ORIGINAL PAPER

HASIMOTO SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

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Abstract. *In this study, we have investigated Hasimoto surfaces with a pointwise 1-type Gauss map, which will be useful in classifying surfaces. Firstly, we have reminded the basic concepts related to the Gauss map of a surface. Later, we have obtained the necessary conditions for surfaces to be the first kind, the second kind, and the harmonic type of the pointwise 1-type Gauss map for Hasimoto surfaces. Finally, we have provided an example of a Hasimoto surface with pointwise 1-type Gauss map and plotted its graphic.*

Keywords: *Hasimoto surface; Gauss map; Laplace operator; circular helix; pointwise 1-type map.*

1. INTRODUCTION

The concept of finite-type submanifolds was developed by Chen in Euclidean and semi-Euclidean spaces, and this finite-type concept was later extended to differentiable maps defined on submanifolds. In this way, the finite-type Gauss map concept, which is very useful in classifying surfaces, was developed [1, 2]. Additionally, Chen and Piccinni studied the submanifolds of Euclidean spaces with the finite-type Gauss map and formed the basis of the theory of submanifolds with finite-type Gauss map and classified compact surfaces with 1-type Gauss map [3]. Besides, the ruled surfaces with a finite type Gauss map in Euclidean space have been studied by many researchers [4-7]. In particular, the point 1-type Gauss map of Darboux ruled surfaces was examined and classified in [8].

On the other hand, the geometric and algebraic properties of Hasimoto surfaces were investigated in detail by [9, 10]. These surfaces were considered as well concerning different frames in Euclidean, Minkowski, and Galilean spaces [10-14]. However, the Gauss maps of the Hasimoto surfaces obtained from integrable curves have, to the best of the authors' knowledge, not been considered in the studies carried out so far.

Our aim in this study is to investigate the Hasimoto surfaces with the pointwise 1-type Gauss map. For this purpose, the condition to be the first kind, the second kind, and the harmonic of the point 1-type Gauss map of these surfaces are given, separately. Also, the graph of an example of these surfaces is drawn.

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2. GEOMETRIC CONCEPTS ASSOCIATED WITH GAUSS MAP IN EUCLIDEAN 3-SPACE

Let M be an oriented regular surface in R^3 parametrized by $r: U \subset R^2 \to M \subset R^3$. In the following, we use (u_1, u_2) to denote coordinates on the parameter plane R^2 and (x_1, x_2, x_3) to denote coordinates in the ambient R^3. Then the Gauss map, which is denoted by Ω, of a surface M is given by

$$\Omega : M \to S^2 \subset R^3$$
$$p \to \Omega(p) = \frac{r_{u_1} \times r_{u_2}}{\left\| r_{u_1} \times r_{u_2} \right\|},$$

where S^2 is the unit sphere in R^3.

Let $g_{ij} = \left\langle \frac{\partial r}{\partial u_i}, \frac{\partial r}{\partial u_j} \right\rangle$, $(1 \le i, j \le 2)$, be an induced metric on M, then the Laplacian operator Δ with respect to the induced metric on M is defined as

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial u_j} \right).$$

Also, the matrix (g_{ij}) consisting of the components of the metric on M and the inverse matrix (g^{ij}) of (g_{ij}) can be written as

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \; g^{ij} = \frac{1}{\det(g_{ij})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix},$$

respectively. Besides, $g = \det(g_{ij})$ denotes the determinant of the matrix (g_{ij}).

Let M be a surface in Euclidean 3-space, the Gauss map Ω of M is 1–type Gauss map if and only if the Gauss map Ω is to satisfy the equation $\Delta\Omega = \lambda(\Omega + C)$ for a non-zero real constant λ and constant vector C. With the studies carried out over time, it has been observed that the Gauss maps of some surfaces provide this equality for a regular function f, not for a constant λ. Conjugate of Enneper surface, catenoid surfaces, and helicoid surfaces can be given as examples in the Euclidean space. This situation led to the emergence of the concept called the Gauss map, which is a pointwise 1-type. If the Gauss map Ω of a surface M in R^3 satisfies the equation $\Delta\Omega = f(\Omega + C)$ for a regular function f and a constant vector C then this Gauss map is called the pointwise 1-type Gauss map.

- If the constant vector vanishes everywhere and f is a regular function, the pointwise 1-type Gauss map of M is called the first kind,
- If the constant vector doesn't vanish anywhere and f is a regular function, the pointwise 1-type Gauss map of M is called the second kind,
- If $\Delta\Omega = 0$, the pointwise 1-type Gauss map of M is called harmonic [7].

3. HASIMOTO SURFACES WITH POINTWISE 1-TYPE GAUSS

In this section, the Gauss map of Hasimoto surfaces is investigated. It can be seen that the definition of Hasimoto surfaces has been given by Abdel-All, Hussien and Youssef in [10].

Let $r(u_1, u_2)$ be a position vector of a moving space curve r which has a unit speed for all u_1 on surface M in R^3. The surface M is called a Hasimoto surface, if the position vector $r(u_1, u_2)$ of a moving space curve on M satisfies the following condition

$$r_{u_2} = r_{u_1} \times r_{u_1 u_1} = \kappa B, \tag{1}$$

where u_2 is the time parameter, u_1 is the arc-length parameter and the subscripts denote the partial differential with respect to parameters. Also, $\{T, N, B, \kappa, \tau\}$ are the Frenet apparatus of the curve r. Differentiating the equation of the Hasimoto surface defined by (1) in terms of u_1 and u_2, the following partial differential equations are found

$$r_{u_1} = T \text{ and } r_{u_2} = \kappa B.$$

Using these partial differentiation equations, the Gauss map Ω of the Hasimoto surface is found as

$$\Omega = \frac{r_{u_1} \times r_{u_2}}{\left\| r_{u_1} \times r_{u_2} \right\|} = -N.$$

For the matrix $\left(g_{ij}\right)$ consisting of the components of the metric on the Hasimoto surface, we denote the determinant and the inverse matrix of the matrix $\left(g_{ij}\right)$ by g and $\left(g^{ij}\right)$, respectively. In this case, we can easily find that

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^2 \end{pmatrix}, \; g = \kappa^2 \text{ and } g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\kappa^2} \end{pmatrix}, \; \kappa \neq 0.$$

Therefore, the formula of the Laplacian Δ on the Hasimoto surface is given in terms of u_1 and u_2 by

$$\Delta = -\frac{\partial^2}{\partial u_1^{\,2}} - \frac{1}{\kappa^2}\frac{\partial^2}{\partial u_2^{\,2}} - \frac{\kappa'}{\kappa}\frac{\partial}{\partial u_1}. \tag{2}$$

Now, let's apply the Laplace operator Δ to the Gauss map Ω. If the derivatives of the Gauss map Ω_{u_1}, $\Omega_{u_1 u_1}$, Ω_{u_2}, $\Omega_{u_2 u_2}$, and $\Omega_{u_1 u_2}$ are substituted in the equation (2), then we can easily calculate that the Laplacian of the Gauss map Ω of M is

$$\Delta\Omega = -\left(\kappa^2+\tau^2\right)\left(\Omega - \frac{2\kappa'}{\left(\kappa^2+\tau^2\right)}T + \frac{\kappa'\tau+\tau'\kappa}{\kappa\left(\kappa^2+\tau^2\right)}B\right). \tag{3}$$

The last equation can be written in the form $\Delta\Omega = f\left(\Omega + C\right)$ such that

$$f = -\left(\kappa^2+\tau^2\right),\ \ C = -\frac{2\kappa'}{\left(\kappa^2+\tau^2\right)}T + \frac{\kappa'\tau+\tau'\kappa}{\kappa\left(\kappa^2+\tau^2\right)}B. \tag{4}$$

Let's examine the following cases for the Gauss map of the Hasimoto surface parametrized by $r(u_1,u_2)$.

3.1. CASE 1: POINTWISE 1-TYPE GAUSS MAP OF THE FIRST KIND

Considering the each equations of (4), for the Gauss map of the Hasimoto surface has a pointwise 1-point map of the first kind, there exists two cases that satisfy the following condition:

- If the curvature κ and the torsion τ of the moving space curve on M are constants, then

$$f = -\left(\kappa^2+\tau^2\right) \text{ and } C = 0.$$

In that case,

$$\Delta\Omega = -\left(\kappa^2+\tau^2\right)\Omega.$$

- If r be a moving curve with constant curvature κ and zero torsion τ on M, then

$$f = -\kappa^2 \text{ and } C = 0.$$

So,

$$\Delta\Omega = -\kappa^2\Omega.$$

We can give the following theorem with respect to the two above cases:

Theorem 1. Let r be a moving curve on the Hasimoto surface defined by (1) in R^3. The Hasimoto surface has a pointwise 1-type Gauss map of the first kind if and only if the curve r is a planar curve or a circular helix.

3.2. CASE 2: POINTWISE 1-TYPE GAUSS MAP OF THE SECOND KIND

The Gauss map of the Hasimoto surface has pointwise 1-point Gauss map of the second kind, if f is a regular function and C is constant vector that is not vanishing. But, considering equation (4), it is impossible for the vector C to be a nonzero constant vector. So, we can express the following theorem.

Theorem 2. Let M be a Hasimoto surface in R^3. Then M has not pointwise 1-type Gauss map of the second kind.

3.3. CASE 3: HARMONIC GAUSS MAP

The Gauss map of the Hasimoto surface is harmonic, if $\Delta\Omega = 0$. But, considering the equations (3) and (4) together, it is not the case that $\Delta\Omega = 0$. Then, the following theorem can be given.

Theorem 3. Let M be a Hasimoto surface in R^3. The Gauss map of the Hasimoto surface M cannot be harmonic.

Example 1. Let us consider a Hasimoto surface given by the parametric equation

$$r(u_1,u_2) = \left(\cos u_1, \sin u_1, u_2\right).$$

One can easily find that the Gauss map of the Hasimoto surface is $\Omega = \left(0,0,-1\right)$. On the other hand, the matrix $\left(g_{ij}\right)$, the determinant g, and the inverse matrix $\left(g^{ij}\right)$ of the matrix $\left(g_{ij}\right)$ of the Hasimoto surface are calculated as

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g = 1 \text{ and } g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ 1 \le i, j \le 2.$$

Therefore, the formula of the Laplacian Δ on the Hasimoto surface is given in terms of u_1 and u_2 by

$$\Delta = -\frac{\partial^2}{\partial u_1^{\,2}} - \frac{\partial^2}{\partial u_2^{\,2}}.$$

Now let's apply the Laplace operator Δ to the Gauss map Ω. If the derivatives of the Gauss map Ω_{u_1}, $\Omega_{u_1u_1}$, Ω_{u_2}, $\Omega_{u_2u_2}$, and $\Omega_{u_1u_2}$ are determined and the derivatives of the Gauss map are substituted in the last equation, then we get

$$\Delta\Omega = \Omega.$$

As a result, we can easily say that the Gauss map Ω of the Hasimoto surface has pointwise 1-type of the first kind.

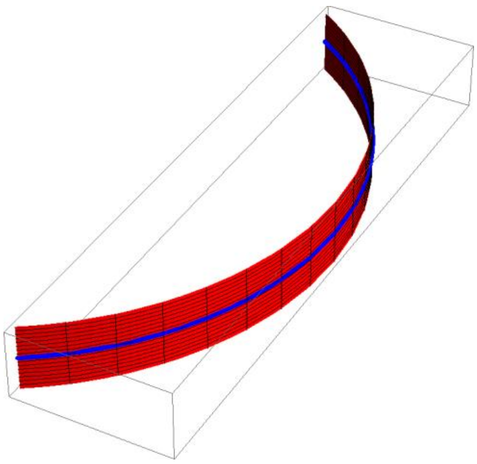

Figure 1. The graph of the Hasimoto surface (in red) and the curve r (in blue) with $u_1 \in (-1,1)$ and $u_2 \in (-0.1, 0.1)$.

4. CONCLUSIONS

In this work, we investigate the Gauss map of the Hasimoto surfaces, which have not been studied until now, and which are generated from integrable curves in Euclidean 3-space. We determine the conditions for the pointwise 1-type Gauss map of the Hasimoto surfaces to be the first type. Also, we stated that the pointwise 1-type Gauss map of the Hasimoto surfaces can't be the second kind and harmonic. Besides, this has been shown with an example that a Hasimoto surface is the pointwise 1-type Gauss map of the first kind.

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