ORIGINAL PAPER

TRAVELLING WAVE SOLUTIONS OF (1+1)-DIMENSIONAL MODIFIED BENJAMIN-BONA-MAHONY EQUATION (MBBME) BY (G′/G,1/G)-EXPANSION METHOD

AZMAT LIAQAT[1], GHULAM FARID[1*], MUHAMMAD BILAL[1]

Manuscript received: 10.01.2024; Accepted paper: 15.07.2024;
Published online: 30.09.2024.

Abstract *In this study, we study the travelling wave solutions of the Modified Benjamin-Bona-Mahony Equation (MBBME) using the* $(G'/G,1/G)$*-expansion method (EM). The expansion method, along with an appropriate transformation, allows us to derive a set of explicit solutions for the MBBME in terms of rational, hyperbolic, and trigonometric functions. The travelling wave profiles that these solutions depict are observable in a variety of physical systems that the MBBME describes. The results obtained by this solution play important role in the mathematical physics. Moreover, the graphical analysis of* $(G'/G,1/G)$*-EM are visualized in 2D and as well as 3D given in this work.*

Keywords: $(G'/G,1/G)$*-expansion method; travelling wave solution; (1+1)-dimensional Modified Benjamin-Bona-Mahony Equation.*

Mathematics Subject Classification: *44A20; 11B37; 32A17.*

1. INTRODUCTION

The (1+1)-dimensional Modified Benjamin-Bona-Mahony Equation (MBBME) is a nonlinear partial differential equation (NLPDE) that arises in the study of various physical phenomena, including water waves and plasma physics. Understanding the solutions of this equation is crucial for predicting and analyzing wave propagation in these systems.

The soliton notion is used to propagate a large-scale kind of wave, and soliton types account for most solutions to nonlinear evaluation equations (NLEEs) [1, 2]. Various analytical techniques, optical solutions, breather waves, periodic waves, and soliton types of solutions, can be used to identify different types of NLEEs solutions [3, 4]. There are various methods available for solving NLPDEs, each with its own advantages and limitations. These methods include the Rieman-Hilbert method [5, 6], , the Lie symmetric analysis [7, 8], the auxiliary equation method [9,10], the Sine-Gordon expansion method (EM) [11, 12], the tan-cot function method [13,14], the Sardar-subequation technique [15, 16], the first integral method for some accurate solutions of the KdV equations for the dimensions (2 + 1) and (3 + 1), Ibrahim E. Inan *et al.* recently suggested the $\exp(-\varphi(\xi))$-EM [17], Perturbation methods[18], Homotopy Analysis Method (HAM) [19] and Adomian Decomposition Method (ADM) [20]. Furthermore, a small number of scientists have searched for an improved method that is more useful and efficient than any previous method. In order to get travelling wave solutions (TWSs) including parameters of the KdV equation, and the Hirota-Satsuma

[1] University of Agriculture, Department of Mathematics and Statistics, Faisalabad, Pakistan.
E-mail: azmatliaqat14@gmail.com; mr.bilal6447@gmail.com.
* Corresponding author: ghulamfarid@uaf.edu.pk.

https://doi.org/10.46939/J.Sci.Arts-24.3-a06

equations, Wang *et al.* originally developed the $(G^{'}/G)$-EM [21] in 2008. Then $(G^{'}/G)$-EM was used by Gao and Zhang in 2011 [22], and since then, it has been extensively applied in various scientific disciplines.

The $(G^{'}/G)$-EM is a powerful mathematical technique widely used to solve NLPDEs. The $(G^{'}/G)$-EM provides a systematic approach to obtain exact solutions for a wide range of NLPDEs. It has been successfully applied in fields such as fluid dynamics, quantum mechanics, solid mechanics and mathematical biology, among others.

As an extension of Wang *et al.*'s $(G^{'}/G)$-EM, Li *et al.* initially presented the $(G^{'}/G,1/G)$-EM [23] in 2010 for creating precise solutions to the Zakharov equation. The $(G^{'}/G,1/G)$-EM was also used by Zayed and Alurrfi in 2016 [24]. The $(G^{'}/G,1/G)$-EM is an advanced mathematical technique that extends the capabilities of the traditional $(G^{'}/G)$-EM in solving NLPDEs. This method introduces two auxiliary functions, $(G^{'}/G)$ and $(1/G)$, to provide a more versatile and powerful approach to finding exact solutions for a wide range of NLPDEs. Then, for analytical solutions, the $(G^{'}/G,1/G)$-EM is applied to a variety of NPDEs [25], including the Gardner-KP equations [26] and the Phi-Four equation [27] and equations of the Boussinesq type [28].

Over the years, several mathematical methods have been employed to investigate the solutions of the MBBME. These methods range from exact analytical techniques to approximate numerical approaches. The TWSs [29], exact and explicit solutions of MBBME equation [30] was also found. Among the analytical methods, the $(G^{'}/G,1/G)$-EM has gained significant attention due to its effectiveness in obtaining exact solutions for a wide range of NLPDEs.

In this study, we focus on the $(G^{'}/G,1/G)$-EM and apply it to the MBBME to explore its TWSs. The MBBM equation are given as:

$$u_t + u_{xxx} + u_x - \alpha u^2 u_{xx} = 0, \tag{1.1}$$

where α is constant.

The $(G^{'}/G,1/G)$-EM is a powerful mathematical technique that combines algebraic and analytical methods to construct exact solutions. It has been successfully applied to various NLEs, providing valuable insights into the behavior and characteristics of the underlying systems.

By utilizing suitable transformations and the $(G^{'}/G,1/G)$-EM, we derive explicit expressions for the TWSs of the MBBME. These solutions are presented in terms of hyperbolic functions, which are commonly observed in wave phenomena. The obtained solutions offer a comprehensive understanding of the wave propagation governed by the MBBME, allowing us to analyze the wave profiles and their dependence on the system parameters.

This is how the remainder of the paper is structured. The two variables $(G'/G,1/G)$-EM is described in Section 2. We use this strategy to (1.1) in Section 3. conclusions are drawn in Section 4.

2. MAIN RESULTS

We are going to discuss some (G′/G,1/G)-expansion method to find the travelling wave solutions of nonlinear (1+1)-DMBBME.

2.1. DESCRIPTION OF THE (G′/G,1/G)-EXPANSION METHOD

The double $(G^{'}/G,1/G)$-EM for locating TWSs to NLEEs is described in this section. First of all, according to Li *et al.* (2010), the second-order linear ordinary differential equation (LODE) is presented as:

$$G^{''}(\zeta)+\lambda G(\zeta)=\eta \tag{2.1}$$

and let

$$\phi=\frac{G^{'}}{G},\ \beta=\frac{1}{G}, \tag{2.2}$$

then from (2.1) and (2.2), we obtain

$$\phi^{'}=-\phi^2+\beta\eta-\lambda \qquad \beta^{'}=-\phi\beta \tag{2.3}$$

Depending on the sign of the parameter λ, there are three possible ways to express the solution to Eqn. (2.1). These examples come from the following:

Case I. When $\lambda<0$, the general solution of LODE (2.1) is available

$$G(\zeta)=E_1\sinh(\zeta\sqrt{-\lambda})+E_2\cosh(\zeta\sqrt{-\lambda})+\frac{\eta}{\lambda}, \tag{2.4}$$

where the integration's constants are represented by E_1 and E_2. Considering

$$\beta^2=\frac{-\lambda}{\lambda^2\rho+\eta^2}(\phi^2-2\eta\beta+\lambda), \tag{2.5}$$

where $\rho=E_1^2-E_2^2$.

Case II. If $\lambda>0$

$$G(\zeta)=E_1\sin(\zeta\sqrt{\lambda})+E_2\cos(\zeta\sqrt{\lambda})+\frac{\eta}{\lambda}, \tag{2.6}$$

where E_1 and E_2 are arbitrary constants. Considering

$$\beta^2=\frac{\lambda}{\lambda^2\rho-\eta^2}(\phi^2-2\eta\beta+\lambda), \tag{2.7}$$

where $\rho=E_1^2+E_2^2$.

Case III. If $\lambda = 0$

$$G(\zeta) = \frac{\eta}{2}\zeta^2 + E_1\zeta + E_2, \tag{2.8}$$

where E_1 and E_2 are reel numbers. Considering

$$\beta^2 = \frac{\lambda}{E_1^2 - 2E_2\eta}(\phi^2 - 2\eta\beta + \lambda). \tag{2.9}$$

Next, we are primarily interested in solving the general nonlinear evolution problem using the double $(G^{'}/G,1/G)$-EM. Suppose that $u = u(x,t)$ is an unknown function that depends on the x, y, and t variables. We define the polynomial Q in $u(x,t)$ and its various order partial derivatives with nonlinear terms as follows:

$$Q(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0. \tag{2.10}$$

The following procedures are used to solve equation (2.10):

Step I. We use an initial transition in order to

$$u(x,t) = u(\zeta),\ \zeta = x - et, \tag{2.11}$$

where k is the travelling wave's velocity. Equation (2.10) is transformed into an ordinary differential equation (ODE) for $u = u(\zeta)$ by equation (2.11) in the manner shown below.

$$Q(u, -vu^{'}, u^{'}, v^2u^{''}, -vu^{''}, u^{''}, ...) = 0, \tag{2.12}$$

where Q the polynomial which have the elements $u(\zeta)$ and its derivatives are with respect to ζ.

Step II. The solution to equation (2.12) is thus assumed to be stated as follows in terms of the polynomials $\phi(\zeta)$ and $\beta(\zeta)$:

$$u(\zeta) = \sum_{k=0}^{m} a_k\phi^k + \sum_{k=1}^{m} b_k\phi^{k-1}\beta, \tag{2.13}$$

where equation (2.1) is satisfied by $G = G(\zeta)$, and the constants $a_k(k = 0,...,m)$, $b_k(k = 0,...,m)$, k, λ, and η are to be found later. We apply the homogeneous balancing approach to determine the value of m.

Step III. After determining the value of m, we replace equation (2.12) with Eqn. (2.13) and, using Eqn. (2.3) with Eqn. (2.5) (as in instance 1), we convert Eqn. (2.12) into a polynomial of ϕ^k and β^k, where the degree is any integer and which are $k \le 1$, $0 \le k \le n$ and n. Consequently, a system of algebraic equations in the following cases can be obtained: $a_i(i = 0,...,m)$, $b_k(k = 0,...,m)$, e, λ ($\lambda < 0$), η, E_1 and E_2.

Step IV. We use the computer software program to quickly retrieve the solution to the algebraic Eqns. that we deduced in Step 3. We get values for a_k, b_k, e, λ ($\lambda < 0$), η, E_1 and E_2, if there is a potential solution. We can now find the TWS for Eqn. (2.12) in terms of hyperbolic functions for case 1 by substituting these values into equation (2.13). The solution to the NLPDE in equation (2.10) is then obtained by reversing the variables that we used in equation (2.11). Similarly, steps 3 and 4 can be used to derive the TWSs of equation (2.12) (i.e., equation (2.12)) in terms of rational functions and trigonometric functions, respectively, in order to obtain solutions for the cases 2 and 3.

2.2. APPLICATION OF SPECIAL EXPANSION METHOD ON (1+1) DIMENSIONAL MBBME

In this section, accurate TWSs of nonlinear (1+1)-DMBBME are found by using the technique outlined in Section 2. As a result, we observe that the travelling wave variables $\zeta = x - et$ decrease to the subsequent ODE in (1.1):

$$3(1-v)u + 3u'' - \alpha u^3 = 0. \tag{3.1}$$

The $m=1$ is obtained by balancing between u'' and u^3 in (3.1). After entering this balance figure into equation (2.13), that form the results is as follows:

$$u(\zeta) = a_0 + a_1\phi + b_1\beta, \tag{3.2}$$

where the constants a_0, a_1 and b_1 are yet to be found. The following three situations are going to be discussed.

Case I. For $\lambda < 0$:

Using (2.3) and (2.5) and substituting (3.2) into (3.1) will result in a polynomial on the left side of (3.1) in ϕ and β. When the coefficients of this polynomial are set to zero, an algebraic system of Eqns. in the following variables arises: a_0, a_1, b_1, η, and e. These equations can be solved to obtain the following answers:

$$\phi^3: \qquad -\alpha a_1^3 + 6a_1 + \frac{3\alpha a_1 b_1^2 \lambda}{\lambda^2\rho + \eta^2} = 0,$$

$$\phi^2\psi: \qquad 6b_1 - 3\alpha a_1^2 b_1 + \frac{\alpha b_1^3 \lambda}{\lambda^2\rho + \eta^2} = 0,$$

$$\phi^2: \qquad -3\alpha a_0 a_1^2 + \frac{3b_1\eta\lambda}{\lambda^2\rho + \eta^2} + \frac{2\alpha b_1^3 \eta\lambda^2}{(\lambda^2\rho + \eta^2)^2} + \frac{3\alpha a_0 b_1^2 \lambda}{\lambda^2\rho + \eta^2} = 0,$$

$$\phi\psi: \qquad -6\alpha a_0 a_1 b_1 - 9\eta a_1 - \frac{6\eta\alpha a_1 b_1^2 \lambda}{\lambda^2\rho + \eta^2} = 0,$$

ϕ : $$3a_1 - 3ea_1 - 3\alpha a_0^2 a_1 + 6\lambda a_1 + \frac{3\alpha a_1 b_1^2 \lambda^2}{\lambda^2 \rho + \eta^2} = 0,$$

ψ : $$3b_1 - 3eb_1 - 3\alpha a_0^2 b_1 + 3\lambda b_1 - \frac{6b_1\eta^2\lambda}{\lambda^2\rho+\eta^2} - \frac{4\alpha b_1^3\eta^2\lambda^2}{(\lambda^2\rho+\eta^2)^2} + \frac{\alpha b_1^3\lambda^2}{\lambda^2\rho+\eta^2} - \frac{6\eta\alpha a_0 b_1^2\lambda}{\lambda^2\rho+\eta^2} = 0,$$

ψ^0 : $$3a_0 - 3ea_0 - \alpha a_0^3 - \frac{3b_1\eta\lambda^2}{\lambda^2\rho+\eta^2} + \frac{2\alpha b_1^3\eta\lambda^3}{(\lambda^2\rho+\eta^2)^2} + \frac{3\alpha a_0 b_1^2\lambda^2}{\lambda^2\rho+\eta^2} = 0.$$

We have the solutions, by solving the above algebraic equations, which are given below:

Case I: $$(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{-6\lambda\sigma}{\alpha}}, 0, 1-\lambda).$$

Case II: $$(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{6}{\alpha}}, 0, 0, 1+2\lambda).$$

Case III: $$(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{3}{2\alpha}}, \pm\sqrt{\frac{3\lambda\sigma}{-2\alpha}}, 0, \frac{1}{2}(2+\lambda)).$$

By entering these values into equation (3.3) and using the cooperation of equations (2.2) and (2.4), we can find the TWSs to the NLMBBM equation (2.1) as follows:

For Case I: That is; for

$$(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{-6\lambda\sigma}{\alpha}}, 0, 1-\lambda)$$

we obtain the exact solution

$$u(\zeta) = \pm\sqrt{\frac{6\lambda\sigma}{-\lambda\alpha}}\left(\frac{1}{E_1\sinh(\zeta\sqrt{-\lambda}) + E_2\cosh(\zeta\sqrt{-\lambda})}\right), \quad (3.4)$$

where α is constant, $\rho = E_1^2 - E_2^2$ and $\zeta = x - (1-\lambda)t$.

If $E_1 \neq 0$ and $E_2 = 0$ in Eqn. (3.4), then we obtain the solitary wave solution (SWS) – Fig. 1 which is given by:

$$u(\zeta) = \pm\sqrt{\frac{-6\lambda}{\alpha}}\operatorname{sec}h(\zeta\sqrt{-\lambda}). \quad (3.4)_a$$

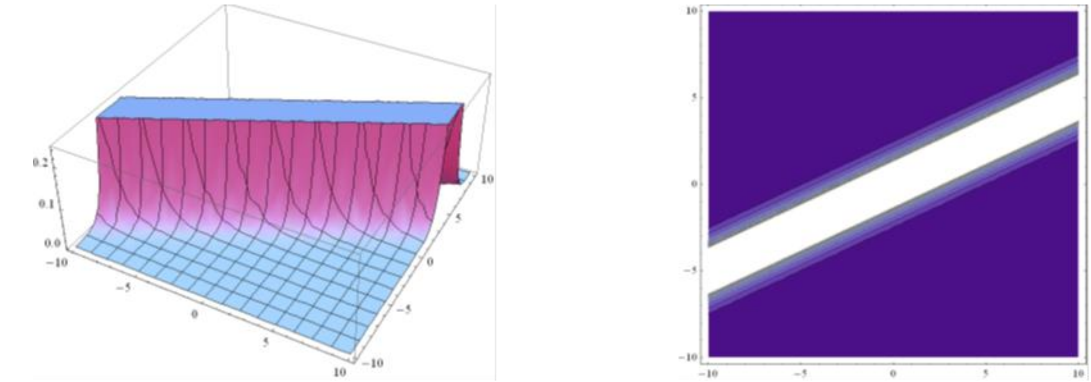

Figure 1(a). The 3D and Contour shape figures are respectively, when $\lambda=-1$ and $\alpha=2$ in Eqn. (3.4)$_a$.

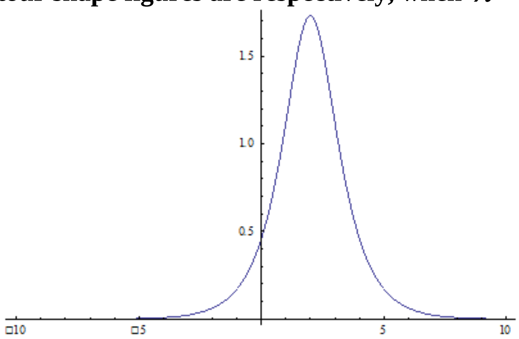

Figure 1(b). The 2D figure of Eqn. (3.4)$_a$, when $\lambda=-1$, $t=1$ and $\alpha=2$.

If $E_1=0$ and $E_2\neq 0$ in Eqn. (3.4), then we obtain the SWS – Fig. 2 which is defined by:

$$u(\zeta)=\pm\sqrt{\frac{-6\lambda}{\alpha}}\operatorname{csch}(\zeta\sqrt{-\lambda}). \tag{3.4$_b$}$$

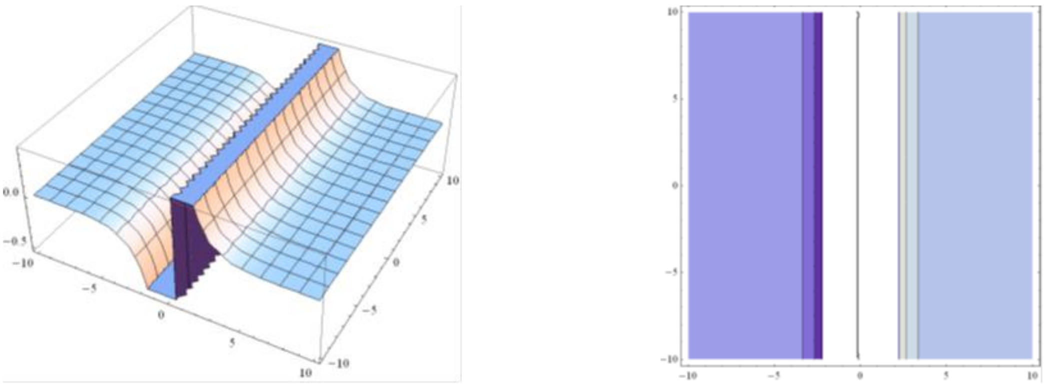

Figure 2(a). The 3D and Contour shape figures are respectively in Eqn. (3.4)$_b$, when $\lambda=-1$ and $\alpha=3$.

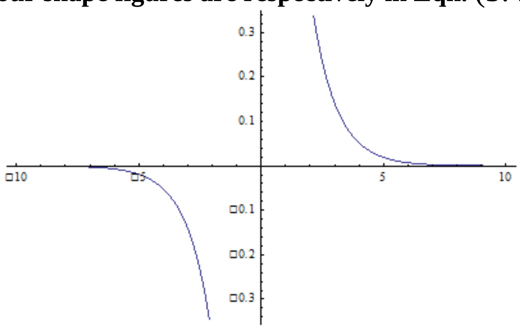

Figure 2(b). The 2D figure of Eqn. (3.4)$_b$, when $\lambda=-1$, $t=1$ and $\alpha=3$.

For case II: That is; for

$$(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{6}{\alpha}}, 0, 0, 1+2\lambda)$$

we have the exact solution

$$u(\zeta) = \pm\sqrt{\frac{6}{\alpha}}\left(\frac{\sqrt{-\lambda}(E_1\cosh(\zeta\sqrt{-\lambda}) + E_2\sinh(\zeta\sqrt{-\lambda}))}{E_1\sinh(\zeta\sqrt{-\lambda}) + E_2\cosh(\zeta\sqrt{-\lambda})}\right), \tag{3.5}$$

where α be constant and $\zeta = x-(1+2\lambda)t$.

If $E_1 \neq 0$ and $E_2 = 0$ in Eqn. (3.5), then we obtain the TWS – Fig. 3 which is defined by:

$$u(\zeta) = \pm\sqrt{\frac{-6\lambda}{\alpha}}\tan h(\zeta\sqrt{-\lambda}). \tag{3.5)$_a$ (}$$

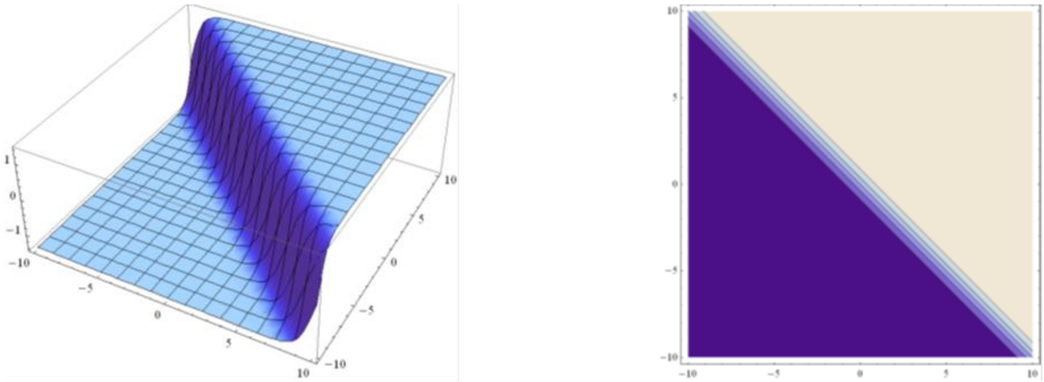

Figure 3(a). The 3D and Contour shape figures are respectively of Eqn. $(3.5)_a$, when $\lambda = -1$ and $\alpha = 3$.

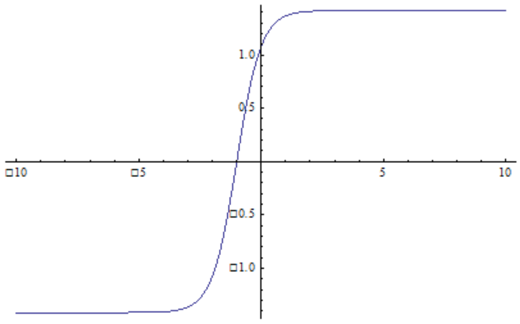

Figure 3(b). The 2D figure of Eqn. $(3.5)_a$, when $\lambda = -1$, $t = 1$ and $\alpha = 3$.

If $E_1 = 0$ and $E_2 \neq 0$ in Eqn. (3.5), then we get the SWS – Fig. 4 which is given as:

$$u(\zeta) = \pm\sqrt{\frac{6\lambda}{-\alpha}}\cot h(\zeta\sqrt{-\lambda}). \qquad (3.5)_b$$

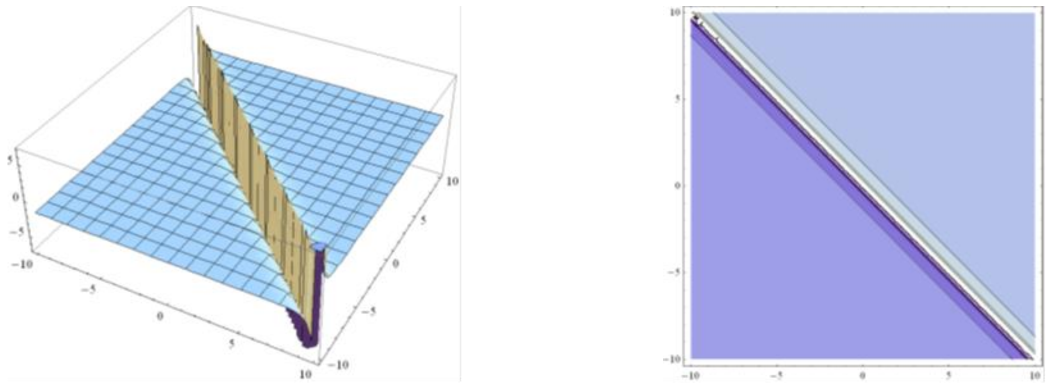

Figure 4(a). The 3D and Contour shape figures are respectively of Eqn. $(3.5)_b$, **when** $\lambda = -1$ **and** $\alpha = 2$.

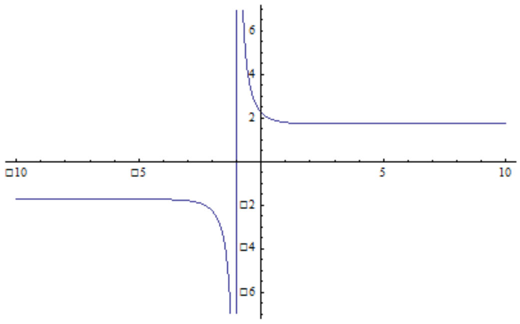

Figure 4(b). The 2D figure of Eqn. $(3.5)_b$, **when** $\lambda = -1$, $t = 1$ **and** $\alpha = 2$.

For case III: That is for

$$(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{3}{2\alpha}}, \pm\sqrt{\frac{3\lambda\sigma}{-2\alpha}}, 0, \frac{1}{2}(2+\lambda))$$

we have the exact solution

$$u(\zeta) = \pm\sqrt{\frac{3}{2\alpha}}\left(\frac{\sqrt{-\lambda}(E_1\cosh(\zeta\sqrt{-\lambda}) + E_2\sinh(\zeta\sqrt{-\lambda}))}{E_1\sinh(\zeta\sqrt{-\lambda}) + E_2\cosh(\zeta\sqrt{-\lambda})}\right)$$
$$\pm\sqrt{\frac{-3\lambda\sigma}{2\alpha}}\left(\frac{1}{E_1\sinh(\zeta\sqrt{-\lambda}) + E_2\cosh(\zeta\sqrt{-\lambda})}\right), \quad (3.6)$$

where α is constant, $\sigma = E_1^2 - E_2^2$ and $\zeta = x - \frac{1}{2}(2+\lambda)t$.

If $E_1 \neq 0$ and $E_2 = 0$ in Eqn. (3.6), then we get the SWS – Fig. 5 which will be

$$u(\zeta) = \pm\sqrt{\frac{-3\lambda}{2\alpha}}\tanh(\zeta\sqrt{-\lambda}) \pm \sqrt{\frac{-3\lambda}{2\alpha}}\text{sech}(\zeta\sqrt{-\lambda}). \quad (3.6)_a$$

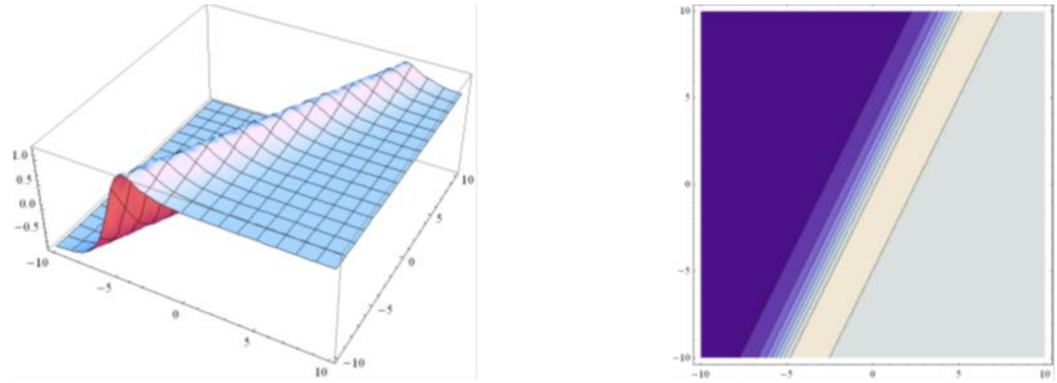

Figure 5(a). The 3D and Contour shape figures are respectively of Eqn. $(3.6)_a$**, when** $\lambda = -1$ **and** $\alpha = 2$**.**

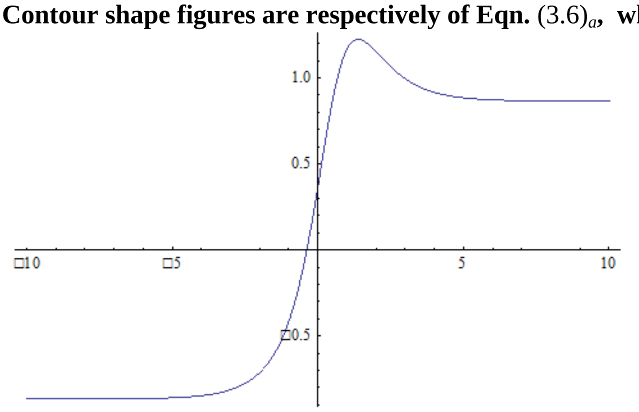

Figure 5(b). The 3D figure of Eqn. $(3.6)_a$**, when** $\lambda = -1$**,** $t = 1$ **and** $\alpha = 2$**.**

If $E_1 \neq 0$ and $E_2 = 0$ in Eqn. (3.6), then we have the TWS – Fig. 6 which is defined by

$$u(\zeta) = \pm\sqrt{\frac{-3\lambda}{2\alpha}}\coth(\zeta\sqrt{-\lambda}) \pm \sqrt{\frac{-3\lambda}{2\alpha}}\text{csch}(\zeta\sqrt{-\lambda}). \qquad (3.6)_b$$

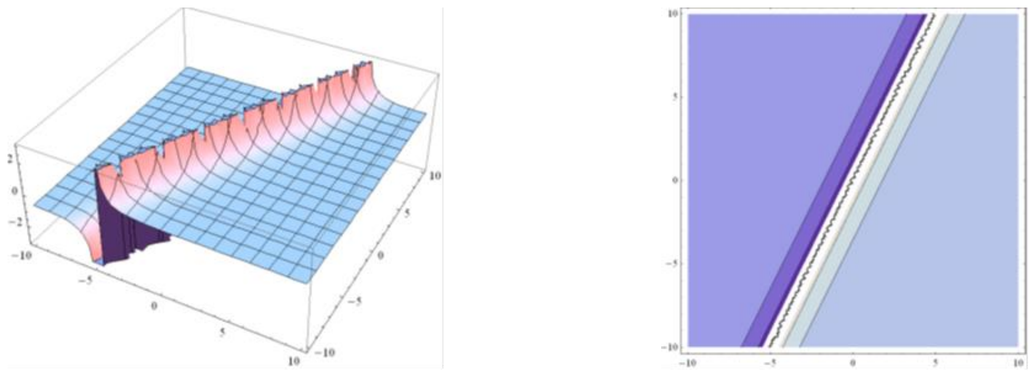

Figure 6(a). The 3D and Contour shape figures are respectively of Eqn. $(3.6)_b$**, when** $\lambda = -1$ **and** $\alpha = 3$**.**

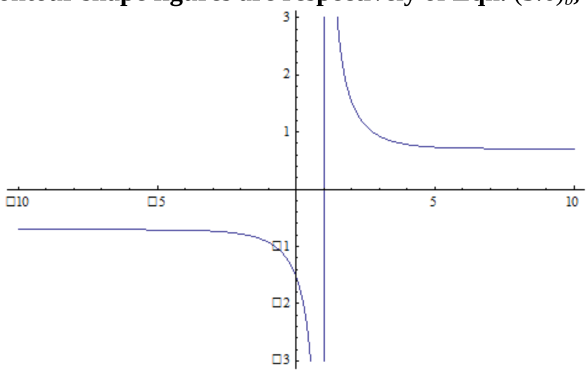

Figure 6(b). The 2D figure of Eqn. $(3.6)_b$**, when** $\lambda = -1$**,** $t = 1$ **and** $\alpha = 3$**.**

Case II. For $\lambda > 0$:

Using (2.3) and (2.5) and substituting (3.2) into (3.1) will result in a polynomial on the left side of (3.1) in ϕ and β. When the coefficients of this polynomial are set to zero, an algebraic system of Eqns. in the following variables arises: a_0, a_1, b_1, η, and e. These Eqns. can be solved to obtain the following answers:

$$\phi^3: \quad -\alpha a_1^3 + 6a_1 - \frac{3\alpha a_1 b_1^2 \lambda}{\lambda^2\rho - \eta^2} = 0,$$

$$\phi^2\psi: \quad 6b_1 - 3\alpha a_1^2 b_1 - \frac{\alpha b_1^3 \lambda}{\lambda^2\rho - \eta^2} = 0,$$

$$\phi^2: \quad -3\alpha a_0 a_1^2 - \frac{3b_1\eta\lambda}{\lambda^2\rho - \eta^2} + \frac{2\alpha b_1^3 \eta\lambda^2}{(\lambda^2\rho - \eta^2)^2} - \frac{3\alpha a_0 b_1^2 \lambda}{\lambda^2\rho - \eta^2} = 0,$$

$$\phi\psi: \quad -6\alpha a_0 a_1 b_1 - 9\eta a_1 + \frac{6\eta\alpha a_1 b_1^2 \lambda}{\lambda^2\rho - \eta^2} = 0,$$

$$\phi: \quad 3a_1 - 3ka_1 - 3\alpha a_0^2 a_1 + 6\lambda a_1 - \frac{3\alpha a_1 b_1^2 \lambda^2}{\lambda^2\rho - \eta^2} = 0,$$

$$\psi: \quad 3b_1 - 3eb_1 - 3\alpha a_0^2 b_1 + 3\lambda b_1 + \frac{6b_1\eta^2\lambda}{\lambda^2\rho - \eta^2} - \frac{4\alpha b_1^3 \eta^2\lambda^2}{(\lambda^2\rho - \eta^2)^2} - \frac{\alpha b_1^3 \lambda^2}{\lambda^2\rho - \eta^2} + \frac{6\eta\alpha a_0 b_1^2 \lambda}{\lambda^2\rho - \eta^2} = 0,$$

$$\psi^0: \quad 3a_0 - 3ea_0 - \alpha a_0^3 - \frac{3b_1\eta\lambda^2}{\lambda^2\rho - \eta^2} + \frac{2\alpha b_1^3 \eta\lambda^3}{(\lambda^2\rho - \eta^2)^2} - \frac{3\alpha a_0 b_1^2 \lambda^2}{\lambda^2\rho - \eta^2} = 0.$$

We have the answers after solving the aforementioned algebraic equations:

Case I: $(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{6\lambda\rho}{\alpha}}, 0, 1-\lambda).$

Case II: $(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{6}{\alpha}}, 0, 0, 1+2\lambda).$

Case III: $(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{3}{2\alpha}}, \pm\sqrt{\frac{3\lambda\rho}{2\alpha}}, 0, \frac{1}{2}(2+\lambda)).$

We obtain the TW solutions of the NLMBBM equation (1.1) as follows by substituting these values into Eqn. (3.3), with collaboration of (2.2) and (2.6):
For Case I: That is; for

$$(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{6\lambda\rho}{\alpha}}, 0, 1-\lambda)$$

we obtain the exact solution

$$u(\zeta)=\pm\sqrt{\frac{6\lambda\rho}{\alpha}}\left(\frac{1}{E_1\sin(\zeta\sqrt{\lambda})+E_2\cos(\zeta\sqrt{\lambda})}\right), \tag{3.7}$$

where α is unknown parameter, $\rho=E_1^2+E_2^2$ and $\zeta=x-(1-\lambda)t$.

If $E_1=0$ and $E_2\neq 0$, then we have the SWS – Fig. 7 of Eqn. (3.7) given by

$$u(\zeta)=\pm\sqrt{\frac{6\lambda}{\alpha}}\sec(\zeta\sqrt{\lambda}). \qquad (3.7)_a$$

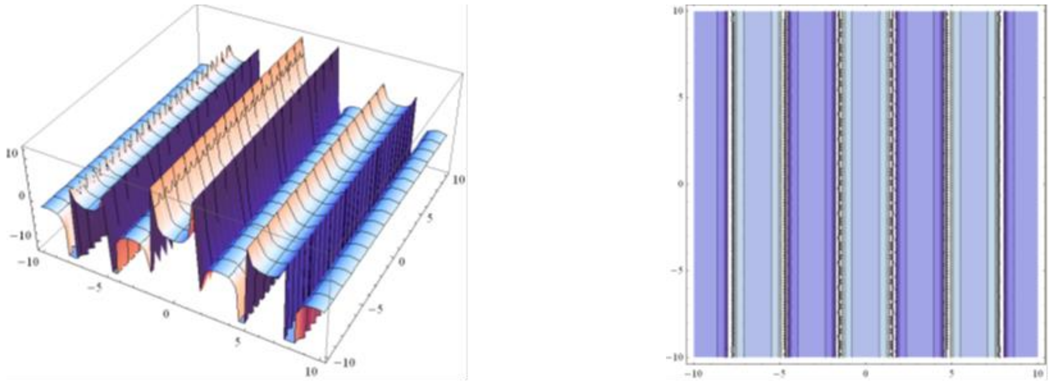

Figure 7(a). The 3D and Contour shape figures are respectively of Eqn. $(3.7)_a$, when $\lambda=1$ and $\alpha=2$.

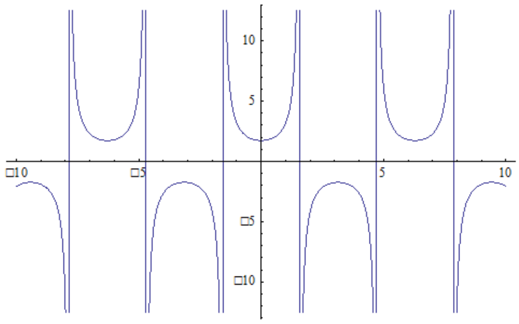

Figure 7(b). The 2D figure of Eqn. $(3.7)_a$, when $\lambda=1$, $t=1$ and $\alpha=2$.

If $E_1\neq 0$ and $E_2=0$, then we have the SWS – Fig. 8 of Eqn. (3.7) given by

$$u(\zeta)=\pm\sqrt{\frac{6\lambda}{\alpha}}\csc(\zeta\sqrt{\lambda}). \qquad (3.7)_b$$

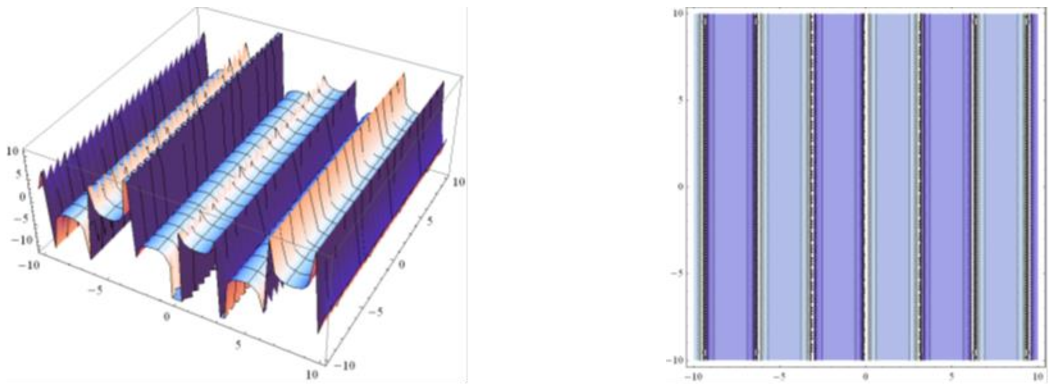

Figure 8(a). The 3D and Contour shape figures are respectively of Eqn. $(3.7)_b$, when $\lambda=1$ and $\alpha=2$.

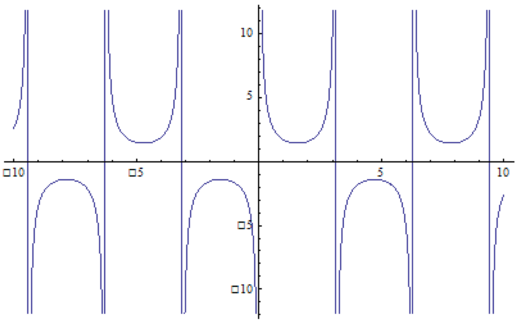

Figure 8(b). The 2D figure of Eqn. $(3.7)_b$, when $\lambda=1$, $t=1$ and $\alpha=2$.

For Case II: That is; for

$$(a_0,a_1,b_1,\eta,e)=(0,\pm\sqrt{\frac{6}{\alpha}},0,0,1+2\lambda)$$

we obtain the exact solution

$$u(\zeta)=\pm\sqrt{\frac{6}{\alpha}}\left(\frac{\sqrt{\lambda}(E_1\cos(\zeta\sqrt{\lambda})+E_2\sin(\zeta\sqrt{\lambda}))}{E_1\sin(\zeta\sqrt{\lambda})+E_2\cos(\zeta\sqrt{\lambda})}\right), \tag{3.8}$$

where α is a unknown parameter and $\zeta=x-(1+2\lambda)t$.

If $E_1=0$ and $E_2\neq 0$, then we have the SWS – Fig. 9 of Eqn. (3.8) given by

$$u(\zeta)=\pm\sqrt{\frac{6\lambda}{\alpha}}\tan(\zeta\sqrt{\lambda}). \qquad (3.8)_a$$

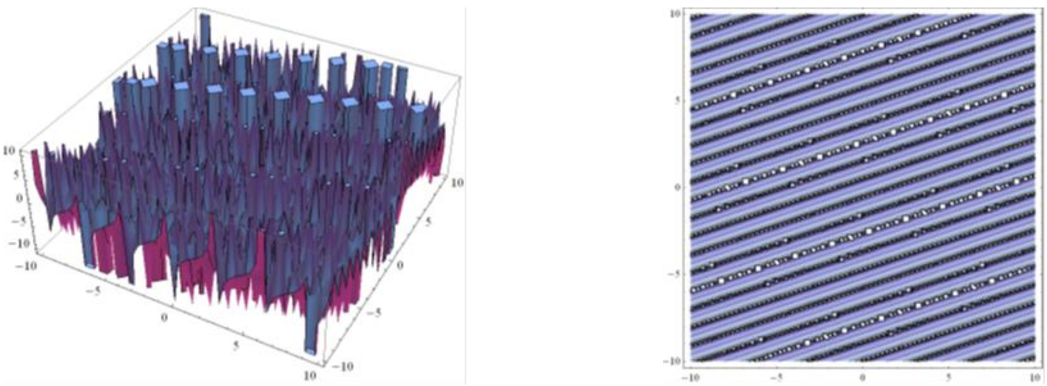

Figure 9(a). The 3D and Contour shape figures are respectively of Eqn. $(3.8)_a$, when $\lambda=1$ and $\alpha=2$.

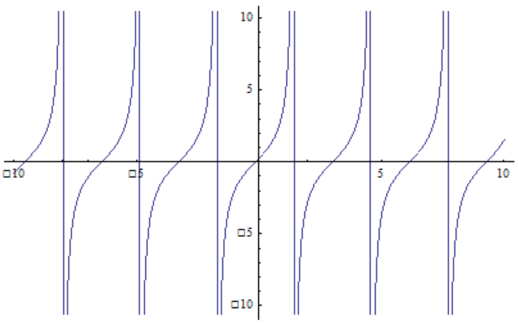

Figure 9(b). The 2D figure of Eqn. $(3.8)_a$, when $\lambda=1$, $t=1$ and $\alpha=2$.

If $E_1 \neq 0$ and $E_2 = 0$, then we have the SWS – Fig. 10 of Eqn. (3.8) defined by

$$u(\zeta) = \pm\sqrt{\frac{6\lambda}{\alpha}}\cot(\zeta\sqrt{\lambda}). \tag{3.8$_b$}$$

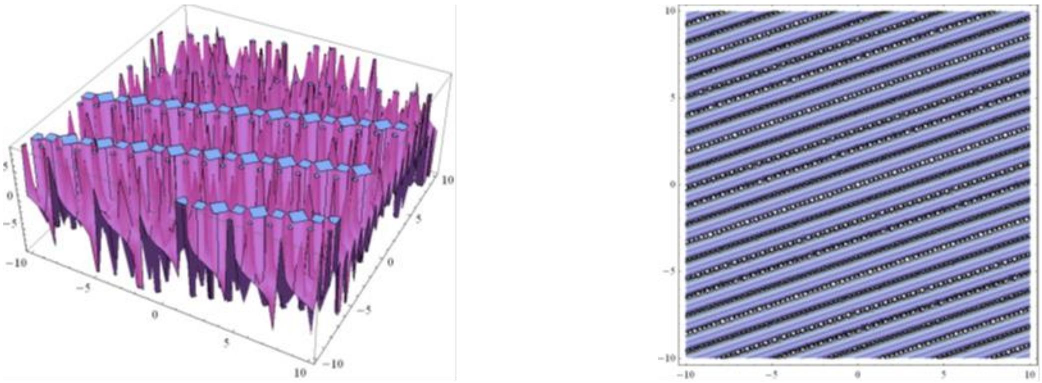

Figure 10(a). The 3D and Contour shape figures are respectively of Eqn. $(3.8)_b$, when $\lambda = 1$ and $\alpha = 3$

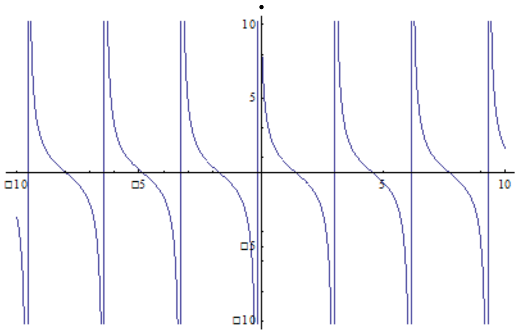

Figure 10(b). The 2D figure of Eqn. $(3.8)_b$, when $\lambda = 1$, $t = 1$ and $\alpha = 3$.

Case III: That is for

$$(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{3}{2\alpha}}, \pm\sqrt{\frac{3\lambda\rho}{2\alpha}}, 0, \frac{1}{2}(2+\lambda))$$

we have the exact solution

$$u(\zeta) = \pm\sqrt{\frac{3}{2\alpha}}\left(\frac{\sqrt{\lambda}(E_1\cos(\zeta\sqrt{\lambda}) + E_2\sin(\zeta\sqrt{\lambda}))}{E_1\sin(\zeta\sqrt{\lambda}) + E_2\cos(\zeta\sqrt{\lambda})}\right) \pm \sqrt{\frac{3\lambda\rho}{2\alpha}}\left(\frac{1}{E_1\sin(\zeta\sqrt{\lambda}) + E_2\cos(\zeta\sqrt{\lambda})}\right), \tag{3.9}$$

where α is unknown parameter, $\rho = E_1^2 + E_2^2$ and $\zeta = x - \frac{1}{2}(2+\lambda)t$.

If $E_1 = 0$ and $E_2 \neq 0$, then we have the SWS – Fig. 11 of (3.9) defined by

$$u(\zeta) = \pm\sqrt{\frac{3\lambda}{2\alpha}}\tan(\zeta\sqrt{\lambda}) \pm \sqrt{\frac{3\lambda}{2\alpha}}\sec(\zeta\sqrt{\lambda}). \tag{3.9$_a$}$$

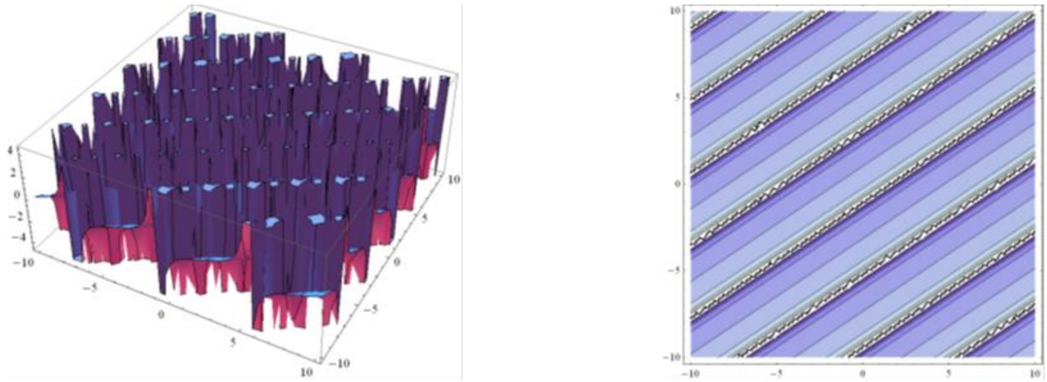

Figure 11(a). The 3D and Contour shape figures are respectively of Eqn. $(3.9)_a$, **when** $\lambda = 1$ **and** $\alpha = 3$.

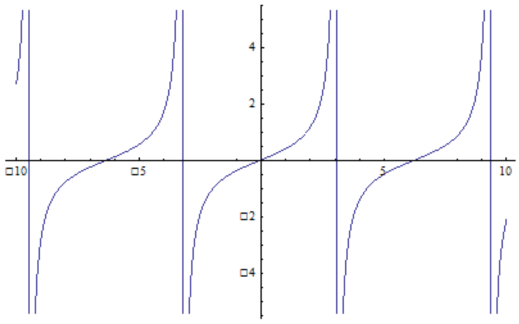

Figure 11(b). The 2D figure of Eqn. $(3.9)_a$, **when** $\lambda = 1$, $t = 1$ **and** $\alpha = 3$.

If $E_1 \neq 0$ and $E_2 = 0$, then we have the SWS of (3.9) is given by

$$u(\zeta) = \pm\sqrt{\frac{3\lambda}{2\alpha}}\cot(\zeta\sqrt{\lambda}) \pm \sqrt{\frac{3\lambda}{2\alpha}}\csc(\zeta\sqrt{\lambda}). \qquad (3.9)_b$$

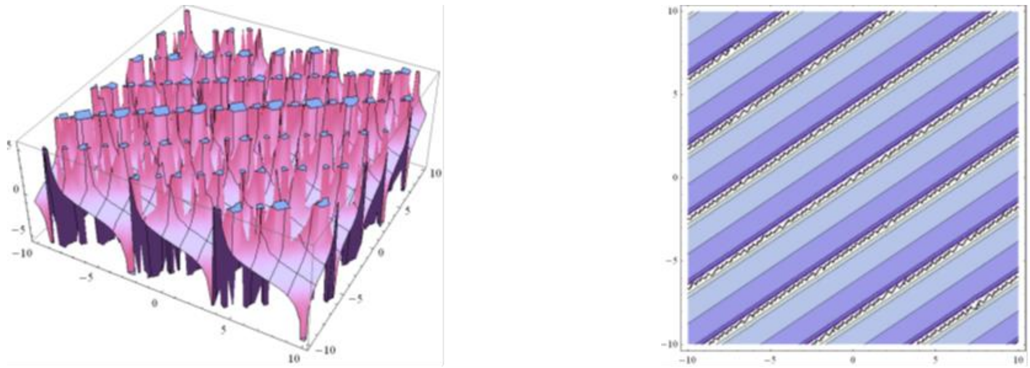

Figure 12(a). The 3D and Contour shape figures are respectively of Eqn. $(3.9)_b$, **when** $\lambda = 1$ **and** $\alpha = 2$.

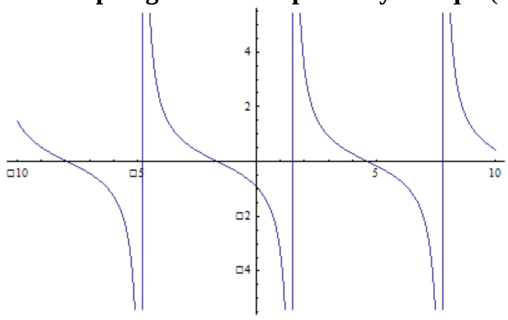

Figure 12(b). The 2D figure of Eqn. $(3.9)_b$, **when** $\lambda = 1$, $t = 1$ **and** $\alpha = 2$.

Case III: When $\lambda = 0$:

Using (2.3) and (2.5) and substituting (3.2) into (3.1) will result in a polynomial on the left side of (3.1) in ϕ and β. When the coefficients of this polynomial are set to zero, an algebraic system of Eqns. in the following variables arises: a_0, a_1, b_1, η, and e. These Eqns. can be solved to obtain the following answers:

$$\phi^3: \quad -\alpha a_1^3 + 6a_1 - \frac{3\alpha a_1 b_1^2}{E_1^2 - 2\eta E_2} = 0,$$

$$\phi^2\psi: \quad 6b_1 - 3\alpha a_1^2 b_1 - \frac{\alpha b_1^3}{E_1^2 - 2\eta E_2} = 0,$$

$$\phi^2: \quad -3\alpha a_0 a_1^2 - \frac{3b_1\eta}{E_1^2 - 2\eta E_2} + \frac{2\alpha b_1^3\eta}{(E_1^2 - 2\eta E_2)^2} - \frac{3\alpha a_0 b_1^2}{E_1^2 - 2\eta E_2} = 0,$$

$$\phi\psi: \quad -6\alpha a_0 a_1 b_1 - 9\eta a_1 + \frac{6\eta\alpha a_1 b_1^2}{E_1^2 - 2\eta E_2} = 0,$$

$$\phi: \quad 3a_1 - 3ea_1 - 3\alpha a_0^2 a_1 = 0,$$

$$\psi: \quad 3b_1 - 3eb_1 - 3\alpha a_0^2 b_1 + \frac{6b_1\eta^2}{E_1^2 - 2\eta E_2} - \frac{4\alpha b_1^3\eta^2}{(E_1^2 - 2\eta E_2)^2} + \frac{6\eta\alpha a_0 b_1^2\lambda}{E_1^2 - 2\eta E_2} = 0,$$

$$\psi^0: \quad 3a_0 - 3ea_0 - \alpha a_0^3 = 0.$$

From the above algebraic Eqns., we get the solution

Case I: $(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{6E_1^2}{\alpha}}, 0, 1).$

Case II: $(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{6}{\alpha}}, 0, 0, 1).$

Case III: $(a_0, a_1, b_1, \eta, e) = (0, \pm\sqrt{\frac{3}{2\alpha}}, \pm\sqrt{\frac{3E_1^2}{2\alpha}}, 0, 1).$

By inserting these values into Eqn. (3.3), with the help of Eqns. (2.2) and (2.8), we can derive the TWSs to the NLMBBM problem (1.1) in the following ways for each case:
Now, we find the general solution of above three cases.
For Case I: That is for

$$(a_0, a_1, b_1, \eta, e) = (0, 0, \pm\sqrt{\frac{6E_1^2}{\alpha}}, 0, 1)$$

we have the exact solution

$$u(\zeta)=\pm\sqrt{\frac{6E_1^2}{\alpha}}\left(\frac{1}{E_1\zeta+E_2}\right), \tag{3.10}$$

where α is unknown parameter and $\zeta = x-(1)t$.
For Case II: That is for

$$(a_0,a_1,b_1,\eta,e)=(0,\ \pm\sqrt{\frac{6}{\alpha}},0,0,1)$$

we have the exact solution

$$u(\zeta)=\pm\sqrt{\frac{6}{\alpha}}\left(\frac{E_1}{E_1\zeta+E_2}\right), \tag{3.11}$$

where α is unknown parameter and $\zeta = x-(1)t$.
For Case III: That is for

$$(a_0,a_1,b_1,\eta,e)=(0,\ \pm\sqrt{\frac{3}{2\alpha}},\pm\sqrt{\frac{3E_1^2}{2\alpha}},0,1)$$

we find the exact solution

$$u(\zeta)=\pm\sqrt{\frac{3}{2\alpha}}\left(\frac{E_1}{E_1\zeta+E_2}\right)\pm\sqrt{\frac{3E_1^2}{2\alpha}}\left(\frac{1}{E_1\zeta+E_2}\right), \tag{3.12}$$

where α is constant and $\zeta = x-(1)t$.

3. CONCLUSIONS

With the help of the $(G'/G,1/G)$-EM, the (1+1)-DMBBME (1.1) has been effectively solved. We derive the following SWSs [31, 32] as the two parameters E_1 and E_2 take on peculiar values. The $(G'/G,1/G)$-EM is the (G'/G)-EM, when in both $b_i=0$ in expansion (2.13) and $\eta=0$ in (2.1). Verifying that the answers (3.5), (3.8), and (3.11) agree completely with the outcomes produced by the (G'/G)-EM is a simple process.

Additionally, we can contrast alternative approaches, such the extended hyperbolic function method, with the $(G'/G,1/G)$-EM. In order to build the solutions for NLEEs, Riccati equations are selected as its subsidiary ODEs. It is important to highlight that, despite the fact that $\phi=G'/G$ and $\beta=1/G$ also satisfy the projective Riccati equations (2.3) in this work, we make no use of the particular solutions provided by Eqn. (2.3). Rather, we generate the solutions of the NLEEs directly using the general solution of the second order LODE (2.1), which is widely known to researchers. As a result, the $(G'/G,1/G)$-EM has its own benefits, which are clear, simple, and straightforward. This method has proven to be effective in finding analytical solutions for NLPDEs. By applying the $(G'/G,1/G)$-EM the solution to the (1+1)-DMBBME has been obtained, providing valuable insights into the behavior of the system. The graph provides a visual representation of the relationship between the variables and helps to interpret the solution in a more intuitive manner. The graphical analysis enhances our comprehension of the MBBM equation's properties and facilitates its application in various scientific and engineering domains. The results obtained through this method contribute to our understanding of the dynamics of the MBBME and its applications in various fields of science and engineering.

REFERENCES

[1] Liu, J. G., Eslami, M., Rezazadeh, H., Mirzazadeh, M., *Nonlinear Dynamics*, **95**, 1027, 2019.
[2] Zahran, E. H., Bekir, A., Alotaibi, M. F., Omri, M., Ahmed, H., *Results in Physics*, **29**, 104730, 2021.
[3] Ismael, H. F., Bulut, H., Park, C., Osman, M. S., *Results in Physics*, **19**, 103329, 2020.
[4] Tala-Tebue, E., Tetchoka-Manemo, C., Rezazadeh, H., Bekir, A., Chu, Y. M., *Results in Physics*, **19**, 103514, 2020.
[5] Yang, J. J., Tian, S. F., Li, Z. Q., *Journal of Mathematical Analysis and Application*, **511**(2), 2022.
[6] Its, A., Its, E., Kaplunov, J., *Letters in Mathematical Physics*, **96**, 53, 2011.
[7] Ali, M. R., Ma, W. X., Sadat, R., *East Asian Journal Applied Mathematics*, **12**, 201, 2012.
[8] Kumar, A., Kumar, S., Kharbanda, H., *Modern Physics Letters B*, **36**(07), 2150609, 2022.
[9] Liu, Y., Wu, G., *Journal of Applied Mathematics and Physics*, **9**(12), 2021.
[10] Akbulut, A., Kaplan, M., *Computers and Mathematics with Applications*, **75**(3), 876, 2018.
[11] Korkmaz, A., Hepson, O. E., Hosseini, K., Rezazadeh, H., Eslami, M., *Journal of King Saud University-Science*, **32**(1), 567, 2020.
[12] Ali, S., *Journal of Pure Applied Mathematics*, **5**, 61, 2021.
[13] Jawad, A. J. A. M., *American Journal of Numerical Analysis*, **1**(1), 32, 2013.
[14] Rasool, T., Hussain, R., Al Sharif, M. A., Mahmoud, W., Osman, M. S., *Optical and Quantum Electronics*, **55**(5), 396, 2023.
[15] Jafari, H., Kadkhoda, N., Khalique, C. M., *Computers & Mathematics with Applications*, **64**(6), 2084, 2012.
[16] Hossain, K. S., Akbar, M. A., *International Journal of Applied Mathematics Theory of Physics*, **3**, 20, 2017.
[17] Inan, I. E., Inc, M., Yepez-Martinez, H., Mahmoud, K. H., *Journal of Ocean Engineering and Science*, 2022.
[18] Sweilam, N. H., Kader, M. M., *Computer & Mathematics with Applications*, **58**(12), 2009.
[19] Nadjafi, J. S., Buzhabad, R., Nik, H. S., *Applied Mathematics*, **3**(8), 2012.
[20] Adomian, G., *Mathematical and Computer Modelling*, **13**, 17, 1998.
[21] Wang, M., Li, X., Zhang, J., *Physics Letters A*, **372**(4), 417, 2008.
[22] Gao, X., Zhang, J., *Physics Letters A*, **375**, 533, 2011.
[23] Li, L. X., Li, E. Q., Wang, M. L., *Applied Mathematics-A Journal of Chinese Universities*, **25**, 454, 2010.
[24] Zayed, E. M. E., Alurrfi, K. A. E., *Optik*, **127**(4), 1581, 2016.
[25] Demiray, S., Unsal, O., Bekir, A., *Journal of the Egyptian Mathematical Society*, **23**, 78, 2015.
[26] Shakeel, M., Mohyud-Din, S. T., *Ain Shams Engineering Journal*, **5**, 951, 2014.
[27] Akbulut, A., Kaplan, M., Tascan, F., *Zeitschrift für Naturforschung A*, **71**(5), 439, 2016.
[28] Demiray, S., Unsal, O., Bekir, A., *Acta Physica Polonica A*, **125**(5), 1093, 2014.
[29] Bekir, A**.,** Hasibun, N**.,** Farah, A. A., *Applied Mathematics and Computation*, **215**, 2857, 2009.
[30] Aslan, I., *Applied Mathematics and Computation*, **215**(2), 857, 2009.