ORIGINAL PAPER

A NOTE ON THE THIRD FOCUS OF THE CARTESIAN OVAL

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Abstract. *The subject of this paper is the determination of the coordinate of the third focus of the Cartesian oval. For this purpose, the Computer Algebra System Maple was used with the use of the pseudo-resultant technique and Groebner bases. The third focus formula and consistency conditions for determining the third focus were obtained. The algebraic formula for the coordinate of the third focus when the focal points with general coordinates p, q, r were obtained and the consistency conditions for defining the Cartesian oval through the third focus were also presented.*
Keywords: *Cartesian ovals; Groebner base; Focal points.*

1. INTRODUCTION

The discussion of Cartesian ovals dates back to 984 when Ibn Sahl in a treatise *On the burning instruments* studied the design of stigmatic lenses [1]. The general equation of the oval was given in the 17th century by René Descartes (lat. Renatus Cartesius) when studying the paths of light rays, the mathematical properties of the oval, and its application in optics. [2].

Shown here is the definition of an oval according to [3, 4]. Let $F_1(p,0)$ and $F_2(q,0)$ be two different fixed points in a plane (for some $p,q \in \mathbb{R}$) and let distances from $M(x,y)$ to $F_1(p,0)$ and $F_2(q,0)$ be marked with R_1 and R_2, respectively. For positive real numbers m_1, n_1, S_1, the geometric locus of the points *M* in the plane, which satisfies

$$m_1 R_1 \pm n_1 R_2 = S_1 \tag{1}$$

determines the Cartesian oval.

This paper considers the case when the Cartesian oval exists and when both equations in (1) are possible. The conditions for the existence of the Cartesian oval are stated in [5] and [6]. The geometric locus of the points in the plane corresponding to the positive sign in (1) will be denoted by ξ_1 and will be called the first branch of the Cartesian oval. The geometric locus of the points in the plane corresponding to the negative sign in (1) will be denoted by ξ_2 and will be called the second branch of the Cartesian oval. The paper considers the Cartesian oval under the assumption that both branches exist (see Fig. 1).

It should be noted that when

- $m_1 = n_1$ and $S_1 > n_1(q-p)$ the Cartesian oval becomes an ellipse,
- $m_1 = -n_1$ and $S_1 < n_1(q-p)$ it becomes a hyperbole, and

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- $m_1 / n_1 = S_1 / (q-p)$ it becomes a limacon of Pascal (Pascal's Snail) [7, 3].

The previously mentioned cases (ellipse, hyperbola, Pascal's snail) are called degenerative cases of the Cartesian oval, and the next section of this paper considers the derivation of the general equation of the Cartesian oval.

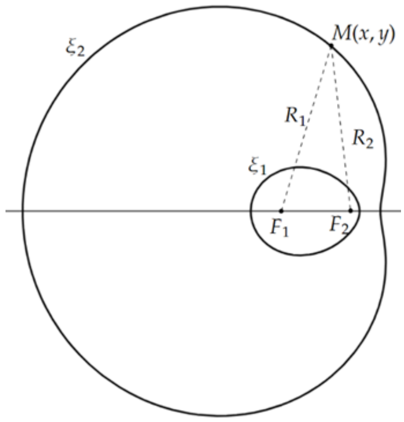

Figure 1. Cartesian oval

2. MAIN RESULTS

2.1. DERIVATION OF THE ALGEBRAIC EQUATION OF THE CARTESIAN OVAL

Without diminishing the generality, instead of (1) the following is considered:

$$m_1 R_1 + n_1 R_2 = S_1 \tag{2}$$

where $m_1 > n_1$ and $q > p$. Thus, only the case of the branch ξ_1 is considered. An analogous consideration can be made for the case of the branch ξ_2.

The equation (2) can be written in the following form:

$$m_1\sqrt{(x-p)^2+y^2} + n_1\sqrt{(x-q)^2+y^2} = S_1 . \tag{3}$$

Starting from (3), it is possible to perform double squaring and determine the algebraic equation of the Cartesian oval in two ways, specifying the appropriate conditions.

If (3) is written in the form:

$$m_1\sqrt{(x-p)^2+y^2} = S_1 - n_1\sqrt{(x-q)^2+y^2} \tag{4}$$

to obtain the algebraic equation of the Cartesian oval, it is necessary to square the equation (4) twice, whereby the following conditions must apply:

$$(x-q)^2 + y^2 < \left(\frac{S_1}{n_1}\right)^2 \tag{5}$$

and

$$\left(x-\frac{m_1^2p-n_1^2q}{m_1^2-n_1^2}\right)^2+y^2<\frac{S_1^2(m_1^2-n_1^2)+m_1^2n_1^2(p-q)^2}{(m_1^2-n_1^2)^2} \tag{6}$$

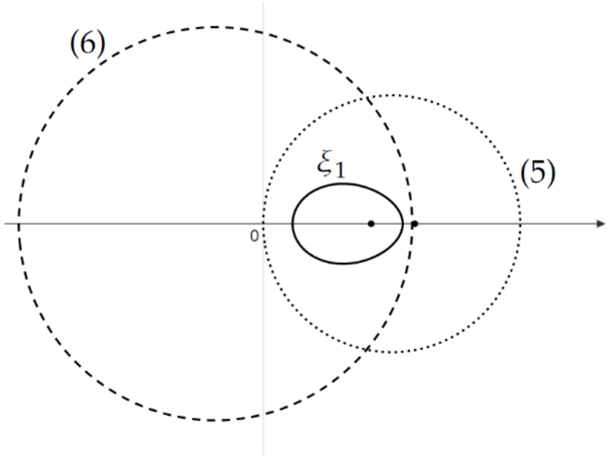

Figure 2. Branch ξ_1 position in relation to conditions (5) and (6)

If (3) is written in the form:

$$n_1\sqrt{(x-q)^2+y^2}=S_1-m_1\sqrt{(x-p)^2+y^2} \tag{7}$$

to obtain the algebraic equation of the Cartesian oval, it is necessary to square the equation (7) twice, whereby the following conditions must apply:

$$(x-p)^2+y^2<\left(\frac{S_1}{m_1}\right)^2 \tag{8}$$

and

$$\left(x-\frac{n_1^2p-m_1^2q}{n_1^2-m_1^2}\right)^2+y^2>\frac{S_1^2(n_1^2-m_1^2)+m_1^2n_1^2(p-q)^2}{(n_1^2-m_1^2)^2} \tag{9}$$

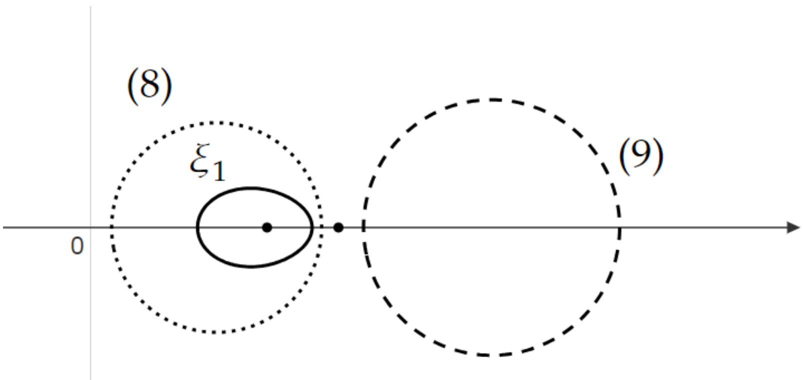

Figure 3. Branch ξ_1 position in relation to conditions (8) and (9)

The result of double squaring in both ways is the algebraic equation of the Cartesian oval:

$$
\begin{aligned}
\boldsymbol{C}_{\mathbf{12}} &= (m_1^4 - 2m_1^2n_1^2 + n_1^4)x^4 + (-4m_1^4p + 4m_1^2n_1^2p + 4m_1^2n_1^2q - 4n_1^4q)x^3 + \\
&+ (2m_1^4 - 4m_1^2n_1^2 + 2n_1^4)x^2y^2 + \\
&+ (6m_1^4p^2 - 2m_1^2n_1^2p^2 - 8m_1^2n_1^2pq - 2m_1^2n_1^2q^2 + 6n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2)x^2 + \\
&+ (-4m_1^4p + 4m_1^2n_1^2p + 4m_1^2n_1^2q - 4n_1^4q)xy^2 + \\
&+ (-4m_1^4p^3 + 4m_1^2n_1^2p^2q + 4m_1^2n_1^2pq^2 - 4n_1^2q^3 + 4m_1^2pS_1^2 + 4n_1^2qS_1^2)x + \\
&+ (m_1^4 - 2m_1^2n_1^2 + n_1^4)y^4 + (2m_1^4p^2 - 2m_1^2n_1^2p^2 - 2m_1^2n_1^2q^2 + 2n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2)y^2 + \\
&+ m_1^4p^4 - 2m_1^2n_1^2p^2q^2 + n_1^4q^4 - 2m_1^2p^2S_1^2 - 2n_1^2q^2S_1^2 + S_1^4 = 0
\end{aligned}
\tag{10}
$$

When determining the algebraic equation of the Cartesian oval in these two ways, based on (5), (6), (8), and (9), it can be concluded that the points of one branch of the Cartesian oval are located inside or outside the corresponding circles (see Figs. 2-4) whereby the result for one branch of the oval is specified according to [8].

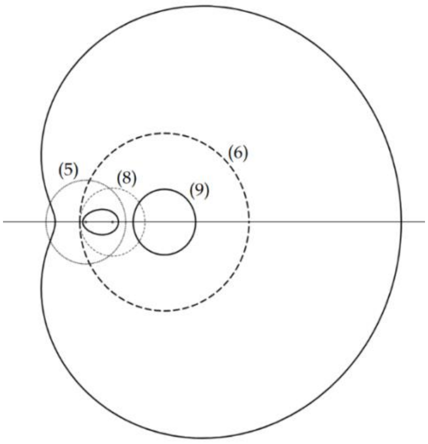

Figure 4. Cartesian oval and conditions for squaring.

The equation of the Cartesian oval (10) can be written in the form:

$$\boldsymbol{C}_{\mathbf{12}} = A_9x^4 + A_8x^3 + A_7x^2y^2 + A_6x^2 + A_5xy^2 + A_4x + A_3y^4 + A_2y^2 + A_1 = 0 \tag{11}$$

where the coefficients A_i, $(i = 1..9)$ are given by:

$$
\begin{aligned}
A_9 &= A_3 = \frac{1}{2}A_7 = (m_1^2 - n_1^2)^2 \\
A_8 &= A_5 = 4(m_1^2 - n_1^2)(n_1^2q - m_1^2p) \\
A_6 &= 6m_1^4p^2 - 2m_1^2n_1^2p^2 - 8m_1^2n_1^2pq - 2m_1^2n_1^2q^2 + 6n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2 \\
A_4 &= -4m_1^4p^3 + 4m_1^2n_1^2p^2q + 4m_1^2n_1^2pq^2 - 4n_1^4q^3 + 4m_1^2pS_1^2 + 4n_1^2qS_1^2 \\
A_2 &= 2m_1^4p^2 - 2m_1^2n_1^2p^2 - 2m_1^2n_1^2q^2 + 2n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2 \\
A_1 &= ((m_1p - n_1q)^2 - S_1^2)((m_1p + n_1q)^2 - S_1^2)
\end{aligned}
\tag{12}
$$

In this way, the Cartesian oval (11) was determined using the foci $F_1(p,0)$, $F_2(q,0)$ and the coefficients m_1, n_1, S_1.

2.2. DETERMINING THE COORDINATE OF THE THIRD FOCUS THROUGH PSEUDO-RESULTANTS

According to [9], Michel Chasles demonstrated, using geometrical methods, that for some $r \in \mathbb{R}$ there is a third focus $F_3(r,0)$ of the Cartesian oval. Foci F_1, F_2 and F_3 are collinear and the literature states that the third focus is obtained through appropriate geometrical constructions [10].

The Cartesian oval (11) is first determined with two foci F_1 and F_2 and with parameters m_1, n_1, S_1. According to the geometric results, it will be demonstrated algebraically that the same branch of the oval can be obtained using the foci F_2 and F_3, for some $m_2,\, n_2,\, S_2$ in the form:

$$m_2 R_2 + n_2 R_3 = S_2 \tag{13}$$

Following the same procedure as in the previous section, the Cartesian oval is given by:

$$\boldsymbol{C_{23}} = B_9 x^4 + B_8 x^3 + B_7 x^2 y^2 + B_6 x^2 + B_5 x y^2 + B_4 x + B_3 y^4 + B_2 y^2 + B_1 = 0 \tag{14}$$

where the coefficients $B_i,\ (i = 1..9)$ are given by:

$$\begin{aligned}
B_9 &= B_3 = \frac{1}{2} AB = (m_2^2 - n_2^2)^2 \\
B_8 &= A_5 = 4(m_2^2 - n_2^2)(n_2^2 q - m_2^2 p) \\
B_6 &= 6m_2^4 p^2 - 2m_2^2 n_2^2 p^2 - 8m_2^2 n_2^2 pq - 2m_2^2 n_2^2 q^2 + 6n_2^4 q^2 - 2S_2^2 m_2^2 - 2S_2^2 n_2^2 \\
B_4 &= -4m_2^4 p^3 + 4m_2^2 n_2^2 p^2 q + 4m_2^2 n_2^2 pq^2 - 4n_2^4 q^3 + 4m_2^2 p S_2^2 + 4n_2^2 q S_2^2 \\
B_2 &= 2m_2^4 p^2 - 2m_2^2 n_2^2 p^2 - 2m_2^2 n_2^2 q^2 + 2n_2^4 q^2 - 2S_2^2 m_2^2 - 2S_2^2 n_2^2 \\
B_1 &= ((m_2 p - n_2 q)^2 - S_2^2)(((m_2 p + n_2 q)^2 - S_2^2)
\end{aligned} \tag{15}$$

According to the works of Chasles, the algebraic equations (11) and (14) determine the same Cartesian oval, whereby the coefficients A_i, B_i with the corresponding monomials in (11) and (14) are proportional. Based on that, for i, j = 1..9 a polynomial system is obtained:

$$A_i B_j - A_j B_i = 0. \tag{16}$$

In this way, a polynomial system of 81 equations with $m_1, n_1, S_1, m_2, n_2, S_2$ is formed. To solve that system, one approach is to determine the solution using a Computer Algebra System. Specifically, in this paper, the software `Maple` is used. Here is the solution procedure. First, a list of equations is formed:

$$L = [A_i B_j - A_j B_i = 0 \,|\, i, j = 1..9]. \tag{17}$$

The solution is determined using the command

`solve`$(L, [m_1, n_1, S_1, r])$.

Initially, `infolevel[solve]:=5` is set so that the methods and steps of the solution procedure can be followed. Based on the execution of the `Maple` software, the information is obtained that the pseudo-resultants method was used, see [11], [12]. The result is a list of 11 solutions, 7 of which are trivial:

$$\begin{array}{llll}
1.\ m_2=0 & n_2=0 & r=r & S_2=0, \\
2.\ m_2=n_2 & n_2=n_2 & r=q & S_2=0, \\
3.\ m_2=-n_2 & n_2=n_2 & r=q & S_2=0, \\
4.\ m_2=\dfrac{n_1n_2}{m_1} & n_2=n_2 & r=p & S_2=-\dfrac{S_1n_2}{m_1}, \\
5.\ m_2=\dfrac{n_1n_2}{m_1} & n_2=n_2 & r=p & S_2=\dfrac{S_1n_2}{m_1}, \\
6.\ m_2=-\dfrac{n_1n_2}{m_1} & n_2=n_2 & r=p & S_2=-\dfrac{S_1n_2}{m_1}, \\
7.\ m_2=-\dfrac{n_1n_2}{m_1} & n_2=n_2 & r=p & S_2=\dfrac{S_1n_2}{m_1}.
\end{array} \tag{18}$$

and four non-trivial:

$$\begin{array}{l}
1.\ m_2=m_2,\ n_2=-\dfrac{m_1m_2(p-q)}{S_1}, S_2=-\dfrac{m_2n_1(p^2m_1^2-2m_1^2pq+q^2m_1^2-S_1^2)}{S_1(m_1^2-n_1^2)}, r=\dfrac{(m_1^2p-n_1^2q)(p-q)-S_1^2}{(m_1^2-n_1^2)(p-q)}; \\
2.\ m_2=m_2,\ n_2=-\dfrac{m_1m_2(p-q)}{S_1}, S_2=\dfrac{m_2n_1(p^2m_1^2-2m_1^2pq+q^2m_1^2-S_1^2)}{S_1(m_1^2-n_1^2)}, r=\dfrac{(m_1^2p-n_1^2q)(p-q)-S_1^2}{(m_1^2-n_1^2)(p-q)}; \\
3.\ m_2=m_2,\ n_2=\dfrac{m_1m_2(p-q)}{S_1}, S_2=-\dfrac{m_2n_1(p^2m_1^2-2m_1^2pq+q^2m_1^2-S_1^2)}{S_1(m_1^2-n_1^2)}, r=\dfrac{(m_1^2p-n_1^2q)(p-q)-S_1^2}{(m_1^2-n_1^2)(p-q)}; \\
4.\ m_2=m_2,\ n_2=\dfrac{m_1m_2(p-q)}{S_1}, S_2=\dfrac{m_2n_1(p^2m_1^2-2m_1^2pq+q^2m_1^2-S_1^2)}{S_1(m_1^2-n_1^2)}, r=\dfrac{(m_1^2p-n_1^2q)(p-q)-S_1^2}{(m_1^2-n_1^2)(p-q)}.
\end{array} \tag{19}$$

Among the non-trivial solutions, it can be seen that a unique solution is determined for the coordinate of the third focus of the Cartesian oval:

$$r=\frac{(m_1^2p-n_1^2q)(p-q)-S_1^2}{(m_1^2-n_1^2)(p-q)} \tag{20}$$

and in addition, the consistency conditions are also obtained:

$$\frac{n_2}{m_2}=\pm\frac{m_1(p-q)}{S_1} \tag{21}$$

and

$$\frac{S_2}{m_2} = \pm \frac{n_1(p^2 m_1^2 - 2m_1^2 pq + q^2 m_1^2 - S_1^2)}{S_1(m_1^2 - n_1^2)} \tag{22}$$

The right-hand sides of the connections (21) and (22) are calculated with a positive and negative sign. If the parameters $m_2,\ n_2,\ S_2,\ (S_2 > 0)$ fulfill those connections, then with (13) both branches of the Cartesian oval are obtained. It should be emphasized that the results for the coordination of the third focus in the general form with consistency conditions are not stated in the available literature.

As an illustration of the above, a simple example is provided.

Example. Let the Cartesian oval be determined by the foci $F_1(2, 0)$, $F_2(5, 0)$ and parameters $m_1 = 4$, $n_1 = 3$ and $S_1 = 15$ with (1). The algebraic equation of the Cartesian oval is

$$\boldsymbol{C_{12}} = 49x^4 + 364x^3 + 98x^2y^2 - 12828x^2 + 364xy^2 + 60928x + 49y^4 - 13504y^2 - 53504 = 0 \tag{23}$$

According to (20), what is obtained is that

$$F_3\left(\frac{62}{7}, 0\right).$$

With every choice m_2, n_2, S_2 so that (21) and (22) hold, we have

$$\frac{n_2}{m_2} = \pm\frac{4}{5}$$

and

$$\frac{S_2}{m_2} = \pm\frac{81}{35}.$$

Then, for a specific choice of signs, the corresponding branch of the same Cartesian oval is obtained from (13). Specifically, when choosing $m_2 = 1,\ n_2 = \frac{4}{5},\ S_2 = \frac{81}{35}$ we have the algebraic equation of the same Cartesian oval

$$\boldsymbol{C_{23}} = \frac{81}{625}x^4 + \frac{162}{625}x^2y^2 + \frac{81}{625}y^4 + \frac{4212}{4375}x^3 + \frac{4212}{4375}y^2x - \frac{1039068}{30625}x^2 - \\ - \frac{1093824}{30625}y^2 + \frac{705024}{4375}x - \frac{4333824}{30625} = 0. \tag{24}$$

Whereby

$$\frac{C_{12}}{C_{23}} = \frac{30625}{81}.$$

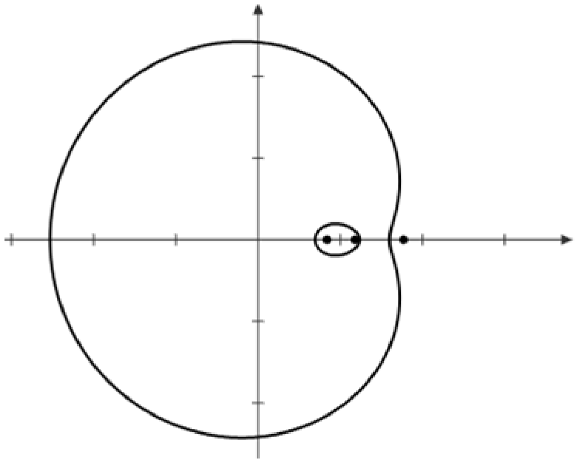

Figure 5. Cartesian oval with foci F_1, F_2, F_3

2.3. CARTESIAN OVAL AND GROEBNER'S BASES

It is assumed that the basic notions of ideals, Groebner bases, and reduced Groebner bases are known at the level of [13] and [14]. In this part, by applying the Groebner bases technique, the equation of the Cartesian oval and the coordinate of the third focus of the Cartesian oval are simply determined.

2.3.1. Determining the equation of the Cartesian oval

The starting point is the polynomial ideal

$$I = \left\langle m_1 R_1 + n_1 R_2 - S_1, (x-p)^2 + y^2 - R_1^2,\ (x-q)^2 + y^2 - R_2^2 \right\rangle \tag{25}$$

where the variables $x, y,\ R_1, R_2$ are in lexicographic order

$$R_1 < R_2 < x < y$$

and where $m_1,\ n_1,\ S_1$ are positive real numbers. Determining the reduced Groebner base using the Computer Algebra System `Maple` for ideal I is accomplished with the command

GB1 := Basis(I, plex(R_1, R_2, p, q, x, y))

and the result is a list of polynomials:

$$\begin{aligned}
\texttt{GB1} = [&(m_1^4 - 2m_1^2n_1^2 + n_1^4)x^4 + (-4m_1^4p + 4m_1^2n_1^2p + 4m_1^2n_1^2q - 4n_1^4q)x^3 + \\
&+(2m_1^4 - 4m_1^2n_1^2 + 2n_1^4)x^2y^2 + \\
&+(6m_1^4p^2 - 2m_1^2n_1^2p^2 - 8m_1^2n_1^2pq - 2m_1^2n_1^2q^2 + 6n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2)x^2 + \\
&+(-4m_1^4p + 4m_1^2n_1^2p + 4m_1^2n_1^2q - 4n_1^4q)xy^2 + \\
&+(-4m_1^4p^3 + 4m_1^2n_1^2p^2q + 4m_1^2n_1^2pq^2 - 4n_1^2q^3 + 4m_1^2pS_1^2 + 4n_1^2qS_1^2)x + \\
&+(m_1^4 - 2m_1^2n_1^2 + n_1^4)y^4 + (2m_1^4p^2 - 2m_1^2n_1^2p^2 - 2m_1^2n_1^2q^2 + 2n_1^4q^2 - 2S_1^2m_1^2 - 2S_1^2n_1^2)y^2 + \\
&+m_1^4p^4 - 2m_1^2n_1^2p^2q^2 + n_1^4q^4 - 2m_1^2p^2S_1^2 - 2n_1^2q^2S_1^2 + S_1^4,\ (m_1^2 - n_1^2)x^2 + (m_1^2 - n_1^2)y^2 + \\
&+(-2m_1^2p + 2n_1^2q)x + m_1^2p^2 - n_1^2q^2 + 2n_1R_2S_1 - S_1^2, (-m_1^2 + n_1^2)x^2 + (-m_1^2 + n_1^2)y^2 + \\
&+(2m_1^2p - 2n_1^2q)x - m_1^2p^2 + n_1^2q^2 + 2m_1R_1S_1 - S_1^2]
\end{aligned} \tag{26}$$

The reduced Groebner basis determined in this way has three elements, where the first element is precisely the polynomial of the Cartesian oval (11) and thus the equation of the Cartesian oval is also determined.

2.3.2 Determination of the coordinates of the third focus

The starting point is the polynomial ideal

$$J = \left\langle A_i \cdot B_j - A_j \cdot B_i \mid i, j = 1..9 \right\rangle, \tag{27}$$

which consists of 81 polynomials by variables $m_2,\ n_2,\ S_2, r$, while the default parameters are $m_1,\ n_1,\ S_1,\ p,\ q$. Let the lexicographic order hold for $m_2,\ n_2,\ S_2, r$

$$m_2 < n_2 < S_2 < r$$

and the remaining parameters $m_1,\ n_1,\ S_1,\ p,\ q$ are positive real numbers. Determining the reduced Groebner base using the Computer Algebra System Maple for the ideal J is accomplished with the command

$$\texttt{GB2} := \texttt{Basis}(J, \texttt{plex}(m_2, n_2, S_2, r))$$

and the result is a list of polynomials:

$$\texttt{GB2} = [g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9] \tag{28}$$

where the polynomials $g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9$ are listed in the Appendix of this paper due to the scope of some of their notations.

Note that the polynomial ideal J determines the system of equations

$$\bigwedge_{i,j=1..9} A_i \cdot B_j - A_j \cdot B_i = 0 \tag{29}$$

The previous system is equivalent to the system obtained from the reduced Groebner basis GB2

$$\begin{cases} g_1 = 0, & g_2 = 0, & g_3 = 0, \\ g_4 = 0, & g_5 = 0, & g_6 = 0, \\ g_7 = 0, & g_8 = 0, & g_9 = 0 \end{cases} \tag{30}$$

By symbolically solving all the equations from (30) in *r*, it can be concluded that the solution given by (20) is their only common solution.

3. CONCLUSION

In this paper, the formula for the coordinate of the third focus of the Cartesian oval was obtained in two ways: by solving a system of polynomial equations using pseudo-resultants and by applying Groebner's bases. It should be noted that the procedure using Groebner's bases yields only the equation of the oval, but not the consistency conditions (21) and (22), so the pseudo-resultant method gives a complete answer. Finally, it is of note that in modern research in optics, the use of the Cartesian oval is present [15-17]. For the obtained formula of the coordinates of the third focus, we expect it to be of interest in all areas where consideration of the focus of the Cartesian oval occurs.

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APPENDIX

$$g_1 = (m_1^2 p - m_1^2 q - n_1^2 p + n_1^2 q) r^2 S_2^6 + (-2m_1^2 p^2 + 2m_1^2 pq + n_1^2 p^2 - n_1^2 q^2 + S_1^2) r S_2^6 +$$
$$+ (m_1^2 p^3 - m_1^2 p^2 q - n_1^2 p^2 q + n_1^2 pq^2 - pS_1^2) S_2^6$$

$$g_2 = (m_1^2 p - m_1^2 q) n_2^2 r^2 S_2^2 + (-m_1^2 p^2 + m_1^2 q^2 + S_1^2) n_2^2 r S_2^2 + (m_1^2 p^2 q - m_1^2 pq^2 - qS_1^2) n_2^2 S_2^2 +$$
$$+ (-m_1^2 p + m_1^2 q) S_2^4$$

$$g_3 = n_1^2 S_2^4 (m_1^4 n_1^2 p^4 S_1^2 - 4m_1^4 n_1^2 p^3 q S_1^2 + 6m_1^4 n_1^2 p^2 q^2 S_1^2 - 4m_1^4 n_1^2 pq^3 S_1^2 + m_1^4 n_1^2 q^4 S_1^2 - 2m_1^2 n_1^2 p^2 S_1^4 +$$
$$+ 4m_1^2 n_1^2 pq S_1^4 - 2m_1^2 n_1^2 q^2 S_1^4 + n_1^2 S_1^6) + r S_2^6 (m_1^8 p^3 - 3m_1^8 p^2 q + 3m_1^8 pq^2 - m_1^8 q^3 - m_1^6 n_1^2 p^3 +$$
$$+ 3m_1^6 n_1^2 p^2 q - 3m_1^6 n_1^2 pq^2 + m_1^6 n_1^2 q^3 - m_1^4 n_1^2 p S_1^2 + m_1^4 n_1^2 q S_1^2 + m_1^4 n_1^4 p S_1^2 - m_1^4 n_1^2 q S_1^2) +$$
$$+ S_2^6 (-m_1^8 p^4 + 3m_1^8 p^3 q - 3m_1^8 p^2 q^2 + m_1^8 pq^3 + m_1^6 n_1^2 p^3 q - 3m_1^6 n_1^2 p^2 q^2 + 3m_1^6 n_1^2 p^2 q^2 -$$
$$- m_1^6 n_1^2 pq^3 + 3m_1^4 n_1^2 p^2 S_1^2 - 5m_1^4 n_1^2 pq S_1^2 + 2m_1^4 n_1^2 q^2 S_1^2 - m_1^2 n_1^4 p^2 S_1^2 + m_1^2 n_1^4 pq S_1^2 - m_1^2 n_1^2 S_1^4)$$

$$g_4 = (m_1^2 p - m_1^2 q) n_2^4 r^3 + (-m_1^2 p^2 + m_1^2 pq + 2m_1^2 q^2 + S_1^2) n_2^4 r^2 + ((2m_1^2 p^2 q - m_1^2 pq^2 - m_1^2 q^3 - 2qS_1^2) n_2^4$$
$$+ (-m_1^2 p + m_1^2 q) n_2^2 S_2^2) r + (-m_1^2 p^2 q^2 + m_1^2 pq^3 + q^2 S_1^2) n_2^4 + (m_1^2 pq - m_1^2 q^2) n_2^2 S_2^2$$

$$g_5 = (m_1^8 n_1^4 p^8 S_1^4 - 8m_1^8 n_1^4 p^7 q S_1^4 + 28m_1^8 n_1^4 p^6 q^2 S_1^4 - 56m_1^8 n_1^4 p^5 q^3 S_1^4 + 70m_1^8 n_1^4 p^4 q^4 S_1^4 - 56m_1^8 n_1^4 p^3 q^5 S_1^4 +$$
$$+ 28m_1^8 n_1^4 p^2 q^6 S_1^4 - 8m_1^8 n_1^4 pq^7 S_1^4 + m_1^8 n_1^4 q^8 S_1^4 - 4m_1^6 n_1^4 p^6 S_1^6 + 24m_1^6 n_1^4 p^5 q S_1^6 - 60m_1^6 n_1^4 p^4 q^2 S_1^6 +$$
$$+ 80m_1^6 n_1^4 p^3 q^3 S_1^6 - 60m_1^6 n_1^4 p^2 q^4 S_1^6 + 24m_1^6 n_1^4 pq^5 S_1^6 - 4m_1^6 n_1^4 q^6 S_1^6 + 6m_1^4 n_1^4 p^4 S_1^8 - 24m_1^4 n_1^4 p^3 q S_1^8 +$$
$$+ 36m_1^4 n_1^4 p^2 q^2 S_1^8 - 24m_1^4 n_1^4 pq^3 S_1^8 + 6m_1^4 n_1^4 q^4 S_1^8 - 4m_1^2 n_1^4 p^2 S_1^{10} + 8m_1^2 n_1^4 pq S_1^{10} - 4m_1^2 n_1^4 q^2 S_1^{10} +$$
$$+ n_1^4 S_1^{12}) n_2^4 S_2^2 + (m_1^{14} n_1^2 p^7 - 7m_1^{14} n_1^2 p^6 q + 21m_1^{14} n_1^2 p^5 q^2 - 35m_1^{14} n_1^2 p^4 q^3 + 35m_1^{14} n_1^2 p^3 q^4 - 21m_1^{14} n_1^2 p^2 q^5 +$$
$$+ 7m_1^{14} n_1^2 pq^6 - m_1^{14} n_1^2 q^7 - m_1^{12} n_1^4 p^7 + 7m_1^{12} n_1^4 p^6 q - 21m_1^{12} n_1^4 p^5 q^2 + 35m_1^{12} n_1^4 p^4 q^3 - 35m_1^{12} n_1^4 p^3 q^4 +$$
$$+ 21m_1^{12} n_1^4 p^2 q^5 - 7m_1^{12} n_1^4 pq^6 + m_1^{12} n_1^4 q^7 + m_1^{14} p^5 S_1^2 - 5m_1^{14} p^4 q S_1^2 + 10m_1^{14} p^3 q^2 S_1^2 - 10m_1^{14} p^2 q^3 S_1^2 +$$
$$+ 5m_1^{14} pq^4 S_1^2 - m_1^{14} q^5 S_1^2 - 5m_1^{12} n_1^2 p^5 S_1^2 + 25m_1^{12} n_1^2 p^4 q S_1^2 - 50m_1^{12} n_1^2 p^3 q^2 S_1^2 + 50m_1^{12} n_1^2 p^2 q^3 S_1^2 -$$
$$- 25m_1^{12} n_1^2 pq^4 S_1^2 + 5m_1^{12} n_1^2 q^5 S_1^2 + 4m_1^{10} n_1^4 p^5 S_1^2 - 20m_1^{10} n_1^4 p^4 q S_1^2 + 40m_1^{10} n_1^4 p^3 q^2 S_1^2 - 40m_1^{10} n_1^4 p^2 q^3 S_1^2 +$$
$$+ 20m_1^{10} n_1^4 pq^4 S_1^2 - 4m_1^{10} n_1^4 q^5 S_1^2 + 4m_1^8 n_1^4 p^3 S_1^4 - 12m_1^8 n_1^4 p^2 q S_1^4 + 12m_1^8 n_1^4 pq^2 S_1^4 - 4m_1^8 n_1^4 q^3 S_1^4 -$$
$$- 5m_1^6 n_1^6 p^3 S_1^4 + 15m_1^6 n_1^6 p^2 q S_1^4 - 15m_1^6 n_1^6 pq^2 S_1^4 + 5m_1^6 n_1^6 q^3 S_1^4 + m_1^4 n_1^8 p^3 S_1^4 - 3m_1^4 n_1^8 p^2 q S_1^4 +$$
$$+ 3m_1^4 n_1^8 pq^2 S_1^4 - m_1^4 n_1^8 q^3 S_1^4 - m_1^6 n_1^4 p S_1^6 + m_1^6 n_1^4 q S_1^6 + m_1^4 n_1^6 p S_1^6 - m_1^4 n_1^6 q S_1^6) r S_2^6 +$$
$$+ (-m_1^{14} n_1^2 p^8 + 7m_1^{14} n_1^2 p^7 q - 21m_1^{14} n_1^2 p^6 q^2 + 35m_1^{14} n_1^2 p^5 q^3 - 35m_1^{14} n_1^2 p^4 q^4 + 21m_1^{14} n_1^2 p^3 q^5 -$$
$$- 7m_1^{14} n_1^2 p^2 q^6 + m_1^{14} n_1^2 pq^7 + m_1^{12} n_1^4 p^7 q - 7m_1^{12} n_1^4 p^6 q^2 + 21m_1^{12} n_1^4 p^5 q^3 - 35m_1^{12} n_1^4 p^4 q^4 + 35m_1^{12} n_1^4 p^3 q^5 -$$
$$- 21m_1^{12} n_1^4 p^2 q^6 + 7m_1^{12} n_1^4 pq^7 - m_1^{12} n_1^4 q^8 - m_1^{14} p^6 S_1^2 + 5m_1^{14} p^5 q S_1^2 - 10m_1^{14} p^4 q^2 S_1^2 + 10m_1^{14} p^3 q^3 S_1^2 -$$
$$- 5m_1^{14} p^2 q^4 S_1^2 + m_1^{14} pq^5 S_1^2 + 5m_1^{12} n_1^2 p^6 S_1^2 - 25m_1^{12} n_1^2 p^5 q S_1^2 + 50m_1^{12} n_1^2 p^4 q^2 S_1^2 - 50m_1^{12} n_1^2 p^3 q^3 S_1^2 +$$
$$+ 25m_1^{12} n_1^2 p^2 q^4 S_1^2 - 5m_1^{12} n_1^2 pq^5 S_1^2 - 4m_1^{10} n_1^4 p^5 q S_1^2 + 20m_1^{10} n_1^4 p^4 q^2 S_1^2 - 40m_1^{10} n_1^4 p^3 q^3 S_1^2 + 40m_1^{10} n_1^4 p^2 q^4 S_1^2 -$$
$$- 20m_1^{10} n_1^4 pq^5 S_1^2 + 4m_1^{10} n_1^4 q^6 S_1^2 - 10m_1^8 n_1^4 p^4 S_1^4 + 36m_1^8 n_1^4 p^3 q S_1^4 - 48m_1^8 n_1^4 p^2 q^2 S_1^4 + 28m_1^8 n_1^4 pq^3 S_1^4 -$$
$$- 6m_1^8 n_1^4 q^4 S_1^4 + 5m_1^6 n_1^6 p^4 S_1^4 - 15m_1^6 n_1^6 p^3 q S_1^4 + 15m_1^6 n_1^6 p^2 q^2 S_1^4 - 5m_1^6 n_1^6 pq^3 S_1^4 - m_1^4 n_1^8 p^4 S_1^4 +$$
$$+ 3m_1^4 n_1^8 p^3 q S_1^4 - 3m_1^4 n_1^8 p^2 q^2 S_1^4 + m_1^4 n_1^8 pq^3 S_1^4 + 5m_1^6 n_1^4 p^2 S_1^6 - 9m_1^6 n_1^4 pq S_1^6 + 4m_1^6 n_1^4 q^2 S_1^6 - m_1^4 n_1^6 p^2 S_1^6 +$$
$$+ m_1^4 n_1^6 pq S_1^6 - m_1^4 n_1^4 S_1^8) S_2^6$$

$$g_6 = (m_1^8 n_1^6 p^8 S_1^6 - 8m_1^8 n_1^6 p^7 q S_1^6 + 28m_1^8 n_1^6 p^6 q^2 S_1^6 - 56m_1^8 n_1^6 p^5 q^3 S_1^6 + 70m_1^8 n_1^6 p^4 q^4 S_1^6 -$$
$$- 56m_1^8 n_1^6 p^3 q^5 S_1^6 + 28m_1^8 n_1^6 p^2 q^6 S_1^6 - 8m_1^8 n_1^6 p q^7 S_1^6 + m_1^8 n_1^6 q^8 S_1^6 - 4m_1^6 n_1^6 p^6 S_1^8 +$$
$$+ 24m_1^6 n_1^6 p^5 q S_1^8 - 60m_1^6 n_1^6 p^4 q^2 S_1^8 + 80m_1^6 n_1^6 p^3 q^3 S_1^8 - 60m_1^6 n_1^6 p^2 q^4 S_1^8 + 24m_1^6 n_1^6 p q^5 S_1^8 -$$
$$- 4m_1^6 n_1^6 q^6 S_1^8 + 6m_1^4 n_1^6 p^4 S_1^{10} - 24m_1^4 n_1^6 p^3 q S_1^{10} + 36m_1^4 n_1^6 p^2 q^2 S_1^{10} - 24m_1^4 n_1^6 p q^3 S_1^{10} +$$
$$+ 6m_1^4 n_1^6 q^4 S_1^{10} - 4m_1^2 n_1^6 p^2 S_1^{12} + 8m_1^2 n_1^6 p q S_1^{12} - 4m_1^2 n_1^6 q^2 S_1^{12} + n_1^6 S_1^{14}) n_2^6 r^2 + ((-2m_1^8 n_1^6 p^8 q S_1^6 +$$
$$+ 16m_1^8 n_1^6 p^7 q^2 S_1^6 - 56m_1^8 n_1^6 p^6 q^3 S_1^6 + 112m_1^8 n_1^6 p^5 q^4 S_1^6 - 140m_1^8 n_1^6 p^4 q^5 S_1^6 + 112m_1^8 n_1^6 p^3 q^6 S_1^6 -$$
$$- 56m_1^8 n_1^6 p^2 q^7 S_1^6 + 16m_1^8 n_1^6 p q^8 S_1^6 - 2m_1^8 n_1^6 q^9 S_1^6 + 8m_1^6 n_1^6 p^6 q S_1^8 - 48m_1^6 n_1^6 p^5 q^2 S_1^8 +$$
$$+ 120m_1^6 n_1^6 p^4 q^3 S_1^8 - 160m_1^6 n_1^6 p^3 q^4 S_1^8 + 120m_1^6 n_1^6 p^2 q^5 S_1^8 - 48m_1^6 n_1^6 p q^6 S_1^8 + 8m_1^6 n_1^6 q^7 S_1^8 -$$
$$- 12m_1^4 n_1^6 p^4 q S_1^{10} + 48m_1^4 n_1^6 p^3 q^2 S_1^{10} - 72m_1^4 n_1^6 p^2 q^3 S_1^{10} + 48m_1^4 n_1^6 p q^4 S_1^{10} - 12m_1^4 n_1^6 q^5 S_1^{10} +$$
$$+ 8m_1^2 n_1^6 p^2 q S_1^{12} - 16m_1^2 n_1^6 p q^2 S_1^{12} + 8m_1^2 n_1^6 q^3 S_1^{12} - 2n_1^6 q S_1^{14}) n_2^6 + (m_1^{16} n_1^4 p^9 - 9m_1^{16} n_1^4 p^8 q +$$
$$+ 36m_1^{16} n_1^4 p^7 q^2 - 84m_1^{16} n_1^4 p^6 q^3 + 126m_1^{16} n_1^4 p^5 q^4 - 126m_1^{16} n_1^4 p^4 q^5 + 84m_1^{16} n_1^4 p^3 q^6 -$$
$$- 36m_1^{16} n_1^4 p^2 q^7 + 9m_1^{16} n_1^4 p q^8 - m_1^{16} n_1^4 q^9 - m_1^{14} n_1^6 p^9 + 9m_1^{14} n_1^6 p^8 q - 36m_1^{14} n_1^6 p^7 q^2 +$$
$$+ 84m_1^{14} n_1^6 p^6 q^3 - 126m_1^{14} n_1^6 p^5 q^4 + 126m_1^{14} n_1^6 p^4 q^5 - 84m_1^{14} n_1^6 p^3 q^6 + 36m_1^{14} n_1^6 p^2 q^7 -$$
$$- 9m_1^{14} n_1^6 p q^8 + m_1^{14} n_1^6 q^9 + m_1^{16} n_1^2 p^7 S_1^2 - 7m_1^{16} n_1^2 p^6 q S_1^2 + 21m_1^{16} n_1^2 p^5 q^2 S_1^2 - 35m_1^{16} n_1^2 p^4 q^3 S_1^2 +$$
$$+ 35m_1^{16} n_1^2 p^3 q^4 S_1^2 - 21m_1^{16} n_1^2 p^2 q^5 S_1^2 + 7m_1^{16} n_1^2 p q^6 S_1^2 - m_1^{16} n_1^2 q^7 S_1^2 - 5m_1^{14} n_1^4 p^7 S_1^2 +$$
$$+ 35m_1^{14} n_1^4 p^6 q S_1^2 - 105m_1^{14} n_1^4 p^5 q^2 S_1^2 + 175m_1^{14} n_1^4 p^4 q^3 S_1^2 - 175m_1^{14} n_1^4 p^3 q^4 S_1^2 + 105m_1^{14} n_1^4 p^2 q^5 S_1^2 -$$
$$- 35m_1^{14} n_1^4 p q^6 S_1^2 + 5m_1^{14} n_1^4 q^7 S_1^2 + 4m_1^{12} n_1^6 p^7 S_1^2 - 28m_1^{12} n_1^6 p^6 q S_1^2 + 84m_1^{12} n_1^6 p^5 q^2 S_1^2 -$$
$$- 140m_1^{12} n_1^6 p^4 q^3 S_1^2 + 140m_1^{12} n_1^6 p^3 q^4 S_1^2 - 84m_1^{12} n_1^6 p^2 q^5 S_1^2 + 28m_1^{12} n_1^6 p q^6 S_1^2 - 4m_1^{12} n_1^6 q^7 S_1^2 +$$
$$+ m_1^{16} p^5 S_1^4 - 5m_1^{16} p^4 S_1^4 + 10m_1^{16} p^3 q S_1^4 - 10m_1^{16} p^2 q^3 S_1^4 + 5m_1^{16} p q^4 S_1^4 - m_1^{16} q^5 S_1^4 - 5m_1^{14} n_1^2 p^5 S_1^4 +$$
$$+ 25m_1^{14} n_1^2 p^4 q S_1^4 - 50m_1^{14} n_1^2 p^3 q^2 S_1^4 + 50m_1^{14} n_1^2 p^2 q^3 S_1^4 - 25m_1^{14} n_1^2 p q^4 S_1^4 + 5m_1^{14} n_1^2 q^5 S_1^4 +$$
$$+ 10m_1^{12} n_1^4 p^5 S_1^4 - 50m_1^{12} n_1^4 p^4 q S_1^4 + 100m_1^{12} n_1^4 p^3 q^2 S_1^4 - 100m_1^{12} n_1^4 p^2 q^3 S_1^4 + 50m_1^{12} n_1^4 p q^4 S_1^4 -$$
$$- 10m_1^{12} n_1^4 q^5 S_1^4 - 6m_1^{10} n_1^6 p^5 S_1^4 + 30m_1^{10} n_1^6 p^4 q S_1^4 - 60m_1^{10} n_1^6 p^3 q^2 S_1^4 + 60m_1^{10} n_1^6 p^2 q^3 S_1^4 -$$
$$- 30m_1^{10} n_1^6 p q^4 S_1^4 + 6m_1^{10} n_1^6 q^5 S_1^4 - m_1^8 n_1^6 p^3 S_1^6 + 3m_1^8 n_1^6 p^2 q S_1^6 - 3m_1^8 n_1^6 p q^2 S_1^6 + m_1^8 n_1^6 q^3 S_1^6 +$$
$$+ m_1^6 n_1^8 p^3 S_1^6 - 3m_1^6 n_1^8 p^2 q S_1^6 + 3m_1^6 n_1^8 p q^2 S_1^6 - m_1^6 n_1^8 q^3 S_1^6) r + (m_1^8 n_1^6 p^8 q^2 S_1^6 - 8m_1^8 n_1^6 p^7 q^3 S_1^6 +$$
$$+ 28m_1^8 n_1^6 p^6 q^4 S_1^6 - 56m_1^8 n_1^6 p^5 q^5 S_1^6 + 70m_1^8 n_1^6 p^4 q^6 S_1^6 - 56m_1^8 n_1^6 p^3 q^7 S_1^6 + 28m_1^8 n_1^6 p^2 q^8 S_1^6 -$$
$$- 8m_1^8 n_1^6 p q^9 S_1^6 + m_1^8 n_1^6 q^{10} S_1^6 - 4m_1^6 n_1^6 p^6 q^2 S_1^8 + 24m_1^6 n_1^6 p^5 q^3 S_1^8 - 60m_1^6 n_1^6 p^4 q^4 S_1^8 +$$
$$+ 80m_1^6 n_1^6 p^3 q^5 S_1^8 - 60m_1^6 n_1^6 p^2 q^6 S_1^8 + 24m_1^6 n_1^6 p q^7 S_1^8 - 4m_1^6 n_1^6 q^8 S_1^8 + 6m_1^4 n_1^6 p^4 q^2 S_1^{10} -$$
$$- 24m_1^4 n_1^6 p^3 q^3 S_1^{10} + 36m_1^4 n_1^6 p^2 q^4 S_1^{10} - 24m_1^4 n_1^6 p q^5 S_1^{10} + 6m_1^4 n_1^6 q^6 S_1^{10} - 4m_1^2 n_1^6 p^2 q^2 S_1^{12} +$$
$$+ 8m_1^2 n_1^6 p q^3 S_1^{12} - 4m_1^2 n_1^6 q^4 S_1^{12} + n_1^6 q^2 S_1^{14}) n_2^6 + (-m_1^{16} n_1^4 p^{10} + 9m_1^{16} n_1^4 p^9 q - 36m_1^{16} n_1^4 p^8 q^2 +$$
$$+ 84m_1^{16} n_1^4 p^7 q^3 - 126m_1^{16} n_1^4 p^6 q^4 + 126m_1^{16} n_1^4 p^5 q^5 - 84m_1^{16} n_1^4 p^4 q^6 + 36m_1^{16} n_1^4 p^3 q^7 - 9m_1^{16} n_1^4 p^2 q^8 +$$
$$+ m_1^{16} n_1^4 p q^9 + m_1^{14} n_1^6 p^9 q - 9m_1^{14} n_1^6 p^8 q^2 + 36m_1^{14} n_1^6 p^7 q^3 - 84m_1^{14} n_1^6 p^6 q^4 + 126m_1^{14} n_1^6 p^5 q^5 -$$
$$- 126m_1^{14} n_1^6 p^4 q^6 + 84m_1^{14} n_1^6 p^3 q^7 - 36m_1^{14} n_1^6 p^2 q^8 + 9m_1^{14} n_1^6 p q^9 - m_1^{14} n_1^6 q^{10} - m_1^{16} n_1^2 p^8 S_1^2 +$$
$$+ 7m_1^{16} n_1^2 p^7 q S_1^2 - 21m_1^{16} n_1^2 p^6 q^2 S_1^2 + 35m_1^{16} n_1^2 p^5 q^3 S_1^2 - 35m_1^{16} n_1^2 p^4 q^4 S_1^2 + 21m_1^{16} n_1^2 p^3 q^5 S_1^2 -$$
$$- 7m_1^{16} n_1^2 p^2 q^6 S_1^2 + m_1^{16} n_1^2 p q^7 S_1^2 + 5m_1^{14} n_1^2 p^8 S_1^2 - 35m_1^{14} n_1^4 p^7 q S_1^2 + 105m_1^{14} n_1^4 p^6 q^2 S_1^2 -$$
$$- 175m_1^{14} n_1^4 p^5 q^3 S_1^2 + 175m_1^{14} n_1^4 p^4 q^4 S_1^2 - 105m_1^{14} n_1^4 p^3 q^5 S_1^2 + 35m_1^{14} n_1^4 p^2 q^6 S_1^2 - 5m_1^{14} n_1^4 p q^7 S_1^2 -$$
$$- 4m_1^{12} n_1^6 p^7 q S_1^2 + 28m_1^{12} n_1^6 p^6 q^2 S_1^2 - 84m_1^{12} n_1^6 p^5 q^3 S_1^2 + 140m_1^{12} n_1^6 p^4 q^4 S_1^2 - 140m_1^{12} n_1^6 p^3 q^5 S_1^2 +$$

$$+84m_1^{12}n_1^6p^2q^6S_1^2-28m_1^{12}n_1^6pq^7S_1^2+4m_1^{12}n_1^6q^8S_1^2-m_1^{16}p^6S_1^4+5m_1^{16}p^5qS_1^4-10m_1^{16}p^4q^2S_1^4+$$
$$+10m_1^{16}p^3q^3S_1^4-5m_1^{16}p^2q^4S_1^4+m_1^{16}pq^5S_1^4+5m_1^{14}n_1^2p^6S_1^4-25m_1^{14}n_1^2p^5qS_1^4+50m_1^{14}n_1^2p^4q^2S_1^4-$$
$$-50m_1^{14}n_1^2p^3q^3S_1^4+25m_1^{14}n_1^2p^2q^4S_1^4-5m_1^{14}n_1^2pq^5S_1^4-10m_1^{12}n_1^4p^6S_1^4+50m_1^{12}n_1^4p^5qS_1^4-$$
$$-100m_1^{12}n_1^4p^4q^2S_1^4+100m_1^{12}n_1^4p^3q^3S_1^4-50m_1^{12}n_1^4p^2q^4S_1^4+10m_1^{12}n_1^4pq^5S_1^4+6m_1^{10}n_1^6p^5qS_1^4-$$
$$-30m_1^{10}n_1^6p^4q^2S_1^4+60m_1^{10}n_1^6p^3q^3S_1^4-60m_1^{10}n_1^6p^2q^4S_1^4+30m_1^{10}n_1^6pq^5S_1^4-6m_1^{10}n_1^6q^6S_1^4+$$
$$+5m_1^8n_1^6p^4S_1^6-19m_1^8n_1^6p^3qS_1^6+27m_1^8n_1^6p^2q^2S_1^6-17m_1^8n_1^6pq^3S_1^6+4m_1^8n_1^6q^4S_1^6-m_1^6n_1^8p^4S_1^6+$$
$$+3m_1^6n_1^8p^3qS_1^6-3m_1^6n_1^8p^2q^2S_1^6+m_1^6n_1^8pq^3S_1^6-m_1^6n_1^6p^2S_1^8+2m_1^6n_1^6pqS_1^8-m_1^6n_1^6q^2S_1^8)S_2^6$$

$$g_7=-2n_2^4r^2n_1^2S_1^2+(4qn_2^4n_1^2S_1^2+n_2^2S_2^2(-m_1^4p+m_1^4q+m_1^2n_1^2p-m_1^2n_1^2q))r+$$
$$+m_2^2S_2^2(m_1^4p^2-2m_1^4pq+m_1^4q^2)-2n_1^2n_2^4q^2S_1^2+n_2^2S_2^2(m_1^4p^2-m_1^4pq-$$
$$-m_1^2n_1^2pq+m_1^2n_1^2q^2)$$

$$g_8=(m_1^2-n_1^2)n_2^4r^2+((m_1^2p-m_1^2q)m_2^2n_2^2+(-m_1^2p-m_1^2q+2n_1^2q)n_2^4)r+$$
$$+(-m_1^2pq+m_1^2q^2)m_2^2n_2^2+(m_1^2pq-n_1^2q^2)n_2^4$$

$$g_9=(m_1^4p^2-2m_1^4pq+m_1^4q^2)m_2^4+(-2m_1^4p^2+4m_1^4pq-2m_1^4q^2)m_2^2n_2^2+$$
$$+(-m_1^4+2m_1^2n_1^2-n_1^4)n_2^4r^2+(2m_1^4q-4m_1^2n_1^2q+2n_1^4q)n_2^4r+$$
$$+(m_1^4p^2-2m_1^4pq+2m_1^2n_1^2q^2-n_1^4q^2)n_2^4$$