}\right]^{1-b_1-b_2-b_3} \end{aligned} \tag{11}$$

for every $t, s \in E - Fixh$.

3. MAIN RESULTS

3.1. INTERPOLATIVE (α, q)-CONVEX CONTRACTIONS

In this section, we introduce the concept of interpolative (α, q)-convex contractions in metric spaces. We establish a fixed results for such contractions. We denote by Z the set of all functions $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following condition:

$$\zeta(t,s) \leq s - t \text{ for all } t,s > 0.$$

Definition 3.1.1. Let (E,ρ) be a metric space. Then a self mapping $h: E \to E$ is an interpolative (α, q)-convex contraction, if for some $\alpha \in (0,1)$ and $q \geq 1$ there exists $\lambda \in (0,1)$ and $b_i \in (0,1)$, $i = 1,2,3,4$ with $b_1 + b_2 + b_3 + b_4 < 1$ such that the following inequality holds

$$\begin{aligned} &\alpha\rho^q(ht,hs) + (1-\alpha)\rho^q(h^2t,h^2s) \\ &\leq \lambda[\rho^q(t,s)]^{b_1}[\rho^q(t,ht)]^{b_2}[\rho^q(s,hs)]^{b_3}[\rho^q(ht,h^2t)]^{b_4} \\ &[\rho^q(hs,h^2s)]^{1-b_1-b_2-b_3-b_4}. \end{aligned} \tag{12}$$

Our first result is the following theorem.

Theorem 3.1.1. Let (E,ρ) be a metric space and $h: E \to E$ be an interpolative (α, q)-convex contraction such that $\lambda + \alpha < 1$. Then h has the AFPP. Moreover, if (E,ρ) is a complete metric space, then h has a fixed point.

Proof: Let $t_0 \in E$ and the sequence $\{t_n\}$ be defined as $t_{n+1} = ht_n = h^{n+1}t_0$ for all $n \geq 0$. If $t_{n_0} = t_{n_0+1}$ for $n_0 \in \mathbb{N}$, then the proof is finished. Let $t_{n_0} \neq t_{n_0+1}$ for $n_0 \in \mathbb{N}$ and $\eta = \max\{\rho(t_0,ht_0), \rho(ht_0,h^2t_0)\}$. If we take $t = t_0$ and $s = ht_0$ in inequality (12), then we have

$$\begin{aligned} (1-\alpha)\rho^q(h^2t_0,h^3t_0) &\leq \alpha\rho^q(ht_0,h^2t_0) + (1-\alpha)\rho^q(h^2t_0,h^3t_0) \\ &\leq \lambda[\rho^q(t_0,ht_0)]^{b_1}[\rho^q(t_0,ht_0)]^{b_2}[\rho^q(ht_0,h^2t_0)]^{b_3}[\rho^q(ht_0,h^2t_0)]^{b_4} \\ &\quad [\rho^q(h^2t_0,h^3t_0)]^{1-b_1-b_2-b_3-b_4} \\ &\leq \lambda[\rho^q(t_0,ht_0)]^{b_1+b_2}[\rho^q(ht_0,h^2t_0)]^{b_3+b_4} \\ &\quad [\rho^q(h^2t_0,h^3t_0)]^{1-b_1-b_2-b_3-b_4}, \end{aligned}$$

and

$$\begin{aligned} (1-\alpha)[\rho^q(h^2t_0,h^3t_0)]^{b_1+b_2+b_3+b_4} &\leq \lambda\eta^{q(b_1+b_2+b_3+b_4)} \\ \rho^q(h^2t_0,h^3t_0) &\leq \frac{\lambda}{1-\alpha}\eta^q \\ \rho(h^2t_0,h^3t_0) &\leq k\eta \end{aligned}$$

where $k^q = \frac{\lambda}{1-\alpha}$ and since $\lambda + \alpha < 1$, we have $k^q < 1$. If we take $t = ht_0$ and $s = h^2t_0$ in (12), then we have

$$\begin{aligned} (1-\alpha)\rho^q(h^3t_0,h^4t_0) &\leq \alpha\rho^q(h^2t_0,h^3t_0) + (1-\alpha)\rho^q(h^3t_0,h^4t_0) \\ \rho^q(h^3t_0,h^4t_0) &\leq \frac{\lambda}{1-\alpha}\eta^q, \end{aligned}$$

and

$$\rho(h^3t_0,h^4t_0) \leq k\eta.$$

Also, if we continue this process, we get

$$\rho(h^4 t_0, h^5 t_0) \leq k^2 \eta \text{ and } \rho(h^5 t_0, h^6 t_0) \leq k^2 \eta$$

and

$$\rho(h^m t_0, h^{m+1} t_0) \leq k^l \eta$$

whenever $m = 2l + 1$ or $m = 2l$. Hence $\rho(h^m t_0, h^{m+1} t_0) \to 0$ as $m \to \infty$. Thus, we can say that h is asymptotically regular at t_0 and so, h has an approximate fixed point. For $m = 2l + 1$ with $j, l \geq 1$, we get

$$\begin{aligned} &\rho\left(h^m t_0, h^{m+j} t_0\right) \\ &\leq \rho(h^{2l+1} t_0, h^{2l+2} t_0) + \rho(h^{2l+2} t_0, h^{2l+3} t_0) + \rho(h^{2l+3} t_0, h^{2l+4} t_0) \\ &+ \cdots + \rho(h^{2l+j-1} t_0, h^{2l+j} t_0) + \rho(h^{2l+j} t_0, h^{2l+j+1} t_0) \\ &\leq k^l \eta + k^{l+1} \eta + k^{l+1} \eta + \cdots \\ &\leq 2k^l \left(\frac{1}{2} + k + k^2 + k^3 \cdots\right) \eta \\ &< 2k^l (1 + k + k^2 + k^3 \cdots) \eta \\ &\leq 2k^l \frac{1}{1-k} \eta. \end{aligned} \tag{13}$$

Again, for $m = 2l$ with $j, l \geq 1$, we get

$$\begin{aligned} &\rho\left(h^m t_0, h^{m+j} t_0\right) \\ &\leq \rho(h^{2l} t_0, h^{2l+1} t_0) + \rho(h^{2l+1} t_0, h^{2l+2} t_0) + \rho(h^{2l+2} t_0, h^{2l+3} t_0) \\ &+ \cdots + \rho(h^{m+j-2} t_0, h^{m+j-1} t_0) + \rho(h^{m+j-1} t_0, h^{m+j} t_0) \\ &\leq k^l \eta + k^l \eta + k^{l+1} \eta + k^{l+1} \eta \cdots \\ &\leq 2k^l (1 + k + k^2 + k^3 \cdots) \eta \\ &\leq 2k^l \frac{1}{1-k} \eta. \end{aligned} \tag{14}$$

Let $l \to \infty$ in the inequalities (13) and (14). Since $k < 1$, we get $\rho(h^m t_0, h^{m+j} t_0) \to 0$. Thus, $\{t_n\}$ is a Cauchy sequence in E. Since (E, ρ) is a complete metric space, therefore $t_n = h^n t_0 \to u \in E$ as $n \to \infty$. Thus, we obtain $hu = u$.

3.2. GENERALIZED ALMOST (α, q)-CONVEX CONTRACTIVE MAPPINGS VIA SIMULATION FUNCTION

In this section, we introduce concept of generalized almost (α, q)-convex contractive mappings via simulation function and establish some fixed results for such contractions.

Definition 3.2.1. Let (E, ρ) be a metric space and $h: E \to E$ be a mapping. If for some $\alpha \in (0,1)$ and $q \geq 1$ there exist $\delta \in (0,1)$ and $L \geq 0$ satisfiying the following inequality

$$\zeta(\alpha \rho^q (ht, hs) + (1-\alpha) \rho^q (h^2 t, h^2 s), \delta M_\iota(t, s) + L N_\iota(t, s) \geq 0 \tag{15}$$

where

$$M_I(t,s) = \max\{\rho^q(t,s), \rho^q(t,ht), \rho^q(s,hs), \rho^q(ht,h^2t), \rho^q(hs,h^2s)\}$$

and

$$N_I(t,s) = \min\{\rho^q(t,ht), \rho^q(s,hs), \rho^q(t,hs), \rho^q(s,ht), \rho^q(ht,h^2t), \rho^q(hs,h^2s)\}.$$

Then we say that h is a generalized almost (α,q)-convex contractive mapping via simulation function. For $q=1$ in (15) inequality, h is called generalized almost α-convex contractive mapping via simulation function.

Theorem 3.2.1. Let (E,ρ) be a metric space and $h: E \to E$ be a generalized almost (α,q)-convex contractive mapping via simulation function such that $\delta + \alpha < 1$. Then h has the AFPP. Furthermore, if (E,ρ) is a complete metric space, then h has a unique fixed point.

Proof: Let $t_0 \in E$ and the sequence $\{t_n\}$ be defined as $t_{n+1} = ht_n = h^{n+1}t_0$ for all $n \geq 0$. If $t_{n_0} = t_{n_0+1}$ for $n_0 \in \mathbb{N}$, then the proof is over. Let $t_{n_0} \neq t_{n_0+1}$ for $n_0 \in \mathbb{N}$ and $\eta = \max\{\rho(t_0,ht_0), \rho(ht_0,h^2t_0)\}$. If we take $t = t_0$ and $s = ht_0$ in (15), then we have

$$\zeta(\alpha\rho^q(ht_0,h^2t_0) + (1-\alpha)\rho^q(h^2t_0,h^3t_0), \delta M_\iota(t_0,ht_0) + LN_\iota(t_0,ht_0)) \geq 0 \tag{16}$$

where

$$\begin{aligned}(1-\alpha)\rho^q(h^2t_0,h^3t_0) &\leq \alpha\rho^q(ht_0,h^2t_0) + (1-\alpha)\rho^q(h^2t_0,h^3t_0)\\ &\leq \delta\max\left\{\begin{matrix}\rho^q(t_0,ht_0), \rho^q(t_0,ht_0), \rho^q(ht_0,h^2t_0),\\ \rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0)\end{matrix}\right\}\\ &\quad + L\min\left\{\begin{matrix}\rho^q(t_0,ht_0), \rho^q(t_0,h^2t_0), \rho^q(ht_0,ht_0),\\ \rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0)\end{matrix}\right\}\\ &\leq \delta\max\{\rho^q(t_0,ht_0), \rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0)\}\\ &\leq \delta\max\{\eta^q, \rho^q(h^2t_0,h^3t_0)\}.\end{aligned}$$

If $\max\{\eta^q, \rho^q(h^2t_0,h^3t_0)\} = \rho^q(h^2t_0,h^3t_0)$, then we have $(1-\alpha)\rho^q(h^2t_0,h^3t_0) \leq \delta\rho^q(h^2t_0,h^3t_0)$. Since $\delta + \alpha < 1$, we get a contradiction and thus, we get $\max\{\eta^q, \rho^q(h^2t_0,h^3t_0)\} = \eta^q$. Therefore, we have

$$\rho^q(h^2t_0,h^3t_0) \leq k^q\eta^q,$$

where $k^q = \frac{\delta}{1-\alpha}$ and since $\delta + \alpha < 1$, we have $k^q < 1$. We get $\rho(h^2t_0,h^3t_0) \leq k\eta$. If we take $t = ht_0$ and $s = h^2t_0$ in (15), then we have

$$\zeta(\alpha\rho^q(h^2t_0,h^3t_0) + (1-\alpha)\rho^q(h^3t_0,h^4t_0), \delta M_\iota(ht_0,h^2t_0) + LN_\iota(ht_0,h^2t_0)) \geq 0$$

where

$$\begin{aligned}(1-\alpha)\rho^q(h^3t_0,h^4t_0) &\leq \alpha\rho^q(h^2t_0,h^3t_0) + (1-\alpha)\rho^q(h^3t_0,h^4t_0)\\ &\leq \delta\max\left\{\begin{matrix}\rho^q(ht_0,h^2t_0), \rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0),\\ \rho^q(h^2t_0,h^3t_0), \rho^q(h^3t_0,h^4t_0)\end{matrix}\right\}\\ &\quad + L\min\left\{\begin{matrix}\rho^q(ht_0,h^2t_0), \rho^q(ht_0,h^3t_0), \rho^q(h^2t_0,h^2t_0),\\ \rho^q(h^2t_0,h^3t_0), \rho^q(h^3t_0,h^4t_0)\end{matrix}\right\}\\ &\leq \delta\max\{\rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0), \rho^q(h^3t_0,h^4t_0)\}.\end{aligned}$$

In case that $\max\{\rho^q(ht_0,h^2t_0), \rho^q(h^2t_0,h^3t_0), \rho^q(h^3t_0,h^4t_0)\} = \rho^q(h^3t_0,h^4t_0)$, we have $(1-\alpha)\rho^q(h^3t_0, h^4t_0) \le \delta\rho^q(h^3t_0, h^4t_0)$. Since $\delta+\alpha<1$, we get a contradiction. Hence we get

$$\rho^q(h^3t_0, h^4t_0) \le \frac{\delta}{1-\alpha}\max\{\rho^q(ht_0, h^2t_0), \rho^q(h^2t_0, h^3t_0)\}.$$

Therefore, we have

$$\rho(h^3t_0, h^4t_0) \le k\eta.$$

Continuing the process, we get

$$\rho(h^4t_0, h^5t_0) \le k^2\eta, \qquad \rho(h^5t_0, h^6t_0) \le k^2\eta, \dots, \rho(h^mt_0, h^{m+1}t_0) \le k^l\eta.$$

If we continue this process, then we can get

$$\eta > \rho^q(h^2t_0, h^3t_0) > \rho^q(h^3t_0, h^4t_0) > \cdots > \rho^q(h^mt_0, h^{m+1}t_0)$$

whenever $m=2l+1$ or $m=2l$. Hence $\rho(h^mt_0, h^{m+1}t_0) \to 0$ as $m\to\infty$. Thus, we can say that h is asymptotically regular at t_0 and so, h has an approximate fixed point. As in the inequalities (13) and (14), for $m=2l+1$ and $m=2l$ with $j,l\ge 1$, we get

$$\rho(h^mt_0, h^{m+j}t_0) \le 2k^l\frac{1}{1-k}\eta.$$

Since $k<1$ and also, if we take $l\to\infty$ in the inequalities (13) and (14), then we get $\rho(h^mt_0, h^{m+j}t_0) \to 0$. Thus, $\{t_n\}$ is a Cauchy sequence in E. Since (E,ρ) be a complete metric space, then $t_n = h^nt_0 \to u \in E$ as $n\to\infty$. Thus, we obtain that $hu=u$. Now, we demonstrate that h has a unique fixed point in E. Let u and u^* are two fixed point of h.

$$\zeta(\alpha\rho^q(hu,hu^*) + (1-\alpha)\rho^q(h^2u,h^2u^*), \delta M_\iota(u,u^*) + LN_\iota(u,u^*) \ge 0$$

Since $\zeta\in Z$, we have

$$\alpha\rho^q(hu,hu^*) + (1-\alpha)\rho^q(h^2u,h^2u^*) \le \delta\max\begin{Bmatrix}\rho^q(u,u^*), \rho^q(u,hu), \rho^q(u^*,hu^*), \\ \rho^q(hu,h^2u), \rho^q(hu^*,h^2u^*)\end{Bmatrix}$$
$$+L\min\begin{Bmatrix}\rho^q(u,hu), \rho^q(u^*,hu^*), \rho^q(u,hu^*), \\ \rho^q(u^*,hu), \rho^q(hu,h^2u), \rho^q(hu^*,h^2u^*)\end{Bmatrix}.$$

We get that

$$(1-\delta)\rho^q(u,u^*) \le 0.$$

Thus, we obtain $\rho(u,u^*)=0$, that is, $u=u^*$. Hence, h has a unique fixed point in E.

Example 3.2.1. Let (E,ρ) be a usual metric space where $E=[0,1]$ and $\rho(t,s)=|t-s|$. Let define the mapping $h\colon E\to E$ by $h(t)=\frac{1-t^2}{3}, t\in[0,1]$. We want to show that h satisfies (15). It is easily seen that $h^2t=\frac{-t^4+2t^2+8}{27}$. For $\alpha=\frac{1}{2}$, $q=1$ and for $t,s\in[0,1]$, we have

$$\rho(ht,hs)=\left|\frac{1-t^2}{3}-\frac{1-s^2}{3}\right|=\frac{1}{3}|t^2-s^2|\leq\frac{2}{3}|t-s|$$

and

$$\begin{aligned}\rho(h^2t,h^2s)&=\left|\frac{-t^4+2t^2}{27}-\frac{-s^4+2s^2}{27}\right|\\&\leq\frac{4}{27}|t-s|+\frac{2}{27}|t-s|\\&=\frac{2}{9}|t-s|.\end{aligned}$$

For $\alpha=\frac{1}{4}\in(0,1)$ and $L=0$, we get that

$$\begin{aligned}\alpha\rho(ht,hs)+(1-\alpha)\rho(h^2t,h^2s)&\leq\frac{2\alpha}{3}|t-s|+\frac{2(1-\alpha)}{9}|t-s|\\&\leq\frac{1}{3}\rho(t,s)\\&\leq\frac{1}{3}M_I(t,s)+LN_I(t,s).\end{aligned}$$

Thus, we get $\delta=\frac{1}{3}$. Also, one can see that $\delta+\alpha<1$. We have

$$\zeta(\alpha\rho^q(ht,hs)+(1-\alpha)\rho^q(h^2t,h^2s),\delta M_\iota(t,s)+LN_\iota(t,s)\geq 0.$$

Therefore, h is a generalized almost (α,q)-convex contractive mapping via simulation function and h satisfies all conditions of Theorem 3.2.1. So h has a unique fixed point in E. This fixed point is $u=\frac{\sqrt[2]{13}-3}{2}$.

Also, for $t,s\in[0,1]$ and $q=2$, we get that

$$\rho(ht,hs)\leq\frac{4}{9}|t-s|^2,$$

and

$$\rho^2(h^2t,h^2s)\leq\frac{4}{81}|t-s|^2.$$

For $\alpha=\frac{1}{3}\in(0,1)$ and $L=0$, we have

$$\begin{aligned}\alpha\rho(ht,hs)+(1-\alpha)\rho(h^2t,h^2s)&\leq\frac{4\alpha}{9}|t-s|^2+\frac{4(1-\alpha)}{81}|t-s|^2\\&\leq\frac{4}{27}\rho^2(t,s)\\&\leq\frac{4}{27}(M_I(t,s))+LN_I(t,s).\end{aligned}$$

Hence, h is a generalized almost (α,q)-convex contractive mapping for $q=2$.

Theorem 3.2.2. Let (E,ρ) be a complete metric space and $h:E\to E$ be a generalized almost α-convex contractive mapping via simulation function such that $2\delta+\alpha<1$ and $\delta<\alpha$. If there exist an asymptotically (h,h^2)-regular sequence in E, then h has a unique fixed point.

Proof: Let $\{t_n\} \subseteq E$ be an asymptotically (h, h^2)-regular sequence. For any $m, n \in \mathbb{N}$ with $m > n$,we get that

$$\zeta(\alpha\rho^q(ht_m, ht_n) + (1-\alpha)\rho^q(h^2t_m, h^2t_n), \delta M_\iota(t_m, t_n) + LN_\iota(t_m, t_n)) \geq 0.$$

Thus, we have

$$\begin{aligned}(1-\alpha)\rho(t_m, t_n) &\leq (1-\alpha)[\rho(t_m, h^2t_m) + \rho(h^2t_m, h^2t_n) + \rho(h^2t_n, t_n)] \\ &\leq (1-\alpha)[\rho(t_m, h^2t_m) + \rho(h^2t_n, t_n)] + \alpha\rho(ht_m, ht_n) \\ &\quad +(1-\alpha)\rho(h^2t_m, h^2t_n) \\ &\leq (1-\alpha)[\rho(t_m, h^2t_m) + \rho(h^2t_n, t_n)] \\ &\quad +\delta\max\begin{Bmatrix}\rho(t_m, t_n), \rho(t_m, ht_m), \rho(t_n, ht_n) \\ \rho(ht_m, h^2t_m), \rho(ht_n, h^2t_n)\end{Bmatrix} \\ &\quad +L\min\begin{Bmatrix}\rho(t_m, ht_m), \rho(t_n, ht_n), \rho(t_m, ht_n), \\ \rho(t_n, ht_m)\rho(ht_m, h^2t_m), \rho(ht_n, h^2t_n)\end{Bmatrix}.\end{aligned} \tag{17}$$

Since $\{t_n\} \subseteq E$ be an asymptotically (h, h^2)-regular sequence. Taking the limit as $n, m \to \infty$ in (17) and from Lemma 2, we get

$$\lim_{n,m\to\infty}(1-\alpha-\delta)\rho(t_m, t_n) \leq 0.$$

Since $\delta + \alpha < 1$, then we get $\lim_{n,m\to\infty}\rho(t_m, t_n) = 0$. Thus $\{t_n\} \subseteq E$ is a Cauchy sequence. Since (E, ρ) is a complete metric space, we have that as $n \to \infty$, $t_n \to u \in E$. Now we demonstrate that u is a fixed point of h. Using the inequality (15), we get

$$\begin{aligned}\alpha\rho(hu, t_n) &\leq \alpha[\rho(hu, ht_n) + \rho(ht_n, t_n)] + (1-\alpha)\rho(h^2u, h^2t_n) \\ &\leq [(1-\alpha)\rho(h^2u, h^2t_n) + \alpha\rho(hu, ht_n)] + \alpha\rho(ht_n, t_n) \\ &\leq \delta\max\{\rho(u, t_n), \rho(u, hu), \rho(hu, h^2u), \rho(t_n, ht_n), \rho(ht_n, h^2t_n)\} \\ &\quad +L\min\{\rho(u, hu), \rho(hu, h^2u), \rho(t_n, ht_n), \rho(ht_n, h^2t_n), \rho(u, ht_n), \rho(t_n, hu)\} \\ &\quad +\alpha\rho(ht_n, t_n).\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\alpha\rho(hu, u) \leq \delta\max\{\rho(u, hu), \rho(hu, h^2u)\}. \tag{18}$$

Case 1. Since we assume that $\max\{\rho(u, hu), \rho(hu, h^2u)\} = \rho(u, hu)$, we have

$$(\alpha - \delta)\rho(hu, u) \leq 0.$$

Since $\alpha > 2\delta$, we have $hu = u$. In this case, u is a fixed point of h.

Case 2. If we take $\max\{\rho(u, hu), \rho(hu, h^2u)\} = \rho(hu, h^2u)$ in (18) inequality, then we get

$$\alpha\rho(hu, u) \leq \delta\rho(hu, h^2u) \leq \delta[\rho(hu, u) + \rho(u, h^2u)]. \tag{19}$$

Now, using the (15) inequality, we have

$$(1-\alpha)\rho(h^2u, t_n) \leq (1-\alpha)[\rho(h^2u, h^2t_n) + \rho(h^2t_n, t_n)]$$

$$\leq (1-\alpha)\rho(h^2u, h^2t_n) + \alpha\rho(hu, ht_n) + (1-\alpha)\rho(h^2t_n, t_n)$$
$$\leq \delta\max\{\rho(u,t_n), \rho(u,hu), \rho(hu,h^2u)\rho(t_n,ht_n), \rho(ht_n,h^2t_n)\}$$
$$+L\min\begin{Bmatrix}\rho(u,hu), \rho(hu,h^2u), \rho(t_n,ht_n), \\ \rho(ht_n,h^2t_n), \rho(u,ht_n), \rho(t_n,hu)\end{Bmatrix} + (1-\alpha)\rho(h^2t_n, t_n).$$

So, we get

$$\begin{aligned}&(1-\alpha)\rho(h^2u,t_n)\\ &\leq \delta\max\{\rho(u,t_n), \rho(u,hu), \rho(hu,h^2u), \rho(t_n,ht_n), \rho(ht_n,h^2t_n)\}\\ &+L\min\begin{Bmatrix}\rho(u,hu), \rho(hu,h^2u), \rho(t_n,ht_n), \\ \rho(ht_n,h^2t_n), \rho(u,ht_n), \rho(t_n,hu)\end{Bmatrix}\\ &+(1-\alpha)\rho(h^2t_n,t_n).\end{aligned} \tag{20}$$

Taking the limit as $n \to \infty$ in (20), we get

$$(1-\alpha)\rho(h^2u,u) \leq \delta\max\{\rho(u,hu), \rho(hu,h^2u)\}.$$

Since $\max\{\rho(u,hu), \rho(hu,h^2u)\} = \rho(hu,h^2u)$, we obtain that

$$(1-\alpha)\rho(h^2u,u) \leq \delta\rho(hu,h^2u) \leq \delta[\rho(hu,u) + \rho(u,h^2u)].$$

Thus, we obtain

$$\rho(h^2u,u) \leq \frac{\delta}{(1-\alpha-\delta)}\rho(hu,u).$$

Since $2\delta + \alpha < 1$, we get $\frac{\delta}{1-\alpha-\delta} < 1$ and so, we get $\rho(h^2u,u) < \rho(hu,u)$. Now, again from (19) inequality, we have

$$\begin{aligned}\alpha\rho(hu,u) &\leq \delta[\rho(hu,u) + \rho(u,h^2u)]\\ &\leq \delta[\rho(hu,u) + \rho(hu,u)].\end{aligned}$$

Hence, we obtain

$$(\alpha - 2\delta)\rho(hu,u) \leq 0.$$

Since $\alpha > 2\delta$, we have $hu = u$. From Case 1 and Case 2 we get that u is a fixed point of h. Now, we demonstrate that h has a unique fixed point in E. Let u and u^* are two fixed point of h. Using the inequality (15), we have

$$\zeta(\alpha\rho^q(hu,hu^*) + (1-\alpha)\rho^q(h^2u,h^2u^*), \delta M_\iota(u,u^*) + LN_\iota(u,u^*) \geq 0.$$

Since $\zeta \in Z$, we have

$$\alpha\rho^q(hu,hu^*) + (1-\alpha)\rho^q(h^2u,h^2u^*) \leq \delta\max\begin{Bmatrix}\rho^q(u,u^*), \rho^q(u,hu), \rho^q(u^*,hu^*), \\ \rho^q(hu,h^2u), \rho^q(hu^*,h^2u^*)\end{Bmatrix}$$
$$+L\min\begin{Bmatrix}\rho^q(u,hu), \rho^q(u^*,hu^*), \rho^q(u,hu^*), \\ \rho^q(u^*,hu), \rho^q(hu,h^2u), \rho^q(hu^*,h^2u^*)\end{Bmatrix}.$$

We get that

$$(1-\delta)\rho^q(u,u^*) \leq 0,$$

which give us that $\rho(u,u^*)=0$, that is, $u=u^*$. Hence, h has a unique fixed point in E.

In this part, we show that some existing results in the fixed point theory can be derived from our main results. Taking $q=1$ in (12), we get following corollary.

Corollary 3.2.1. Let (E,ρ) be a metric space and $h: E\to E$ be a mapping. If for some $\alpha\in(0,1)$ there exist constants $\lambda\in(0,1)$ and $b_i\in(0,1)$, $i=1,2,3,4$ with $b_1+b_2+b_3+b_4<1$ satisfiying the inequality

$$\alpha\rho(ht,hs)+(1-\alpha)\rho(h^2t,h^2s)\leq\lambda[\rho(t,s)]^{b_1}[\rho(t,ht)]^{b_2}[\rho(s,hs)]^{b_3}[\rho(ht,h^2t)]^{b_4}\\ [\rho(hs,h^2s)]^{1-b_1-b_2-b_3-b_4}$$

with $\lambda+\alpha<1$, then h has the *AFPP*. Moreover, if (E,ρ) is a complete metric space, then h has a fixed point.

Corollary 3.2.2. Let (E,ρ) be a metric space and $h: E\to E$ be a mapping. Let for some $\alpha\in(0,1)$ and $q\geq1$ there exist constants $\lambda\in(0,1)$ and $b\in(0,1)$ satisfiying the following inequality

$$\alpha\rho^q(ht,hs)+(1-\alpha)\rho^q(h^2t,h^2s)\leq\lambda^{b_3}[\rho(ht,h^2t)]^{b}[\rho(hs,h^2s)]^{1-b}$$

with $\lambda+\alpha<1$. Then h has the AFPP. Moreover, if (E,ρ) is a complete metric space, then h has a fixed point.

Corollary 3.2.3. Let (E,ρ) be a complete metric space and $h: E\to E$ be a mapping. If for some $\alpha\in(0,1)$ and $q\geq1$ there exist two constants $\delta\in(0,1)$ and $L\geq0$ satisfying the following inequality

$$\alpha\rho^q(ht,hs)+(1-\alpha)\rho^q(h^2t,h^2s)\leq\delta M_\iota(t,s)+LN_\iota(t,s)$$

where

$$M_I(t,s)=\max\{\rho^q(t,s),\rho^q(t,ht),\rho^q(s,hs),\rho^q(ht,h^2t),\rho^q(hs,h^2s)\},$$
$$N_I(t,s)=\min\{\rho^q(t,ht),\rho^q(s,hs),\rho^q(t,hs),\rho^q(s,ht),\rho^q(ht,h^2t),\rho^q(hs,h^2s)\},$$

and $\delta+\alpha<1$, then h has the AFPP, and also h has a unique fixed point, that is, $u=hu$, $u\in E$.

If we take $L=0$ in (15) inequality, then we get the following corollary.

Corollary 3.2.4. Let (E,ρ) be a complete metric space, $\zeta\in Z$ and $h: E\to E$ be two mappings. If for some $\alpha\in(0,1)$ and $q\geq1$ there exists a constant $\delta\in(0,1)$ satisfying the following inequality

$$\zeta(\alpha\rho^q(ht,hs)+(1-\alpha)\rho^q(h^2t,h^2s),\delta M_\iota(t,s))\geq0$$

where

$$M_I(t,s)=\max\{\rho^q(t,s),\rho^q(t,ht),\rho^q(s,hs),\rho^q(ht,h^2t),\rho^q(hs,h^2s)\},$$
$$N_I(t,s)=\min\{\rho^q(t,ht),\rho^q(s,hs),\rho^q(t,hs),\rho^q(s,ht),\rho^q(ht,h^2t),\rho^q(hs,h^2s)\},$$

and $\delta+\alpha<1$, then h has the AFPP, and also h has a unique fixed point, that is, $u=hu$, $u\in E$.

Corollary 3.2.5. Let (E,ρ) be a complete metric space and $h: E\to E$ be a mapping. If for some $\alpha \in (0,1)$ and $q \geq 1$ there exists a constant $\delta \in (0,1)$ satisfying the following inequality

$$\alpha\rho^q(ht,hs) + (1-\alpha)\rho^q(h^2t,h^2s) \leq \delta M_\iota(t,s)$$

where

$$M_I(t,s) = \max\{\rho^q(t,s), \rho^q(t,ht), \rho^q(s,hs), \rho^q(ht,h^2t), \rho^q(hs,h^2s)\},$$
$$N_I(t,s) = \min\{\rho^q(t,ht), \rho^q(s,hs), \rho^q(t,hs), \rho^q(s,ht), \rho^q(ht,h^2t), \rho^q(hs,h^2s)\},$$

and $\delta + \alpha < 1$, then h has the AFPP, and also, h has a unique fixed point, that is, $u = hu$, $u \in E$.

Taking $q = 1$ in (15) inequality, then we get following corollary.

Corollary 3.2.6. Let (E,ρ) be a complete metric space, $\zeta \in Z$ and $h: E \to E$ be a mapping. If for some $\alpha \in (0,1)$ there exists a constant $\delta \in (0,1)$ and satisfying the following inequality

$$\zeta(\alpha\rho(ht,hs) + (1-\alpha)\rho(h^2t,h^2s), \delta M_\iota(t,s) + LN_\iota(t,s)) \geq 0$$

where

$$M_I(t,s) = \max\{\rho(t,s), \rho(t,ht), \rho(s,hs), \rho(ht,h^2t), \rho(hs,h^2s)\},$$
$$N_I(t,s) = \min\{\rho(t,ht), \rho(s,hs), \rho(t,hs), \rho(s,ht), \rho(ht,h^2t), \rho(hs,h^2s)\},$$

and $\delta + \alpha < 1$, then h has the AFPP, and also h has a unique fixed point, that is, $u = hu$, $u \in E$.

4. CONCLUSION

One can easily see that the results of Khan et al. [8] and Goebel and Pineda [13] can be expressed as a corollary of our main result. Corollary 3.2.1. and Corollary 3.2.2. provide generalizations of the results in [4], [9-11]. In addition to these, it is easily seen that Theorem 3.1.1. and Theorem 3.2.1. are two real generalizations of Banach contraction principle.

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