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NEW ASYMPTOTIC EXPANSIONS AND INEQUALITIES FOR MILLS RATIO

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Abstract. *This paper gives new asymptotic expansions for Mills ratio, and then establishes new inequalities for Mills ratio.*
Keywords: *Mills ratio; asymptotic expansion; inequality.*

1. INTRODUCTION

1.1. SUMMARY OF PREVIOUS RESULTS

Let $\varphi(x)$ and $\bar{\phi}(x)$ represent the normal density function and the normal tail probability function, respectively. Specifically

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$\bar{\phi}(x) = \int_x^\infty \varphi(t)dt.$$

The function

$$r(x) = \frac{\bar{\phi}(x)}{\varphi(x)} = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt \tag{1.1}$$

is known in the literature as Mill's ratio of the standard normal distribution [1, Section 2.26]. Its reciprocal, $1/r(x) = \varphi(x)/\bar{\phi}(x)$, is referred to as the failure (hazard) rate. It is well known that Mills ratio is convex and strictly decreasing on $\mathbb{R} := (-\infty, \infty)$. At the origin, it takes the value $r(0) = \sqrt{\pi/2}$, and its tails are described by the asymptotic expansion (see [2, p. 44]):

$$r(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^n n!} x^{-2n-1} = \frac{1}{x} - \frac{1}{x^3} + \frac{1\cdot 3}{x^5} - \frac{1\cdot 3\cdot 5}{x^7} + \cdots \quad as\ x \to \infty \tag{1.2}$$

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Gordon [3] proved the following inequality:

$$\frac{x}{x^2+1} < r(x) < \frac{1}{x} \quad for \ \ x > 0. \tag{1.3}$$

In fact, it is easy to see from Shenton [4]

$$\frac{x}{x^2+1} = r_2(x) < r_4(x) < \cdots < r(x) < \cdots < r_3(x) < r_1(x) = \frac{1}{x} \quad for \ \ x > 0,$$

where $r_n(x)$ is the nth approximation of the continued fraction

$$r(x) = \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \ddots}}}}.$$

It is known in the literature that

$$\frac{2}{\sqrt{x^2+4}+x} < r(x) < \frac{4}{\sqrt{x^2+8}+3x} \quad for \ \ x > 0, \tag{1.4}$$

which improves the inequality (1.3). The first inequality in (1.4) was proved by Birnbaum [5] and Komatu [6], while the second inequality in (1.4) is due to Sampford [7]. Baricz [8] presented a different proof of (1.4).

The inequalities

$$\frac{2}{\sqrt{x^2+\alpha}+x} < r(x) < \frac{2}{\sqrt{x^2+\beta}+x}\ , \quad x > 0 \tag{1.5}$$

hold, where the constants $\alpha = 4$ and $\beta = 8/\pi$ are the best possible (see [1, p. 239] and [9]). Boyd [10] proved that

$$\frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x} < r(x) < \frac{\pi}{\sqrt{(\pi-2)^2x^2+2\pi}+2x}\ , \quad x > 0. \tag{1.6}$$

Gasull and Utzet [11, Theorem 10] proved that

$$\begin{aligned} \max\left\{\frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x}, \frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x}\right\} &< r(x) \\ < \min\left\{\frac{\pi}{\sqrt{(\pi-2)^2x^2+2\pi}+2x}, \frac{4}{\sqrt{x^2+8}+3x}\right\}&, \qquad x > 0. \end{aligned} \tag{1.7}$$

The left-hand side of inequalities (1.7) shows that, for $x > 0$,

$$\frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x} \quad \text{and} \quad \frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x}$$

are both the lower bounds of $r(x)$, there is no strict comparison between the two lower bounds. Using Maple software, we find

$$\frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x} < \frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x}, \quad 0<x<1.115,$$

which shows that, for small $x(0<x<1.115)$, the lower bound of the inequality

$$\frac{\pi}{\sqrt{2(4-\pi)x^2+2\pi}+2x} < r(x) \tag{1.8}$$

is better than the one in (1.6).

1.2. ANALYSIS OF INEQUALITIES (1.4), (1.5), (1.6) AND (1.8)

In view of the inequalities (1.4), (1.5), (1.6) and (1.8), we consider the approximation families of $r(x)$ in two cases ($x \to 0$ and $x \to \infty$),

$$\frac{a}{\sqrt{x^2+b}+c}.$$

Case 1. $x \to \infty$

We are interested in finding the values of the parameters a, b and c such that

$$r(x) - \frac{a}{\sqrt{x^2+b}+c}$$

is the fastest function which would converge to zero (as $x \to \infty$), and then obtain the best approximation of $r(x)$ for large x. Using Maple software, we find that, as $x \to \infty$,

$$\begin{aligned} r(x) - \frac{a}{\sqrt{x^2+b}+c} &= \frac{1+c-a}{(1+c)x} + \frac{-2-4c-2c^2+ab}{2(1+c)^2x^3} \\ &+ \frac{-24-72c-72c^2-24c^3+3ab^2+ab^2c}{(1+c)^3x^5} + O\left(\frac{1}{x^7}\right). \end{aligned} \tag{1.9}$$

We get the best approximation when the first three terms of the right-hand side of (1.9) vanish. This produces the best approximation from (1.9):

$$\begin{cases} 1+c-a=0 \\ -2-4c-2c^2+ab=0 \\ -24-72c-72c^2-24c^3+3ab^2+ab^2c=0 \end{cases}$$

so

$$a=4, \quad b=8, \quad c=3,$$

and therefore, we obtain the best approximation

$$r(x) \approx \frac{4}{\sqrt{x^2+8}+3x}, \quad x \to \infty \tag{1.10}$$

The right-hand side of approximation formula (1.10) happens to be the upperbound of inequality (1.4). That is to say, in the upper and lower bounds of (1.4), (1.5), (1.6) and (1.8), the upper bound of (1.4) (for large x) is the best approximation, which can also be seen from the following asymptotic formulas:

$$r(x) = \frac{4}{\sqrt{x^2+8}+3x}\left(1+O\left(\frac{1}{x^6}\right)\right) \tag{1.11}$$

$$r(x) = \frac{2}{\sqrt{x^2+4}+x}\left(1+O\left(\frac{1}{x^4}\right)\right) \tag{1.12}$$

$$r(x) = \frac{4}{\sqrt{x^2+\frac{8}{\pi}}+x}\left(1+O\left(\frac{1}{x^2}\right)\right) \tag{1.13}$$

$$r(x) = \frac{\pi}{\sqrt{x^2+2\pi}+(\pi-1)x}\left(1+O\left(\frac{1}{x^4}\right)\right) \tag{1.14}$$

$$r(x) = \frac{4}{\sqrt{(\pi-2)^2x^2+2\pi}+2x}\left(1+O\left(\frac{1}{x^2}\right)\right) \tag{1.15}$$

$$r(x) = \frac{2}{\sqrt{2(4-\pi)x^2+2\pi}+2x}\left(1+O\left(\frac{1}{x}\right)\right) \tag{1.16}$$

as $x \to \infty$. The asymptotic formulas (1.11) – (1.16) are given by Maple software.

Case 2. $x \to 0$

We introduce the following class of approximations for $a, b, c \in \mathbb{R}$:

$$r(x) \approx \frac{a}{\sqrt{x^2+b}+cx}, \quad x \to 0. \tag{1.17}$$

Using the Maple software, we find, upon letting $x \to 0$, that

$$\frac{r(x)}{\frac{a}{\sqrt{x^2+b}+cx}} - 1 = \frac{\sqrt{2\pi}b-2a}{2a} + \frac{\sqrt{2\pi}c-2\sqrt{b}}{2a}x + \frac{\sqrt{2\pi}-4c\sqrt{b}+\sqrt{2\pi}b}{4a\sqrt{b}}x^2 + O(x^3). \tag{1.18}$$

We get the best approximation when the first three terms of the right-hand side in (1.18) vanish. When

$$a = \frac{\pi}{\sqrt{2(4-\pi)}} \ , \ b = \frac{\pi}{4-\pi} \ , \ c = \sqrt{\frac{2}{4-\pi}},$$

we obtain the best approximation of $r(x)$ among all approximations given by (1.17), namely,

$$r(x) \approx \frac{\pi}{\sqrt{2(4-\pi)x^2 + 2\pi} + 2x} \ , \quad x \to 0. \tag{1.19}$$

The right-hand side of approximation formula (1.19) happens to be the lower bound of inequality (1.8).

Remark 1.1. It follows from the upper bound of (1.6) and the lower bound of (1.8) that

$$\frac{\pi}{\sqrt{\lambda x^2 + 2\pi} + 2x} < r(x) < \frac{\pi}{\sqrt{\mu x^2 + 2\pi} + 2x} \ , \quad x > 0, \tag{1.20}$$

where the constants

$$\lambda = 2(4-\pi) = 1.7168 \ldots \ , \quad \mu = (\pi - 2)^2 = 1.3032 \ldots$$

are the best possible, in the sense that $\lambda = 2(4-\pi)$ cannot be replaced by a smaller number, and $\mu = (\pi-2)^2$ cannot be replaced by a larger number.

Remark 1.2. For $x > 0$, we have

$$\frac{\pi}{\sqrt{2(4-\pi)x^2 + 2\pi} + ax} < r(x) < \frac{\pi}{\sqrt{2(4-\pi)x^2 + 2\pi} + bx} \ , \tag{1.21}$$

where the constants

$$a = 2 \, , \quad b = \pi - \sqrt{2(4-\pi)} = 1.8313 \ldots$$

are the best possible, in the sense that $a = 2$ cannot be replaced by a smaller number, and $b = \pi - \sqrt{2(4-\pi)}$ cannot be replaced by a larger number. We here point out that the upper bound in (1.20) is better than the one in (1.21), since

$$\frac{\pi}{\sqrt{(\pi-2)^2 x^2 + 2\pi} + 2x} < \frac{\pi}{\sqrt{2(4-\pi)x^2 + 2\pi} + \left(\pi - \sqrt{2(4-\pi)}\right)x} \ , \quad x > 0.$$

In this paper, we give new asymptotic expansions for Mills ratio, and then establish new inequalities for Mills ratio. The numerical calculations presented in this work were performed by using the Maple software for symbolic computations.

2. ASYMPTOTIC EXPANSIONS

Write (1.2) as

$$xr(x)\sim\sum_{j=0}^{\infty}(-1)^j\frac{(2j)!}{2^j j!}x^{-2j}=1-\frac{1}{x^2}+\frac{1\cdot 3}{x^4}-\frac{1\cdot 3\cdot 5}{x^6}+\cdots \quad as \quad x\to\infty. \tag{2.22}$$

By using the Maple software, we find that

$$\ln(xr(x))\sim-\frac{1}{x^2}+\frac{5}{2x^4}-\frac{37}{3x^6}+\frac{353}{4x^8}-\frac{4081}{5x^{10}}+\cdots \quad as \quad x\to\infty. \tag{2.23}$$

Since every term in the right-hand side of (2.22) is even function, we see that all odd terms in the asymptotic expansion (2.23) vanish. The asymptotic expansion (2.23) can be written as

$$r(x)\sim\frac{1}{x}\exp\left(-\frac{1}{x^2}+\frac{5}{2x^4}-\frac{37}{3x^6}+\frac{353}{4x^8}-\frac{4081}{5x^{10}}+\cdots\right) \quad as \quad x\to\infty. \tag{2.24}$$

Even though as many coefficients as we please in the right-hand side of (2.24) can be obtained by using Mathematica, here we aim at giving a formula for determining these coefficients (see Theorem 2.1). Our formula is mainly based on the partition function.
For our later use, we introduce the following set of partitions of an integer $n\in\mathbb{N}$:

$$\mathcal{A}_n:=\{(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^n : k_1+2k_2+\cdots+nk_n=n\}, \tag{2.25}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. In number theory, the partition function $p(n)$ represents the number of possible partitions of $n\in\mathbb{N}$ (*e.g.*, the number of distinct ways of representing n as a sum of natural numbers regardless of order).

By convention, $p(0)=1$ and $p(n)=0$ for n a negative integer. The first several values of the partition function $p(n)$ are (the starting with $p(0)=1$):

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots\ .$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$. Now we are ready to present a formula which determines the coefficients in the expansion (2.24) with the help of the partition function asserted by the following theorem.

Theorem 2.1. Mills ratio $r(x)$ has the following asymptotic expansion:

$$r(x)\sim\frac{1}{x}exp\left(\sum_{j=1}^{\infty}\frac{a_{2j}}{x^{2j}}\right) \quad as \quad x\to\infty, \tag{2.26}$$

with the coefficients a_{2j} ($j\in\mathbb{N}$) given by the following relations:

$$\sum_{(k_1,k_2,\dots k_j)\in\mathcal{A}_j} \frac{1}{k_1!\,k_2!\dots k_j!} a_2^{k_1} a_4^{k_2} \dots a_{2j}^{k_j} = (-1)^j \frac{(2j)!}{2^j j!} \quad for \quad j \in \mathbb{N}, \tag{2.27}$$

where the $\mathcal{A}_j$ (for $j \in \mathbb{N}$) are given in (2.25).

Proof: To determine the coefficients a_{2j}, we first express (2.26) as follows:

$$xr(x) = \exp\left(\sum_{k=1}^{m} \frac{a_{2k}}{x^{2k}} + R_m(x)\right),$$

where $R_m(r) = O(r^{-2m-2})$. Further, we have:

$$\begin{aligned} xr(x) &= e^{R_m(x)} \exp\left(\sum_{k=1}^{m} \frac{a_{2k}}{x^{2k}}\right) = e^{R_m(x)} \prod_{k=1}^{m} \left[1 + \frac{a_{2k}}{x^{2k}} + \frac{1}{2!}\left(\frac{a_{2k}}{x^{2k}}\right)^2 + \cdots\right] \\ &= e^{R_m(x)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \frac{a_2^{k_1} a_4^{k_2} \dots a_{2m}^{k_m}}{k_1!\,k_2!\dots k_m!} \cdot \frac{1}{x^{2(k_1+2k_2+\cdots+mk_m)}}. \end{aligned} \tag{2.28}$$

Equating the coefficients by the equal powers of x in (2.22) and (2.28), we see that (2.27) holds. This completes the proof of Theorem 2.1.

Remark 2.1. Here we give explicit numerical values of some first terms of a_{2j} by using the partition set (2.25) and the formula (2.27). This shows how easily we can determine a_{2j} in (2.27). Obviously,

$$\sum_{k_1=1} \frac{1}{k_1!} a_2^{k_1} = -1 \Rightarrow a_2 = -1.$$

For $k_1 + 2k_2 = 2$, since $p(2) = 2$, the partition set $\mathcal{A}_2$ in (2.25) is seen to have 2 elements:

$$\mathcal{A}_2 = \{(0,1),(2,0)\}.$$

From (2.27), we have:

$$\sum_{(k_1,k_2)\in\mathcal{A}_2} \frac{1}{k_1!\,k_2!} a_2^{k_1} a_4^{k_2} = 3 \Rightarrow a_4 = 3 - \frac{1}{2!\,0!} a_2^2 = \frac{5}{2}.$$

For $k_1 + 2k_2 + 3k_3 = 3$, since $p(3) = 3$, as above, the partition set $\mathcal{A}_3$ in (2.25) contains 3 elements:

$$\mathcal{A} = \{(0,0,1),(1,1,0),(3,0,0)\}.$$

From (2.27), we have:

$$\sum_{(k_1,k_2,k_3)\in\mathcal{A}_3} \frac{1}{k_1!\,k_2!\,k_3!} a_2^{k_1} a_4^{k_2} a_6^{k_3} = -15 \Rightarrow a_6 = -15 - \frac{1}{1!\,1!\,0!} a_2 a_4 - \frac{1}{3!\,0!\,0!} a_2^3 = -\frac{37}{7}.$$

Likewise, the partition sets $\mathcal{A}_4$ and $\mathcal{A}_5$ have $5 = p(4)$ and $7 = p(5)$ elements, respectively, and so

$$\mathcal{A}_4 = \{(0,0,0,1), (1,0,1,0), (0,2,0,0), (2,1,0,0), (4,0,0,0)\}$$

and

$$\mathcal{A}_5 = \{(0,0,0,0,1), (1,0,0,1,0), (0,1,1,0,0), (2,0,1,0,0), (1,2,0,0,0), (3,1,0,0,0), (5,0,0,0,0)\},$$

which yields

$$a_8 = \frac{353}{4}\ ,\quad a_{10} = -\frac{4081}{5}.$$

We note that the explicit numerical values of a_{2j} (for $j = 1, 2, 3, 4, 5$) here correspond with the cofficients of $1/x^{2j}$ (for $j = 1, 2, 3, 4, 5$) in (2.24), respectively.

Remark 2.2. The asymptotic expansion (2.26) can be written as

$$r(x) \sim \frac{1}{x}\exp\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{x^j}\right) \quad \text{as} \quad x \to \infty, \tag{2.29}$$

with the coefficients α_j given by

$$\alpha_j = \begin{cases} a_{2k}\ , & j = 2k \\ 0\ , & j = 2k-1 \end{cases}, \tag{2.30}$$

where a_{2j} can be calculated using the relations (2.27).

Theorem 2.2 gives a unified treatment of asymptotic expansions for the Mills ratio $r(x)$ and the failure (hazard) rate $1/r(x)$.

Theorem 2.2. Let $p = 0$ be a given real number and $r(x)$ has the following asymptotic expansion:

$$r(x) \sim \frac{1}{x}\left(1 + \sum_{j=1}^{\infty}\frac{b_j}{x^j}\right)^{\frac{x^l}{p}} \quad as \quad x \to \infty, \tag{2.31}$$

with the coefficients $b_j = b_j(l, p)$ (for $j \in \mathbb{N}$) given by

$$b_j = \sum_{(1+l)k_1+(2+l)k_2+\cdots+(j+l)k_l} \frac{p^{k_1+k_2+\cdots+k_j}}{k_1!\,k_2!\ldots k_j!}\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_j^{k_j}\ , \tag{2.32}$$

where the α_j are defined by (2.30), summed over all nonnegative integers k_j satisfying the equation

$$(1 + l)k_1 + (2 + l)k_2 + \cdots + (j + l)k_l = j.$$

Proof: In view of (2.29), we can let

$$\left(xr(x)\right)^{p/x^l} = 1 + \sum_{j=1}^{m} \frac{b_j}{x^j} + O(x^{-m-1}) \quad as \quad x \to \infty, \tag{2.33}$$

where b_j are real numbers to be determined. Write (2.29) as

$$xr(x) \sim \exp\left(\sum_{k=1}^{m} \frac{\alpha_k}{x^k} + \mathcal{R}_m(x)\right) \quad as \quad x \to \infty, \tag{2.34}$$

with $\mathcal{R}_m(x) \sim O(x^{-m-1})$. Further, we have

$$\begin{aligned}\left(xr(x)\right)^{\frac{p}{x^l}} &= e^{\frac{p\mathcal{R}_m(x)}{x^l}} \exp\left(\sum_{k=1}^{m} \frac{p\alpha_k}{x^{k+l}}\right) = e^{\frac{p\mathcal{R}_m(x)}{x^l}} \prod_{k=1}^{m} \left(1 + \frac{p\alpha_k}{x^{k+l}} + \frac{1}{2!}\left(\frac{p\alpha_k}{x^{k+l}}\right)^2 + \cdots\right) \\ &= e^{\frac{p\mathcal{R}_m(x)}{x^l}} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \ldots \sum_{k_m=0}^{\infty} \frac{p^{k_1+k_2+\cdots+k_m}}{k_1!\, k_2! \ldots k_m!} \cdot \alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_m^{k_m} \\ &\times \frac{1}{x^{(1+l)k_1+(2+l)k_2+\cdots+(m+l)k_m}}.\end{aligned}$$

Equating the coefficients by the equal powers of x in (2.33) and (2.34), we obtain the formula (2.32). This completes the proof of Theorem 2.2.

Remark 2.3. In particular, setting $(l, p) = (0,1)$ in (2.31) yields (1.2). Setting $(l, p) = (0, -1)$ in (2.31), we obtain the following asymptotic expansion of the failure (hazard) rate

$$\frac{1}{r(x)} \sim x + \frac{1}{x} - \frac{2}{x^3} + \frac{10}{x^5} - \frac{74}{x^7} + \cdots \quad \text{as} \quad x \to \infty. \tag{2.35}$$

Setting $(l, p) = (1,1)$ and $(l, p) = (1, -1)$ in (2.31), respectively, we obtain the following asymptotic expansions for the Mills ratio and the failure (hazard) rate

$$r(x) \sim \frac{1}{x}\left(1 - \frac{1}{x^3} + \frac{5}{2x^5} + \frac{1}{2x^6} - \frac{37}{3x^7} - \frac{5}{2x^8} + \cdots\right)^x \quad \text{as} \quad x \to \infty \tag{2.36}$$

$$\frac{1}{r(x)} \sim x\left(1 + \frac{1}{x^3} - \frac{5}{2x^5} + \frac{1}{2x^6} + \frac{37}{3x^7} - \frac{5}{2x^8} - \frac{1057}{12x^9} + \cdots\right)^x \quad \text{as} \quad x \to \infty. \tag{2.37}$$

Using Maple software, we find that, as $x \to \infty$, $r(x)$ has the following asymptotic expansions:

$$\begin{aligned}r(x) \sim \frac{4}{\sqrt{x^2+8}+3x}\bigg\{1 &- \frac{2}{x^6} + \frac{36}{x^8} + \frac{522}{x^{10}} + \frac{7432}{x^{12}} - \frac{110946}{x^{14}} + \frac{1788588}{x^{16}} \\ &- \frac{31551154}{x^{18}} + \frac{611219040}{x^{20}} - \frac{12977945082}{x^{22}} + \cdots\bigg\},\end{aligned} \tag{2.38}$$

$$r(x) \sim \frac{4}{\sqrt{x^2+8}+3x} \times \left\{1 - \frac{2}{x^6+18x^4+63x^2+152-\frac{2292}{x^2}+\frac{48960}{x^4}-\frac{1128760}{x^6}+\cdots}\right\}. \tag{2.39}$$

Even though we can obtain as many coefficients as we please in the right-hand sides of (2.38) and (2.39) by using Maple software, here we aim at giving a formula for determining the coefficients of each asymptotic expansion. Theorem 2.3 gives a formula to determine the coefficients of expansion (2.38).

Theorem 2.3. As $x \to \infty$, we have

$$\begin{aligned} r(x) &\sim \frac{4}{\sqrt{x^2+8}+3x}\sum_{n=0}^{\infty}\frac{\mu_n}{x^{2n}} \\ &= \frac{4}{\sqrt{x^2+8}+3x}\left\{1-\frac{2}{x^6}+\frac{36}{x^8}-\frac{522}{x^{10}}+\frac{7432}{x^{12}}-\frac{110946}{x^{14}}\right. \\ &\quad +\frac{1788588}{x^{16}}-\frac{31551154}{x^{18}}+\frac{611219040}{x^{20}}-\frac{12977945082}{x^{22}} \\ &\quad \left.+\cdots\right\}, \end{aligned} \tag{2.40}$$

where

$$\mu_0 = 1\,, \quad \mu_n = \sum_{k=0}^{n-1}(-1)^k\frac{(2k)!}{4^{2k+1}k!}\binom{1/2}{n-k}8^n + (-1)^n\frac{(2n)!}{2^n n!}\,, \quad n \in \mathbb{N}. \tag{2.41}$$

Proof: Using the power series expansion of $(1+t)^\alpha$,

$$(1+t)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}t^k\,, \quad -1<t<1,$$

we obtain that, as $x \to \infty$,

$$\frac{1}{4}\left\{3+\left(1+\left(\frac{2\sqrt{2}}{x}\right)^2\right)^{1/2}\right\} = \frac{1}{4}\left\{4+\sum_{k=1}^{\infty}\binom{1/2}{k}\left(\frac{2\sqrt{2}}{x}\right)^{2k}\right\} = 1+\sum_{k=1}^{\infty}\binom{1/2}{k}\frac{8^k}{4x^{2k}},$$

which can be rewritten as

$$\frac{1}{4}\left\{3+\left(1+\left(\frac{2\sqrt{2}}{x}\right)^2\right)^{1/2}\right\} = \sum_{k=1}^{\infty}\frac{b_k}{x^{2k}}\,, \quad x \to \infty \tag{2.42}$$

where

$$b_0 = 1\,, \quad b_k = \binom{1/2}{k}\frac{8^k}{4}\,, \quad k \in \mathbb{N}.$$

Write (1.2) as

$$xr(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \ , \qquad x \to \infty, \tag{2.43}$$

where

$$a_n = (-1)^n \frac{(2n)!}{2^n n!} \ , \quad n \in \mathbb{N}_0.$$

Using (2.42) and (2.43), we get

$$\begin{aligned} r(x)\frac{\sqrt{x^2+8}+3x}{4} &= xr(x)\frac{1}{4}\left\{3+\left(1+\left(\frac{2\sqrt{2}}{x}\right)^2\right)^{\frac{1}{2}}\right\} \\ &\sim \sum_{n=0}^{\infty}\frac{a_n}{x^{2n}}\sum_{k=0}^{\infty}\frac{a_k}{x^{2k}} = \sum_{n=0}^{\infty}\frac{\mu_n}{x^{2n}} \ , \ \ x \to \infty, \end{aligned} \tag{2.44}$$

where

$$\mu_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n-1} a_k b_{n-k} + a_n b_0 = \sum_{k=0}^{n-1} (-1)^k \frac{(2k)!}{4^{2k+1}k!}\binom{\frac{1}{2}}{n-k} 8^k + (-1)^n \frac{(2n)!}{2^n n!},$$

$n \in \mathbb{N}_0$ (an empty sum is understood to be zero). The expansion (2.44) can be written as (2.40). The proof of Theorem 2.3 is complete.

Using Lemma 2.1 below, the expansion (2.38) can be converted to (2.39).

Lemma2.1. (see [12, Corollary 2.3]) Let $\mu_3 \neq 0$ and

$$F(x) \sim \sum_{j=3}^{\infty} \frac{\mu_j}{x^j} \ , \quad x \to \infty$$

be a given asymptotic expansion. Define the function $G(x)$ by

$$F(x) = \frac{\mu_3}{G(x)}.$$

Then the function $G(x) = 3/F(x)$ has asymptotic expansion of the following form

$$G(x) \sim x^3 + a_{-2}x^2 + a_{-1}x + a_0 + \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \quad x \to \infty,$$

where

$$a_{-2} = -\frac{\mu_4}{\mu_3} \ , \ \ a_{-1} = -\frac{\mu_3\mu_5 - \mu_4^2}{\mu_3^2} \ , \ \ a_0 = -\frac{\mu_3^2\mu_6 - 2\mu_3\mu_4\mu_5 + \mu_4^3}{\mu_3^3},$$

$$a_j = -\frac{1}{\mu_3}\left(\mu_{j+6} + \mu_{j+5}a_{-2} + \mu_{j+4}a_{-1} + \sum_{k=1}^{j} \mu_{k+3}a_{j-k}\right), \qquad j \in \mathbb{N}.$$

Theorem 2.4 gives a formula to determine the coefficients of expansion (2.39).

Theorem 2.4. As $x \to \infty$, we have

$$\begin{aligned} r(x) &\sim \frac{4}{\sqrt{x^2+8}+3x}\left\{1-\frac{2}{x^6+18x^4+63x^2+\sum_{j=0}^{\infty}\frac{a_j}{x^{2j}}}\right\} \\ &= \frac{4}{\sqrt{x^2+8}+3x} \\ &\times\left\{1-\frac{2}{x^6+18x^4+63x^2+152-\frac{2292}{x^2}+\frac{48960}{x^4}-\frac{1128760}{x^6}+\cdots}\right\} \end{aligned} \tag{2.45}$$

where

$$a_0 = 152\,, \qquad a_j = \frac{1}{2}\left(\mu_{j+6}+18\mu_{j+5}+63\mu_{j+4}+\sum_{k=1}^{j}\mu_{k+3}a_{j-k}\right), \tag{2.46}$$

and μ_n are given in (2.41).

Proof: Let $y = x^2$. Write (2.40) as

$$\begin{aligned} \frac{r(x)}{\frac{4}{\sqrt{x^2+8}+3x}} - 1 &\sim \sum_{n=3}^{\infty}\frac{\mu_n}{y^n} \\ &= -\frac{2}{y^3}+\frac{36}{y^4}-\frac{522}{y^5}+\frac{7432}{y^6}-\frac{110946}{y^7}+\frac{1788588}{y^8} \\ &\quad -\frac{31551154}{y^9}+\frac{611219040}{y^{10}}-\frac{12977945082}{y^{11}}+\cdots =: F(y), \end{aligned} \tag{2.47}$$

where the coefficients μ_n are given in (2.41). Define the function $G(y)$ by

$$F(y) = \frac{\mu_3}{G(y)}, \tag{2.48}$$

where $\mu_3 = -2$. By Lemma 2.1, the function $G(y) = \mu_3/F(y)$ has asymptotic expansion of the following form

$$G(y) \sim y^3 + a_{-2}y^2 + a_{-1}y + a_0 + \sum_{j=1}^{\infty}\frac{a_j}{y^j}\ , \quad y \to \infty \tag{2.49}$$

where

$$a_{-2} = -\frac{\mu_4}{\mu_3} = 18\,, \qquad a_{-1} = -\frac{\mu_3\mu_5-\mu_4^2}{\mu_3^2} = 63\,, \ \ a_0 = -\frac{\mu_3^2\mu_6-2\mu_3\mu_4\mu_5+\mu_4^3}{\mu_3^3} = 152,$$

$$a_j = -\frac{1}{\mu_3}\left(\mu_{j+6} + \mu_{j+5}a_{-2} + \mu_{j+4}a_{-1} + \sum_{k=1}^{j}\mu_{k+3}a_{j-k}\right)$$
$$= \frac{1}{2}\left(\mu_{j+6} + 18\mu_{j+5} + 63\mu_{j+4} + \sum_{k=1}^{j}\mu_{k+3}a_{j-k}\right), \quad j \in \mathbb{N}.$$

By (2.47), (2.48) and (2.49), and noting that $a_2 = 18, a_1 = 63, a_0 = 152$, we obtain

$$\frac{r(x)}{\frac{4}{\sqrt{x^2+8}+3x}} - 1 \sim -\frac{2}{y^3 + 18y^2 + 63y + \sum_{j=0}^{\infty}\frac{a_j}{y^j}}, \quad x \to \infty,$$

which can be written as (2.45). The proof of Theorem 2.4 is complete.

Theorems 2.3 and 2.4 develop the asymptotic formula (1.11) in to the complete asymptotic expansions.

3. INEQUALITIES

The expansion (2.38) motivates us to establish Theorem 3.1.

Theorem 3.1. For $x > 0$, we have

$$\frac{4}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6}\right) < r(x) < \frac{4}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6} + \frac{36}{x^8}\right). \tag{3.50}$$

Proof: In order to prove the left-hand side of (3.50), it suffices to show that

$$f_1(x) > 0, \quad x > 0,$$

where

$$f_1(x) = \int_x^{\infty} e^{-t^2/2}dt - \frac{4e^{-\frac{x^2}{2}}}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6}\right).$$

Differentiation yields

$$\frac{e^{\frac{x^2}{2}}}{2}f_1'(x) = -\frac{g_1(x) - h_1(x)}{x^7\sqrt{x^2+8}\left(\sqrt{x^2+8}+3x\right)^2},$$

where

$$g_1(x) = x^{10} + 6x^8 + 4x^4 + 60x^2 + 192 + (12x^3 + 84x)\sqrt{x^2+8}$$
$$h_1(x) = (x^9 + 2x^7)\sqrt{x^2+8}.$$

Noting that

$$g_1^2(x) - h_1^2(x) = 168x^{12} + 1104x^{10} + 2464x^8 + 3648x^6 + 28320x^4 + 79488x^2 + 36864 + (312x^{11} + 1008x^9 + 96x^7 + 2112x^5 + 14688x^3 + 32256x)\sqrt{x^2+8} + 24x^{13}\left(\sqrt{x^2+8} - x\right) > 0,$$

we obtain $g_1(x) > h_1(x)$. Thus $f_1' < 0$. Hence, $f_1(x)$ is strictly decreasing on $(0, \infty)$, and then

$$f_1(x) > \lim_{y\to\infty} f_1(y) = 0\,, \quad x > 0.$$

In order to prove the right-hand side of (3.50), it suffices to show that

$$f_2(x) < 0\,, \quad x > 0,$$

where

$$f_2(x) = \int_x^\infty e^{-t^2/2}dt - \frac{4e^{-\frac{x^2}{2}}}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6} + \frac{36}{x^8}\right).$$

Differentiation yields

$$\frac{e^{\frac{x^2}{2}}}{2} f_2'(x) = \frac{g_2(x) - h_2(x)}{x^9\sqrt{x^2+8}\left(\sqrt{x^2+8}+3x\right)^2}\,,$$

where

$$g_2(x) = 12x^4 + 1032x^2 + 4608 + (x^{11} + 2x^9 + 132x^3 + 1944x)\sqrt{x^2+8}$$

$$h_2(x) = x^{12} + 6x^{10} + 4x^6 + 12x^5\sqrt{x^2+8}\,.$$

Let $g_2^2(x) - h_2^2(x) = p(x) - q(x)$, where

$$p(x) = 24x^{18} + 216x^{16} + 6528x^{14} + 42944x^{12} + 61056x^{10} + 17568x^8 + 677376x^6 + 9060480x^4 + 39744000x^2 + 21233664 + (2112x^{13} + 13248x^{11} + 18432x^9 + 3168x^7 + 319104x^5 + 5228928x^3 + 17915904x)\sqrt{x^2+8}$$

$$q(x) = (24x^{17} + 120x^{15})\sqrt{x^2+8}.$$

Further,

$$p^2(x) - q^2(x) = 299520x^{32} + 4766208x^{30} + 68557824x^{28} + 679540224x^{26} + 3382480384x^{24} + 8729561088x^{22} + 25687996416x^{20} + 241637326848x^{18} + 1860547479552x^{16} + 7298401370112x^{14} + 13605379964928x^{12} + 26077970497536x^{10} + 203062659416064x^8 + 1246533817466880x^6 + 3784235081269248x^4 + 4255658413129728x^2 + 450868486864896$$

$$+(101376x^{31} + 1548288x^{29} + 34182144x^{27} + 362476032x^{25} + 1653078528x^{23} + 3705237504x^{21} + 13134661632x^{19} + 160669458432x^{17} + 1155005872128x^{15} + 3669533466624x^{13} + 4888914444288x^{11} + 14530440462336x^{9} + 144524277153792x^{7} + 753841902845952x^{5} + 1646157977616384x^{3} + 760840571584512x)\sqrt{x^2+8} > 0.$$

We obtain $g_2(x) > h_2(x)$. Thus $f_2' > 0$. Hence, $f_2(x)$ is strictly increasing on $(0,\infty)$, and then

$$f_2(x) < \lim_{y\to\infty} f_2(y) = 0\,, \quad x > 0.$$

The proof of Theorem 3.1 is complete.

The expansion (2.39) motivates us to establish Theorem 3.2.

Theorem 3.2. For $x > 0$, we have

$$r(x) < \frac{4}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6+18x^4+63x^2+152}\right). \tag{3.51}$$

Proof: In order to prove (3.51), it suffices to show that

$$f(x) < 0\,, \quad x > 0,$$

where

$$f(x) = \int_x^\infty e^{-t^2/2}dt - \frac{4e^{-\frac{x^2}{2}}}{\sqrt{x^2+8}+3x}\left(1 - \frac{2}{x^6+18x^4+63x^2+152}\right).$$

Differentiation yields

$$\frac{e^{\frac{x^2}{2}}}{2} f'(x) = -\frac{g(x)-h(x)}{(x^6+18x^4+63x^2+152)^2\sqrt{x^2+8}\left(\sqrt{x^2+8}+3x\right)^2},$$

where

$$g(x) = (x^{14} + 38x^{12} + 522x^{10} + 3460x^8 + 14285x^6 + 36198x^4 + 57316x^2 + 44384)\sqrt{x^2+8}$$

$$h(x) = x^{15} + 42x^{13} + 666x^{11} + 5276x^9 + 25005x^7 + 77178x^5 + 143700x^3 + 148128x.$$

Noting that

$$g^2(x) - h^2(x) = 73344x^{16} + 2980608x^{14} + 45939584x^{12} + 366147072x^{10} + 1868266368x^8 + 6324212480x^6 + 14502622848x^4 + 20730648576x^2 + 15759515648 > 0,$$

we obtain $g(x) > h(x)$, so $f'(x) > 0$, $x > 0$.

Hence, $f(x)$ is strictly increasing on $(0,\infty)$, and we have

$$f(x) < \lim_{y\to\infty} f(y) = 0\,, \quad x > 0.$$

The proof of Theorem 3.2 is complete.
Using Maple software, we find

$$r(x) = \frac{4}{\sqrt{x^2+8}+3x}\left(1-\frac{2}{x^6+18x^4+63x^2+152}\right)\left\{1+O\left(\frac{1}{x^{14}}\right)\right\}, \tag{3.52}$$

as $x\to\infty$. Obviously, the asymptotic formula (3.52) is much better than the asymptotic formulas (1.11) to (1.16).

Competing interest. The authors declare that they have no competing interest. All data included in this study are available upon request by contact with the corresponding author.

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