ORIGINAL PAPER

NON-INVARIANT SUBMANIFOLDS OF ALMOST POLY-NORDEN RIEMANNIAN MANIFOLDS

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Abstract. *In this paper, we study a special class of submanifolds of almost poly-Norden manifolds in the Riemannian setting. We examine the sufficient conditions for such kind of submanifolds to be totally geodesic or minimal.*

Keywords: *bronze mean; almost poly-Norden manifold; Riemannian submanifold.*

1. INTRODUCTION

The geometry of metallic manifolds and their submanifolds was initially studied by C.E. Hreţcanu and M.C. Crâşmăreanu in [1]. In [2], M. Özkan and F. Yılmaz studied such a class of manifolds with the help of the corresponding almost product manifolds. In the metallic Riemannian setting, the properties of invariant, anti-invariant, non-invariant, slant, semi-slant, hemi-slant, and bi-slant submanifolds were investigated by C. E. Hreţcanu and A. M. Blaga (see, e.g., [3-5]). In [6], some classification theorems were given by M. Gök and E. Kılıç for totally umbilical proper semi-invariant submanifolds of a locally decomposable metallic Riemannian manifold. Additionally, the de Rham cohomology groups of semi-invariant, hemi-slant, and semi-slant submanifolds of metallic Riemannian manifolds were examined by M. Gök in [7-9].

On the other hand, using S. Kalia's definition of the bronze mean in [10], B. Şahin [11] introduced and studied the concept of an almost poly-Norden semi-Riemannian manifold. In the Riemannian setting, a particular class of such kind of manifolds, namely almost bronze Riemannian manifolds, was investigated by M. Özkan and S. Doğan [12] in terms of the parallelism and integrability conditions. An almost bronze Riemannian (or almost poly-Norden semi-Riemannian) manifold is not a member of a metallic Riemannian (or metallic semi-Riemannian) manifold, but it is a special class of framed metric $f_{(a,b)}(3,2,1)$-manifolds, studied by M. Gök, E. Kılıç, and C. Özgür in [13]. It is also worth noting that framed metric $f_{(a,b)}(3,2,1)$-manifolds include both metallic and almost poly-Norden semi-Riemannian manifolds. Moreover, S. Y. Perktaş [14] analyzed invariant, anti-invariant, and non-invariant submanifolds of almost poly-Norden Riemannian manifolds.

The main goal of this paper is to go on studying submanifolds in almost poly-Norden Riemannian manifolds, particularly in almost bronze Riemannian manifolds. The plan of the paper is constructed as follows: The first section is introduction giving a brief literature review on metallic and almost poly-Norden semi-Riemannian manifolds. Section 2 deals with basic notions, definitions, and formulas to make other ones understandable. Section 3 consists of an investigation of non-invariant submanifolds of almost poly-Norden Riemannian manifolds in terms of the totally geodesicity and minimality.

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2. PRELIMINARIES

We begin with a short review of the geometry of almost poly-Norden manifolds. For more details, we refer the reader to [11]. The bronze mean [10], denoted by B_m, is the positive solution of the quadratic equation

$$\mathrm{x}^2 - m\mathrm{x} + 1 = 0, \tag{2.1}$$

i.e., it is given by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2}. \tag{2.2}$$

Also, its continued fraction expansion and iterated square root are given by

$$B_m = m - \cfrac{1}{m - \cfrac{1}{m - \cfrac{1}{m - \cdots}}} \tag{2.3}$$

and

$$B_m = \sqrt{-1 + m\sqrt{-1 + m\sqrt{-1 + m\sqrt{\ldots}}}}, \tag{2.4}$$

respectively.

In [11], inspired by the bronze mean, B. Şahin defined a new type of structure on a differentiable manifold, namely a poly-Norden structure, as follows: A poly-Norden structure $\overline{\varphi}$ on a differentiable manifold $\overline{M}$ is an endomorphism of the tangent bundle $T\overline{M}$ such that it satisfies the equation

$$\overline{\varphi}^2 = m\overline{\varphi} - I, \tag{2.5}$$

where I is the identity map on the Lie algebra $\Gamma(T\overline{M})$ of differentiable vector fields on $\overline{M}$. In this case, the pair $(\overline{M}, \overline{\varphi})$ is called an almost poly-Norden manifold. The eigenvalues of the poly-Norden structure $\overline{\varphi}$ are B_m and $m - B_m$. In particular, a poly-Norden structure (or an almost poly-Norden manifold) with $m \in \mathbb{R}\backslash[-2,2]$ is named as a new almost bronze structure (or a new almost bronze manifold) by M. Özkan and S. Doğan [12]. For brevity, such a structure (or a manifold) is referred to as an almost bronze structure (or manifold). The inverse of the poly-Norden structure $\overline{\varphi}$, denoted by $\overline{\varphi}^{-1}$, is given by

$$\overline{\varphi}^{-1} = -\overline{\varphi} + mI. \tag{2.6}$$

Thus, it follows that $\overline{\varphi}^{-1}$ is not a poly-Norden structure and $\overline{\varphi}$ is an isomorphism on the tangent space $T_p\overline{M}$ for each point $p \in \overline{M}$. If there exists a semi-Riemannian metric $\overline{g}$ such that

$$\overline{g}(\overline{\varphi}X, Y) = \overline{g}(X, \overline{\varphi}Y), \tag{2.7}$$

or equivalently

$$\overline{g}(\overline{\varphi}X, \overline{\varphi}Y) = m\overline{g}(\overline{\varphi}X, Y) - \overline{g}(X, Y) \tag{2.8}$$

for any vector fields $X, Y \in \Gamma(T\overline{M})$, then the pair $(\overline{g}, \overline{\varphi})$ is said to be an almost poly-Norden semi-Riemannian structure and the triple $(\overline{M}, \overline{g}, \overline{\varphi})$ is called an almost poly-Norden semi-

Riemannian manifold. Particularly, if $\overline{\nabla}\overline{\varphi} = 0$, then $(\overline{M}, \overline{g}, \overline{\varphi})$ is named a poly-Norden semi-Riemannian manifold, where $\overline{\nabla}$ stands for the Riemannian connection on $\overline{M}$.

Let $\overline{\varphi}$ be a poly-Norden structure on a differentiable manifold $\overline{M}$. The sign of the discriminant of the structure polynomial of the poly-Norden structure $\overline{\varphi}$ characterizes the geometry of the almost poly-Norden manifold $(\overline{M}, \overline{\varphi})$. If $m^2 < 4$ ($m^2 = 4$ or $m^2 > 4$), then the poly-Norden structure $\overline{\varphi}$ on $\overline{M}$ induces two almost complex structures (almost tangent structures or almost product structures) on the same manifold. Conversely, for a given almost complex structure (almost tangent structure or almost product structure) on $\overline{M}$, we have two poly-Norden structures. Such a relationship between poly-Norden structures and almost complex structures implies that if $m^2 < 4$, then $\overline{M}$ is an even-dimensional manifold. In addition, if $(\overline{M}, \overline{g}, \overline{\varphi})$ is an almost poly-Norden semi-Riemannian manifold with $m^2 < 4$, then $\overline{g}$ must be a neutral metric.

3. SUBMANIFOLDS OF ALMOST POLY-NORDEN RIEMANNIAN MANIFOLDS

In this section, we first mention the fundamental properties of submanifolds in almost poly-Norden Riemannian manifolds. Later, we give some sufficient conditions for non-invariant submanifolds of almost poly-Norden Riemannian manifolds to be totally geodesic or minimal.

Let M be an n-dimensional submanifold of codimension k, isometrically immersed in an almost poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Denoting by $\{N_1, \dots, N_k\}$ a local orthonormal frame of the normal bundle $TM^{\perp}$, it is well known that for each vector field $X \in \Gamma(TM)$, the vector fields $\overline{\varphi}(i_* X)$ and $\overline{\varphi}(N_\alpha)$ on $\overline{M}$ are decomposed into tangential and normal components as follows:

$$\overline{\varphi}(i_* X) = i_*(fX) + \sum_{\alpha=1}^{k} \upsilon_\alpha (X) N_\alpha \tag{3.1}$$

and

$$\overline{\varphi} N_\alpha = i_*(\zeta_\alpha) + \sum_{\beta=1}^{k} \theta_{\alpha\beta} N_\beta, \tag{3.2}$$

where i_* is the differential of the immersion $i: M \to \overline{M}$, f is a tensor field of type $(1,1)$ on M, ζ_α's are the tangent vector fields on M, υ_α's are the 1-forms on M, and $(\theta_{\alpha\beta})$ is a matrix of type $k \times k$ of real functions on M for any $\alpha, \beta \in \{1, \dots, r\}$.

Let $\overline{\nabla}$ be the Riemannian connection on $\overline{M}$. In this case, Gauss and Weingarten formulas of M in $\overline{M}$ are given, respectively, by

$$\overline{\nabla}_{i_* X} i_* Y = i_* \nabla_X Y + \sum_{\alpha=1}^{k} h_\alpha (X, Y) N_\alpha \tag{3.3}$$

and

$$\overline{\nabla}_{i_* X} N_\alpha = -i_* A_\alpha X + \sum_{\beta=1}^{k} \sigma_{\alpha\beta} (X) N_\beta \tag{3.4}$$

for any vector fields $X, Y \in \Gamma(TM)$, where ∇ is the induced connection on M, h_α's are the second fundamental tensors corresponding to N_α's, i.e., $h(X,Y) = \sum_{\alpha=1}^{k} h_\alpha(X,Y)N_\alpha$, A_α's are the Weingarten maps in the direction of N_α's, and $\sigma_{\alpha\beta}$'s are the 1-forms on M corresponding to the normal connection $\nabla^\perp$ for any $\alpha, \beta \in \{1, \dots, k\}$, i.e., $\nabla_X^\perp N_\alpha = \sum_{\beta=1}^{k} \sigma_{\alpha\beta}(X) N_\beta$. We also note that

$$\sigma_{\alpha\beta} = -\sigma_{\beta\alpha} \tag{3.5}$$

for any $\alpha, \beta \in \{1, \dots, k\}$.

Furthermore, if $h = 0$, or equivalently $h_\alpha = 0$ for any $\alpha \in \{1, \dots, r\}$, then M is said to be a totally geodesic submanifold; if $H = 0$, then M is called a minimal submanifold; if $h(X,Y) = g(X,Y)H$ for any vector fields $X, Y \in \Gamma(TM)$, then M is named a totally umbilical submanifold, where H denotes the mean curvature vector of M.

Proposition 3.1. [14, Lemma 3.4 and Proposition 3.5] Let M be an n-dimensional submanifold of codimension k, isometrically immersed in an almost poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then there is a structure $\left(f, g, \upsilon_\alpha, \zeta_\alpha, (\theta_{\alpha\beta})_{k\times k}\right)$ on M induced by the poly-Norden structure $\overline{\varphi}$ such that it has the following properties:

$$f^2X = mfX - X - \sum_{\alpha=1}^{k} \upsilon_\alpha(X)\zeta_\alpha, \tag{3.6}$$

$$\upsilon_\alpha(fX) = m\upsilon_\alpha(X) - \sum_{\beta=1}^{k} \theta_{\alpha\beta}\, \upsilon_\beta(X), \tag{3.7}$$

$$\theta_{\alpha\beta} = \theta_{\beta\alpha}, \tag{3.8}$$

$$\upsilon_\beta(\zeta_\alpha) = m\theta_{\alpha\beta} - \delta_{\alpha\beta} - \sum_{\gamma=1}^{k} \theta_{\alpha\gamma}\, \theta_{\beta\gamma}, \tag{3.9}$$

$$f(\zeta_\alpha) = m\zeta_\alpha - \sum_{\beta=1}^{k} \theta_{\alpha\beta}\, \zeta_\beta, \tag{3.10}$$

$$\upsilon_\alpha(X) = g(X, \zeta_\alpha), \tag{3.11}$$

$$g(fX, Y) = g(X, fY), \tag{3.12}$$

and

$$g(fX, fY) = mg(fX, Y) - g(X,Y) - \sum_{\alpha=1}^{k} \upsilon_\alpha(X)\upsilon_\alpha(Y) \tag{3.14}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Proposition 3.2. [14, Propositions 3.3 and 3.6] Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then we have the following relations:

$$(\nabla_X f)Y = \sum_{\alpha=1}^{k} \upsilon_\alpha(Y)A_\alpha X + \sum_{\alpha=1}^{k} h_\alpha(X,Y)\zeta_\alpha, \tag{3.15}$$

$$(\nabla_X \upsilon_\alpha)Y = -h_\alpha(X,fY) + \sum_{\beta=1}^{k} \upsilon_\beta(Y)\sigma_{\alpha\beta}(X) + \sum_{\beta=1}^{k} h_\beta(X,Y)\theta_{\alpha\beta}, \tag{3.16}$$

$$\nabla_X \zeta_\alpha = -f(A_\alpha X) + \sum_{\beta=1}^{k} \theta_{\alpha\beta} A_\beta X + \sum_{\beta=1}^{k} \sigma_{\alpha\beta}(X)\zeta_\beta, \tag{3.17}$$

and

$$X(\theta_{\alpha\beta}) = -h_\beta(X,\zeta_\alpha) - h_\alpha(X,\zeta_\beta) - \sum_{\gamma=1}^{k} \theta_{\alpha\gamma}\,\sigma_{\gamma\beta}(X) - \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\,\sigma_{\gamma\alpha}(X) \tag{3.18}$$

for any vector fields $X, Y \in \Gamma(TM)$.

Now, we present an example of a non-invariant submanifold in a Euclidean space admitting a poly-Norden structure.

Example 1. We consider the $(2a+b)$-dimensional Euclidean space E^{2a+b}, where a and b are positive integers. Let us define a tensor field $\overline{\varphi}$ of type $(1,1)$ by

$$\overline{\varphi}(X^i,Y^i,Z^j) = \left(\frac{m}{2}X^i - \frac{\sqrt{m^2-4}}{2}Y^i, \frac{m}{2}Y^i - \frac{\sqrt{m^2-4}}{2}X^i, B_m Z^j\right) \tag{3.19}$$

for any tangent vector $(X^i,Y^i,Z^j) \in T_{(x^i,y^i,z^j)}E^{2a+b}$, where $m \in \mathbb{R}\setminus[-2,2]$, $(X^i,Y^i,Z^j) = (X^1,\dots,X^a,Y^1,\dots,Y^a,Z^1,\dots,Z^b)$ and $(x^i,y^i,z^j) = (x^1,\dots,x^a,y^1,\dots,y^a,z^1,\dots,z^b)$. Then it is seen that $(\langle,\rangle,\overline{\varphi})$ is an almost bronze Riemannian structure and $(E^{2a+b},\langle,\rangle,\overline{\varphi})$ is a bronze Riemannian manifold.

By reason of the fact that $E^{2a+b} = E^{2a} \times E^b$, it can be mentioned the following two hyperspheres:

$$S^{2a-1}(r) = \left\{(x^1,\dots,x^a,y^1,\dots,y^a): \sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 = r^2\right\} \text{ in } E^{2a} \tag{3.20}$$

and

$$S^{b-1}(r_3) = \left\{(z^1,\dots,z^b): \sum_{j=1}^{b}(z^j)^2 = r_3^2\right\} \text{ in } E^b. \tag{3.21}$$

In an analogous way as in [15, Example 3], we can build the product manifold $M = S^{2a-1}(r) \times S^{b-1}(r_3)$ such that its every point has the coordinates (x^i,y^i,z^j) verifying the equation

$$\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + \sum_{j=1}^{b}(z^j)^2 = R^2, \tag{3.22}$$

where $R^2 = r^2 + r_3^2$. Then M is a submanifold of codimension 2 in the Euclidean space E^{2a+b} and M is a submanifold of codimension 1 in the sphere $S^{2a+b-1}(R)$. Thus, there are successive embeddings such that

$$M \hookrightarrow S^{2a+b-1}(R) \hookrightarrow E^{2a+b}. \tag{3.23}$$

In addition, its tangent space $T_{(x^i,y^i,z^j)}M$ at any point (x^i, y^i, z^j) is given by

$$T_{(x^i,y^i,z^j)}M = T_{(x^i,y^i,0^j)}S^{2a-1}(r) \oplus T_{(0^i,0^i,z^j)}S^{b-1}(r_3). \tag{3.24}$$

Hence, it follows that any tangent vector $(X^i, Y^i, Z^j) \in T_{(x^i,y^i,z^j)}E^{2a+b}$ belongs to $T_{(x^i,y^i,z^j)}M$ for every point $(x^i, y^i, z^j) \in M$ if and only if

$$\sum_{i=1}^{a} x^i X^i + \sum_{i=1}^{a} y^i Y^i = \sum_{j=1}^{b} z^j Z^j = 0. \tag{3.25}$$

Moreover, (X^i, Y^i, Z^j) is a tangent vector on the sphere $S^{2a+b-1}(R)$, so it is seen that

$$T_{(x^i,y^i,z^j)}M \subset T_{(x^i,y^i,z^j)}S^{2a+b-1}(R) \tag{3.26}$$

for every point $(x^i, y^i, z^j) \in M$.

If $\{N_1, N_2\}$ is a local orthonormal basis for the normal space $T_{(x^i,y^i,z^j)}M^{\perp}$ at any point (x^i, y^i, z^j), then the normal vectors N_1 and N_2 can be chosen as follows:

$$N_1 = \frac{1}{R}(x^i, y^i, z^j) \tag{3.27}$$

and

$$N_2 = \frac{1}{R}\left(\frac{r_3}{r}x^i, \frac{r_3}{r}y^i, -\frac{r}{r_3}z^j\right). \tag{3.28}$$

For any tangent vector $X \in T_{(x^i,y^i,z^j)}M$, we identify i_*X with X. From (3.2), we have the following decomposition:

$$\overline{\varphi}N_\alpha = \zeta_\alpha + \sum_{\beta=1}^{2} \theta_{\alpha\beta} N_\beta \tag{3.29}$$

for any $\alpha \in \{1,2\}$, so we get

$$\theta_{\alpha\beta} = \langle \overline{\varphi}N_\alpha, N_\beta \rangle \tag{3.30}$$

for any $\alpha, \beta \in \{1,2\}$. Hence, by a straightforward computation, it is found that the matrix $\mathcal{A} = (\theta_{\alpha\beta})_{2\times 2}$ is given by

$$\mathcal{A} = \begin{pmatrix} \dfrac{mR^2 - (2\lambda - {r_3}^2)\sqrt{m^2-4}}{2R^2} & -\dfrac{r_3(2\lambda - r^2)\sqrt{m^2-4}}{2rR^2} \\ -\dfrac{r_3(2\lambda - r^2)\sqrt{m^2-4}}{2rR^2} & \dfrac{mr^2R^2 - (2{r_3}^2\lambda - r^4)\sqrt{m^2-4}}{2r^2R^2} \end{pmatrix}, \tag{3.31}$$

where $\lambda = \sum_{i=1}^{a} x^i y^i$. Thus, by means of the entries of the matrix $\mathcal{A}$, we derive from (3.29) that the tangent vector fields ζ_1 and ζ_2 are calculated as follows:

$$\zeta_1 = \frac{\sqrt{m^2-4}}{R}\left(-\frac{1}{2}y^i + \frac{\lambda R^2 - r^2 {r_3}^2}{r^2R^2}x^i, -\frac{1}{2}x^i + \frac{\lambda R^2 - r^2 {r_3}^2}{r^2R^2}y^i, \frac{r^2}{R^2}z^i\right) \tag{3.32}$$

and

$$\zeta_2 = \frac{r_3\sqrt{m^2-4}}{rR}\left(-\frac{1}{2}y^i + \frac{\lambda R^2 - r^4}{r^2R^2}x^i, -\frac{1}{2}x^i + \frac{\lambda R^2 - r^4}{r^2R^2}y^i, -\frac{r^2}{R^2}z^i\right). \tag{3.33}$$

As is seen from (3.11) that

$$\upsilon_\alpha(X^i, Y^i, Z^j) = \langle (X^i, Y^i, Z^j), \zeta_\alpha \rangle \tag{3.34}$$

for any $\alpha \in \{1,2\}$, so from (3.32) and (3.33), we have

$$\upsilon_1(X^i, Y^i, Z^j) = -\frac{\mu\sqrt{m^2-4}}{2R} \tag{3.35}$$

and

$$\upsilon_2(X^i, Y^i, Z^j) = -\frac{r^3\mu\sqrt{m^2-4}}{2rR}, \tag{3.36}$$

where $\mu = \sum_{i=1}^{a}(x^iY^i + y^iX^i)$. On the other hand, from (3.1), the vector field $\overline{\varphi}(X^i, Y^i, Z^j)$ is decomposed into the tangential and normal components as follows:

$$\overline{\varphi}(X^i, Y^i, Z^j) = f(X^i, Y^i, Z^j) + \sum_{\alpha=1}^{2} \upsilon_\alpha\,(X^i, Y^i, Z^j)N_\alpha. \tag{3.37}$$

Hence, it is deducible from (3.35), (3.36), and (3.37) that

$$f(X^i, Y^i, Z^j) = \left(\frac{mr^2+\mu\sqrt{m^2-4}}{2r^2}X^i - \frac{\sqrt{m^2-4}}{2}Y^i, \frac{mr^2+\mu\sqrt{m^2-4}}{2r^2}Y^i - \frac{\sqrt{m^2-4}}{2}X^i, B_mZ^j\right), \tag{3.39}$$

where $m \in \mathbb{R}\backslash[-2,2]$.

Therefore, we obtain an induced structure $(f, \langle,\rangle, \upsilon_\alpha, \zeta_\alpha, \mathcal{A})$ on M by the almost bronze Riemannian structure $(\langle,\rangle, \overline{\varphi})$ on the Euclidean space E^{2a+b}. As a result, M is a non-invariant submanifold of codimension 2 in the Euclidean space E^{2a+b}.

Lemma 3.1. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in an almost poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. If the tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent, then the trace of the tensor field f of the induced structure $\left(f, g, \upsilon_\alpha, \zeta_\alpha, (\theta_{\alpha\beta})_{k\times k}\right)$ is given by

$$tr(f) = \begin{cases} mk - tr(\theta_{\alpha\beta}) + \sum_{A=k+1}^{n} \rho_A, \ k < n \\ mk - tr(\theta_{\alpha\beta}), k = n \end{cases}, \tag{3.40}$$

where $\rho_A \in \{B_m, m - B_m\}$.

Proof: We denote by (f) the matrix associated with the tensor field f. Taking $U = (\zeta_1 \cdots \zeta_k)$, then it is deducible from (3.10) that

$$(f)U = U(m\delta_{\alpha\beta} - \theta_{\alpha\beta}) \tag{3.41}$$

for any $\alpha, \beta \in \{1, \dots, k\}$, where $\delta_{\alpha\beta}$ denotes the Kronecker delta. Hence, if $k = n$, then the matrix (f) is given by

$$(f) = U(m\delta_{\alpha\beta} - \theta_{\alpha\beta})U^{-1}, \tag{3.42}$$

from which have

$$tr(f) = mk - tr(\theta_{\alpha\beta}). \tag{3.43}$$

If $k < n$, then we consider two matrices $\overline{U}$ and L defined by

$$\overline{U} = (\zeta_1 \cdots \zeta_k \eta_{k+1} \cdots \eta_n) \tag{3.44}$$

and

$$L = \begin{pmatrix} m\delta_{\alpha\beta} - \theta_{\alpha\beta} & 0 \\ 0 & \rho_A \delta_{AB} \end{pmatrix}, \tag{3.45}$$

respectively. In this case, we obtain from (3.10) that

$$(f)\overline{U} = \overline{U}L. \tag{3.46}$$

Since $\det(\overline{U}) \neq 0$, we get

$$(f) = \overline{U}L\overline{U}^{-1}. \tag{3.47}$$

Thus, it follows that

$$tr(f) = mk - tr(\theta_{\alpha\beta}) + \sum_{A=k+1}^{n} \rho_A, \tag{3.48}$$

which completes the proof.

Lemma 3.2. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in an almost poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then $tr(f)$ is constant if the following assertions hold:
a) The tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent,
b) $\nabla f = 0$.

Proof: Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space $T_P M$ at a point $p \in M$. We extend e_i's to the local vector fields, denoted by E_i's, which are orthonormal and covariantly constant at the point $p \in M$ for any $i \in \{1, \dots, n\}$. It is well known that the trace of the tensor field f is given by

$$tr(f) = \sum_{i=1}^{n} g\,(fe_i, e_i). \tag{3.49}$$

Using the fact that $\nabla g = 0$, it is easy to verify that

$$\nabla_X tr(f) = \left\{\sum_{i=1}^{n} g(\nabla_X f E_i, E_i)\right\}_p + \left\{\sum_{i=1}^{n} g(fE_i, \nabla_X E_i)\right\}_p. \tag{3.50}$$

From the definition of the covariant derivative of f, (3.50) turns into

$$\nabla_X tr(f) = \left\{\sum_{i=1}^{n} g\big((\nabla_X f)E_i, E_i\big)\right\}_p + 2\left\{\sum_{i=1}^{n} g(\nabla_X E_i, fE_i)\right\}_p. \tag{3.51}$$

Since E_i is the local extension of e_i for each $i \in \{1, \dots, n\}$, we obtain

$$\nabla_X tr(f) = \sum_{i=1}^{n} g\left((\nabla_X f)e_i, e_i\right) \tag{3.52}$$

for any vector field $X \in \Gamma(TM)$, which ends the proof from the assumption that $\nabla f = 0$.

Remark 3.1. Taking account of (3.40), $tr(f)$ is constant if and only if so is $tr\big(\theta_{\alpha\beta}\big)$.

Lemma 3.3. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $\big(\overline{M}, \overline{g}, \overline{\varphi}\big)$. If the tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent and $tr(f)$ is constant, then we have

$$\sum_{\alpha=1}^{k} h_\alpha(X, \zeta_\alpha) = 0, \text{or equivalently} \sum_{\alpha=1}^{k} A_\alpha \zeta_\alpha = 0 \tag{3.53}$$

for any vector field $X \in \Gamma(TM)$.

Proof: Taking $\alpha = \beta$ in (3.18), we get

$$2h_\alpha(X, \zeta_\alpha) + \nabla_X \theta_{\alpha\alpha} + 2\sum_{\gamma=1}^{k} \theta_{\alpha\gamma}\, \sigma_{\gamma\alpha}(X) = 0 \tag{3.54}$$

for any vector field $X \in \Gamma(TM)$, from which we have

$$2\sum_{\alpha=1}^{k} h_\alpha(X, \zeta_\alpha) + \nabla_X\, tr(f) + 2\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k} \theta_{\alpha\gamma}\, \sigma_{\gamma\alpha}(X) = 0. \tag{3.55}$$

On the other hand, it follows from (3.5) and (3.8) that

$$\sum_{\alpha=1}^{k}\sum_{\gamma=1}^{k} \theta_{\alpha\gamma}\, \sigma_{\gamma\alpha}(X) = 0. \tag{3.56}$$

Thus, (3.55) becomes

$$2\sum_{\alpha=1}^{k} h_\alpha(X, \zeta_\alpha) + \nabla_X tr(f) = 0. \tag{3.57}$$

Consequently, the assumption of the constancy of $tr(f)$ finishes the proof.

Theorem 3.1. [14, Theorem 3.7] Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then M is a totally geodesic submanifold if the following assertions are valid:
a) The tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent,
b) $\nabla f = 0$.

Theorem 3.2. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then M is a totally geodesic submanifold if the following assertions are satisfied:
a) The tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent,
b) $tr(f)$ is constant,
c) M is a totally umbilical submanifold.

Proof: From the totally umbilicality of M, the second fundamental tensors h_α's are written in the following forms:

$$h_\alpha = c_\alpha g \tag{3.58}$$

for each $\alpha = 1, \dots, k$, where c_α's are constants. Taking account of Lemma 3.3, if we substitute (3.58) into (3.53), then a direct calculation gives us

$$\sum_{\alpha=1}^{k} c_\alpha \zeta_\alpha = 0. \tag{3.59}$$

At the same time, the linear independence of ζ_α's implies that $c_\alpha = 0$ for each $\alpha = 1, \dots, k$, which is the desired relation. Hence, we get that M is a totally geodesic submanifold.

Theorem 3.3. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then M is a minimal submanifold if the following assertions are verified:
a) The tangent vector fields $\zeta_1, \dots, \zeta_k$ are linearly independent,
b) $tr(f)$ is constant,
c) $\sum_{i=1}^{n} (\nabla_{e_i} f) e_i = 0$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space $T_p M$ at a point $p \in M$.

Proof: Taking $X_p = Y_p = e_i$ at the point $p \in M$ in (3.15), we obtain

$$(\nabla_{e_i} f) e_i = \sum_{\alpha=1}^{k} h_\alpha(e_i, e_i) \zeta_\alpha + \sum_{\alpha=1}^{k} \upsilon_\alpha(e_i) A_\alpha e_i. \tag{3.60}$$

By means of the assumption that $\sum_{i=1}^{n} (\nabla_{e_i} f) e_i = 0$, we derive

$$\sum_{\alpha=1}^{k} \left(\sum_{i=1}^{n} h_\alpha(e_i, e_i) \zeta_\alpha + A_\alpha \zeta_\alpha \right) = 0. \tag{3.61}$$

Hence, by Lemma 3.3, (3.61) takes the form

$$\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h_{\alpha}(e_i,e_i)\right)\zeta_{\alpha}=0. \tag{3.62}$$

Since the tangent vector fields $\zeta_1,\dots,\zeta_k$ are linearly independent, it follows from (3.62) that

$$\sum_{i=1}^{n} h_{\alpha}\,(e_i,e_i)=0, \tag{3.63}$$

which is equivalent to the minimality of M.

Lemma 3.4. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in an almost poly-Norden Riemannian manifold $(\overline{M},\overline{g},\overline{\varphi})$. If $\theta_{\alpha\beta}=\rho_{\alpha}\delta_{\alpha\beta}$, $\rho_{\alpha}\in(m-B_m,B_m)$ with $m\in\mathbb{R}\backslash[-2,2]$ for any $\alpha,\beta\in\{1,\dots,k\}$, then the tangent vector fields $\zeta_1,\dots,\zeta_k$ are linearly independent such that $f\zeta_{\alpha}=(m-\rho_{\alpha})\zeta_{\alpha}$.

Proof: We assume that $\theta_{\alpha\beta}=\rho_{\alpha}\delta_{\alpha\beta}$, $\rho_{\alpha}\in(m-B_m,B_m)$ with $m\in\mathbb{R}\backslash[-2,2]$ for any $\alpha,\beta\in\{1,\dots,k\}$. From (3.8) and (3.9), we derive

$$\upsilon_{\beta}(\zeta_{\alpha})=\delta_{\alpha\beta}(-1+m\rho_{\alpha}-\rho_{\alpha}^{2}). \tag{3.64}$$

Thus, it is understood from (3.11) that we get

$$g(\zeta_{\alpha},\zeta_{\beta})=\delta_{\alpha\beta}(-1+m\rho_{\alpha}-\rho_{\alpha}^{2}). \tag{3.65}$$

Moreover, from the assumption that $\rho_{\alpha}\in(m-B_m,B_m)$ with $m\in\mathbb{R}\backslash[-2,2]$, we have

$$-1+m\rho_{\alpha}-\rho_{\alpha}^{2}\neq 0. \tag{3.66}$$

Hence, (3.66) implies that the tangent vector fields $\zeta_1,\dots,\zeta_k$ are linearly independent. In addition, it can be easily deducible from (3.10) that

$$f(\zeta_{\alpha})=(m-\rho_{\alpha})\zeta_{\alpha}. \tag{3.67}$$

Therefore, the proof has been completed.

Using Lemma 3.4 in Theorems 3.1, 3.2, and 3.3, respectively, it is clear to see that the following three theorems are valid:

Theorem 3.4. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M},\overline{g},\overline{\varphi})$. Then M is a totally geodesic submanifold if the following assertions are satisfied:
a) $\theta_{\alpha\beta}=\rho_{\alpha}\delta_{\alpha\beta}$, $\rho_{\alpha}\in(m-B_m,B_m)$ with $m\in\mathbb{R}\backslash[-2,2]$ for any $\alpha,\beta\in\{1,\dots,k\}$,
b) $\nabla f=0$.

Theorem 3.5. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M},\overline{g},\overline{\varphi})$. Then M is a totally geodesic submanifold if the following assertions are verified:

a) $\theta_{\alpha\beta} = \rho_\alpha \delta_{\alpha\beta}$, $\rho_\alpha \in (m - B_m, B_m)$ with $m \in \mathbb{R}\backslash[-2,2]$ for any $\alpha, \beta \in \{1, \dots, k\}$,
b) $tr(f)$ is constant,
c) M is a totally umbilical submanifold.

Theorem 3.6. Let M be an n-dimensional submanifold of codimension k, isometrically immersed in a poly-Norden Riemannian manifold $(\overline{M}, \overline{g}, \overline{\varphi})$. Then M is a minimal submanifold if the following assertions hold:
a) $\theta_{\alpha\beta} = \rho_\alpha \delta_{\alpha\beta}$, $\rho_\alpha \in (m - B_m, B_m)$ with $m \in \mathbb{R}\backslash[-2,2]$ for any $\alpha, \beta \in \{1, \dots, k\}$,
b) $tr(f)$ is constant,
c) $\sum_{i=1}^{n} (\nabla_{e_i} f)\, e_i = 0$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space T_pM at a point $p \in M$.

Remark 3.2. Since every almost bronze Riemannian manifold is an almost poly-Norden Riemannian manifold with $m \in \mathbb{R}\backslash[-2,2]$, under the assumption that the ambient manifold is an almost bronze Riemannian manifold, there is no need to write the condition $m \in \mathbb{R}\backslash[-2,2]$ in Lemma 3.4, Theorems 3.4, 3.5, and 3.6.

4. CONCLUSION

In this work, we have studied non-invariant submanifolds of almost poly-Norden Riemannian manifolds. New sufficient conditions for the totally geodesicity and minimality of an arbitrary non-invariant submanifold of an almost poly-Norden Riemannian manifold are obtained via the induced structure on the submanifold by the poly-Norden structure of the ambient manifold.

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