ORIGINAL PAPER

DIFFERENT PERSPECTIVE OF DARBOUX SLANT RULED SURFACES AND THEIR CONSTRUCTIONS VIA QUATERNIONS

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Abstract. *In this paper, based on the definition of Darboux slant ruled surface in $\mathbb{R}^3$, a novel method for the Darboux slant ruled surface obtained from the striction curve of the natural lift curve is described in $\mathbb{R}^3$. Then, an isomorphism between the unit dual sphere, DS^2, and the subset of the unit 2-sphere, $T\bar{M}$ is used to define this surface. Hence, the striction curve of the natural lift curve is corresponded to Darboux slant ruled surface in $\mathbb{R}^3$by considering the isomorphism. Additionally, some fundamental theorems are proved to characterize this surface. Using the quaternion product of the quaternionic operator and a pure unit quaternion, the quaternionic expression for the Darboux slant ruled surface is defined. In the last, 2-parameter homothetic motion is expressed for this surface.*
Keywords: *Slant ruled surface; quaternions; Darboux vectors.*

1. INTRODUCTION

The endpoints of the unit tangent vectors to the main curve produce the natural lift curve, which was defined in [1]. Next, in $\mathbb{R}^3$, a few geometric characteristics of this curve were examined in [2]. The Frenet vectors of the natural lift curve were computed and the correspondence with the Frenet vectors of the main curve were denoted in [3]. The correspondence between natural lift curves and geodesic curvatures of the spherical indicatrix for some special curves were examined in [4], [5].

The dual numbers are a hypercomplex number system in algebra that was initially presented in the 1800's. Moreover, some novel methods for these numbers were improved in kinematics, statics and dynamics in [6]. Then, E. Study provided a relation between oriented lines in $\mathbb{R}^3$ and the points on unit dual sphere in [7].

In geometry, the set of points swept by a moving straight line is known as a ruled surface, for more specific details see [8]. Also, Frenet frames of the ruled surfaces were constructed in terms of their types in Minkowski space in [9]. Inspring several characterizations for slant helices in [10], a new class of ruled surfaces, defined by slant ruled surfaces, was examined in [11]. A new approach for slant ruled surfaces were given in [12]. Moreover, some theorems were proved to examine the characterizations of these surfaces.

Combining the properties of dual numbers and ruled surfaces, the correspondence between them was given by considering the curve on tangent bundle of unit 2-sphere and pseudo-spheres in [13-15]. Namely, a curve on these tangent bundles were corresponded to ruled surfaces in Euclidean and Minkowski spaces.

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William Rowan Hamilton, an Irish mathematician, initially defined quaternions in 1843 and used them in three-dimensional mechanics. Moreover, initial sketch of biquaternions were denoted in [16]. Exploiting these datum, the algebra of quaternions and Cayley numbers were constructed in [17]. Furthermore, due to extensive applications in geometry, the quaternionic approach were represented for special surfaces such as canal, circular, constant slope surfaces, ruled and slant ruled surfaces by using the quaternion product of the pure quaternion and the quaternionic operator in [18-23].

There is a gap in the literature on how to take advantage of the relation between the tangent bundle of the unit 2-sphere and the unit dual sphere for Darboux slant ruled surfaces. Taking the definition of Darboux slant ruled surface in [24] into account, we present an innovative definition for these surfaces, which are derived from the striction curve of the natural lift curve. Then, we give some theorems for determining the characterization. Furthermore, we use a quaternionic operator for constructing this surface. Finally, we provide an example to confirm what has been found. In order to comprehend these characterizations, the framework of this study is as follows: Section 2 is about some background data for differential geometry of slant ruled surfaces in $\mathbb{R}^3$ and quaternion theory. In Section 3, we give the relation between ruled surfaces and unit dual sphere. In Section 4, we give a different perspective for Darboux slant ruled surfaces. Moreover, we give an example as Figure 1 to validate the results. In Section 5, we mention the quaternionic approach of these surfaces.

2. PRELIMINARIES

In this section, firstly, the definition of the natural lift curve, the geometry of slant ruled surfaces, and the tangent bundle of unit 2-sphere are given. Secondly, some essential characteristics of the theory for quaternions are discussed.

2.1. DIFFERENTIAL GEOMETRY OF SLANT RULED SURFACES

In $\mathbb{R}^3$, let S^2 represent the unit 2-sphere. The set

$$TS^2 = \{(q,\vartheta) \in \mathbb{R}^3 \times \mathbb{R}^3 : |q| = 1, \langle q,\vartheta \rangle = 0\} \tag{1}$$

is called the tangent bundle of S^2, see [14]. The set $T\bar{M}$, which is the subset of TS^2, is denoted as

$$T\bar{M} = \{(\bar{q},\bar{\vartheta}) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\bar{q}| = 1, \langle \bar{q},\bar{\vartheta} \rangle = 0\}, \tag{2}$$

where the derivatives of q and ϑ, respectively, are $\bar{q}$ and $\bar{\vartheta}$, see [12].

Definition 2.1. The natural lift of Γ on $\mathrm{T\bar{M}}$ for the curve Γ is denoted by $\bar{\Gamma}$, which gives the equation that follows:

$$\bar{\Gamma}(u) = \left(\bar{q}(u), \bar{\vartheta}(u)\right) = \left(\dot{q}(u)|_{\gamma(u)}, \dot{\vartheta}(u)|_{\vartheta(u)}\right). \tag{3}$$

Therefore, we have

$$\frac{d\bar{\Gamma}(u)}{du} = \frac{d}{du}\left(\dot{\Gamma}(u)|_{\Gamma(u)}\right) = D_{\dot{\Gamma}(u)}\dot{\Gamma}(u),$$

where D indicates the Levi-Civita connection in $\mathbb{R}^3$. Also, we write

$$T\bar{M} = \bigcup T_p\,\bar{M}\,, p \in \bar{M}$$

see [1].

The parametric expression of ϕ obtained by $\{\vec{k}(u), \vec{\vec{q}}(u)\}$ for a one-paramerter family lines $\{\vec{k}(u), \vec{\vec{q}}(u)\}$ is

$$\vec{\vec{r}}(u) = \vec{k}(u) + v\vec{\vec{q}}(u), u \,\epsilon\, I, v \,\epsilon\, \mathbb{R}. \tag{4}$$

$k = \vec{k}(u)$ denotes a point and $\bar{q} = \vec{\vec{q}}(u)$ also denotes a non-null vector in $\mathbb{R}$. Additionally, the base curve and director vector for the ruled surface ϕ, respectively, are $\vec{k}(u)$ and $\vec{\vec{q}}(u)$, see [8].

$$m = \frac{\vec{\vec{r}}_u \times \vec{\vec{r}}_v}{|\vec{\vec{r}}_u \times \vec{\vec{r}}_v|} = \frac{(\dot{\vec{k}} + v(\dot{\vec{\vec{q}}}) \times \vec{\vec{q}}}{\sqrt{\left|\langle \dot{\vec{k}}, \vec{\vec{q}}\rangle^2 - \langle \vec{\vec{q}}, \vec{\vec{q}}\rangle \langle \dot{\vec{k}} + v\dot{\vec{\vec{q}}}, \dot{\vec{k}} + v\dot{\vec{\vec{q}}}\rangle\right|}}.$$

Definition 2.2. Assume that $\bar{\mathrm{M}}$ is a surface in $\mathbb{R}^3$. For every $\mathrm{X} \in \chi(\bar{\mathrm{M}})$,

$$S_P : \chi(\bar{M}) \to \chi(\bar{M}),$$
$$x \to S_P(X) = D_x\bar{M}.$$

is called Weingarten map (shape operator) for each point p on $\bar{\mathrm{M}}$. Here D refers Riemannian connection in $\mathbb{R}^3$, see [8]. Assume that $\bar{M}$ is a surface in $\mathbb{R}^3$. Gauss map is defined by

$$K : \bar{M} \to \mathbb{R},$$
$$p \to K(p) = detS_p.$$

Moreover, the mean curvature on $\bar{M}$ is denoted by

$$K = \bar{M} \to \mathbb{R},$$
$$p \to K(p) = \frac{1}{2}Tr(S_p),$$

see [8]. At a ruling $u = u_1$, we define

$$\vec{\vec{a}} = \lim_{v\to\infty} \vec{m}\,(u_1, v) = \frac{\dot{\vec{\vec{q}}} \times \vec{\vec{q}}}{\left|\dot{\vec{\vec{q}}}\right|}. \tag{5}$$

On the ruling u_1 , a striction point (or central point) β is defined as the point where $\vec{m}$ forms a straight angle with $\vec{\vec{a}}$. The striction curve of the ruled surface is the set of central points of all rulings. The orthogonality of the vectors $\vec{\vec{q}}, \dot{\vec{\vec{q}}}$, and correspondence Eq. (5) gives the unit vector $\vec{\vec{h}}$:

$$\vec{\vec{h}} = \frac{\dot{\vec{\vec{q}}}}{\left|\dot{\vec{\vec{q}}}\right|}. \tag{6}$$

The striction curve is defined in terms of the arc length parameter u by

$$\vec{c}(u) = \vec{k}(u) - \frac{\langle \dot{\vec{\vec{q}}}(u), \dot{\vec{k}}(u) \rangle}{\langle \dot{\vec{\vec{q}}}(u), \dot{\vec{\vec{q}}}(u) \rangle} \vec{\vec{q}}(u). \tag{7}$$

As a Frenet frame of ϕ along the striction curve, the orthonormal system $\{\beta, \vec{\vec{q}}, \vec{\vec{h}}, \vec{\vec{a}}\}$ is defined. A cone known as the directing cone of the surface is made up of the set of all bounded vectors. The spherical curve k_1 on the unit sphere is represented by the end points of $\vec{q}$. This curve is defined as the spherical image of a ruled surface whose arc length is indicated by s_1.

The Frenet formulas of the ruled surface and of its directing cone in terms of s_1 are provided as

$$\begin{pmatrix} \dot{\vec{\vec{q}}} \\ \dot{\vec{\vec{h}}} \\ \dot{\vec{\vec{a}}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa \\ 0 & -\kappa & 0 \end{pmatrix} \begin{pmatrix} \vec{\vec{q}} \\ \vec{\vec{h}} \\ \vec{\vec{a}} \end{pmatrix},$$

where the directing cone's canonical curvature is denoted by κ . The Frenet formulas make sense in the following kinematic context: The motion that $\vec{q}$ traverses the directing cone is moved by the moving frame $\{\beta, \vec{\vec{q}}, \vec{\vec{h}}, \vec{\vec{a}}\}$, where s is the time parameter. The motionalso contains an instantaneous notation with an angular velocity vector ascertained by the Darboux vector, in addition to an instantaneous translation:

$$\vec{\vec{W}} = \kappa \vec{\vec{q}} + \vec{\vec{a}}\,, \tag{8}$$

where $\left|\vec{\vec{W}}\right| = \sqrt{\kappa^2 + 1}$. Therefore, Frenet formulas can be founded as

$$\dot{\vec{\vec{q}}} = \vec{\vec{W}} \times \vec{\vec{q}}, \; \dot{\vec{\vec{h}}} = \vec{\vec{W}} \times \vec{\vec{h}}, \\ \dot{\vec{\vec{a}}} = \vec{\vec{W}} \times \vec{\vec{a}}\,. \tag{9}$$

In the remainder of the paper, $\left\{\vec{\vec{q}}, \vec{\vec{h}}, \vec{\vec{a}}\right\}$ is the Frenet frame of the slant ruled surface obtained by the striction curve of the natural lift curve. $\vec{\vec{q}}, \vec{\vec{h}}$ and $\vec{\vec{a}}$ represent the unit vectors of ruling, central normal and central tangent of the slant ruled surface formed by the striction curve of natural lift curve, respectively.

2.2. QUATERNION THEORY

Quaternions have the form shown by

$$q = a_0 + a_1 i + a_2 j + a_3 k, \tag{10}$$

where i, j, k are symbols that can be understood as unit vectors pointing along the three spatial axes, and a_0, a_1, a_2, a_3 are real numbers. The following multiplication rules are satisfied by these unit vectors:

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Moreover, a quaternion can be expressed by

$$q = S(q) + V(q) \tag{11}$$

where $V(q) = a_1 i + a_2 j + a_3 k$ and $S(q) = a_0$ stand fort he vector and scalar parts of q, respectively. A quaternion is said to as pure if $S(q) = 0$. The quaternion product for every given pair of quaternions, $q = S(q) + V(q)$ and $p = S(p) + V(p)$, is

$$q * p = S(q)S(p) - \langle V(q), V(p) \rangle + S(q)V(p) + S(p)V(q) + V(q) \times V(p) \tag{12}$$

Two quaternion addition and quaternion multiplication with a scalar $\lambda \in$ R are

$$q + p = \big(S(q) + S(p)\big) + (V(q) + V(p)),$$
$$\lambda q = \lambda S(q) + \lambda V(q).$$

Additionally, the following are the conjugate, norm, modulus, and inverse, in that order:

$$C_q = S(q) - V(q) = a_0 - a_1 i - a_2 j - a_3 k,$$
$$N_q = q * C_q = C_q * q = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$
$$|q| = \sqrt{N_q},$$
$$q^{-1} = \frac{C_q}{N_q}, N_q \neq 0.$$

q is referred to as o unit quaternion if $N_q = 1$. Additionally, $q = \cos\theta + \sin\theta \nu$, where $\nu \epsilon \mathbb{R}^3$ and $\|\nu\| = 1$, can be used to express the unit quaternion. Please, see [17] for additional details regarding quaternions. For $a_1^2 + a_2^2 + a_3^2 \neq 0$ and $\theta \in \mathbb{R}$, we write

$$cos\,\theta = a_0,$$
$$sin\,\theta = \sqrt{a_1^2 + a_2^2 + a_3^2},$$
$$\nu = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

Consider the linear mapping $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\phi(\vec{w}) = p$ for $\vec{w} * p^{-1}$. In this case, $\vec{w}$ is a pure quaternion and q is a unit quaternion. Therefore, the matrix representation of the linear mapping ϕ for the unit quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ is indicated by

$$M = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_3 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_2^2 - a_1^2 \end{bmatrix},$$

where M is orthogonal matrix. As a result, in $\mathbb{R}^3$, a rotation matrix is indicated by the linear mapping ϕ. Furthermore, ϕ is provided as follows:

$$\phi(\vec{w}) = p * \vec{w} * p^{-1} = M\vec{w} \tag{13}$$

The formula for 2-parameter homothetic motion in $\mathbb{R}^3$ is as follows:

$$\vec{y}(s,t) = h(s,t)M(s,t)\vec{x}(s,t) + \vec{n}(s,t) \tag{14}$$

where $\vec{x}$ and $\vec{y}$ represent the position vectors of a point in the fixed space $\mathbb{R}^3$ and the moving space $\mathbb{R}$, respectively. In this case, h is a homothetic scalar, $\vec{n}$ is the translation vector, and M is an orthogonal matrix.

3. THE CORRESPONDENCE BETWEEN RULED SURFACES AND UNIT DUAL SPHERE

In this section, some fundamental concepts and theorems related to dual vectors are given. Moreover, the relationship between DS^2 and $T\overline{M}$ is provided.

The definition of the dual number set is

$$D = \{X = x + \varepsilon x^*; (x, x^*) \in \mathbb{R} \times \mathbb{R}, \varepsilon^2 = 0\}.$$

In $\mathbb{R}^3$, the $\vec{x}$ and $\overrightarrow{x^*}$ is referred to as a dual vector. The real and dual parts of $\vec{X}$ are represented by these vectors, respectively. In $\mathbb{R}^3$, if $\vec{x}$ and $\overrightarrow{x^*}$ are vectors, then $\vec{X} = \vec{x} + \varepsilon\overrightarrow{x^*}$ is a dual vector. Suppose there are two dual vectors $\vec{Y} = \vec{y} + \varepsilon\overrightarrow{y^*}$ and $\vec{X} = \vec{x} + \varepsilon\overrightarrow{x^*}$. The following is about some basic operations:

- The addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon\left(\overrightarrow{x^*} + \overrightarrow{y^*}\right),$$

- The inner product is

$$\langle \vec{X}, \vec{Y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon\left(\langle \overrightarrow{x^*}, \vec{y} \rangle + \langle \vec{x}, \overrightarrow{y^*} \rangle\right).$$

- The vector product is

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon\left(\vec{x} \times \overrightarrow{y^*} + \overrightarrow{x^*} \times \vec{y}\right).$$

- The norm of $\vec{X}$ is

$$\left|\vec{X}\right| = \sqrt{\langle \vec{x}, \vec{x} \rangle} + \varepsilon \frac{\langle \vec{x}, \overrightarrow{x^*} \rangle}{\sqrt{\langle \vec{x}, \vec{x} \rangle}}. \tag{15}$$

The dual vector is referred to as a unit dual vector if X has a norm of 1. The definition of the unit dual sphere, which is made up of every unit dual vector, is

$$DS^2 = \left\{\vec{X} = \vec{x} + \varepsilon\overrightarrow{x^*} : \left|\vec{X}\right| = 1\right\}. \tag{16}$$

See [6] for comprehensive information on dual vectors. The connection between the unit dual sphere and the tangent bundle of the unit 2-sphere of the natural lift curve is given by equations (2) and (16):

$$T\overline{M} \to DS^2,$$
$$\overline{\Gamma} = (\overline{q}, \overline{\vartheta}) \to \vec{\overline{\Gamma}} = \vec{\overline{q}} + \mathcal{E}\vec{\overline{\vartheta}}.$$

Theorem 3.1. (E. Study mapping) According to [7], there is a one-to-one correlation between the points of DS^2 and the oriented lines in $\mathbb{R}^3$.

Theorem 3.2. Assume that a natural lift curve on DS^2 with parameter u is defined as $\vec{\bar{\Gamma}}(\mathrm{u}) = \vec{\bar{\mathrm{q}}}(\mathrm{u}) + \varepsilon\vec{\bar{\vartheta}}(\mathrm{u})$. The ruled surface in $\mathbb{R}^3$ that is produced by the natural lift curve $\bar{\Gamma}(\mathrm{u})$ can be written in this way:

$$\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u) \tag{17}$$

where

$$\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) \tag{18}$$

is the base curve of $\bar{\phi}$.

As a result, the following representation is about the isomorphism between $T\bar{M}, DS^2$ and $\mathbb{R}^3$:

$$T\bar{M} \to DS^2 \to \mathbb{R}^3,$$

$$\bar{\Gamma}(u) = \left(\bar{q}(u), \bar{\vartheta}(u)\right) \to \vec{\bar{\Gamma}}(u) = \vec{\bar{q}}(u) + \varepsilon\vec{\bar{\vartheta}}(u) \to \bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u).$$

$\bar{\phi}(u,v)$, which is the ruled surface in $\mathbb{R}^3$, correspands to the natural lift curve $\bar{\Gamma}(u) \in T\bar{M}$, or dual curve $\bar{\Gamma}(u) = \vec{\bar{q}}(u) + \varepsilon\vec{\bar{\vartheta}}(u) \in DS^2$, as mentioned in [12].

4. $\vec{\bar{\bar{W}}}$ – SLANT RULED SURFACE AND TANGENT BUNDLE OF UNIT 2-SPHERE

The definition and characterization of slant ruled surface in $\mathbb{R}^3$, which are described by $\vec{\bar{\bar{W}}}$- slant ruled surfaces, are discussed in this section.

Definition 4.1. Assume that $\bar{\phi}(u,v) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) + v\vec{\bar{q}}(u)$ is a ruled surface $\mathbb{R}^3$. If all three criteria are satisfed, then $\bar{\phi}(u,v)$ is referred to be a $\vec{\bar{W}}$ slant ruled surface:

(i)The base curve $\bar{\beta}(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u)$ of $\bar{\phi}(u,v)$ must be

$$\bar{\beta}(u) = \left(\vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u)\right) - \frac{\langle\left(\vec{\bar{q}}(u)\times\vec{\bar{\vartheta}}(u)\right)'\dot{\vec{\bar{q}}}(u)\rangle}{\langle\dot{\vec{\bar{q}}}(u),\dot{\vec{\bar{q}}}(u)\rangle}\vec{\bar{q}}(u) = \vec{\bar{q}}(u) \times \vec{\bar{\bar{\vartheta}}}^*(u), \vec{\bar{\bar{\vartheta}}}^* \subseteq \vec{\bar{\vartheta}}.$$

(ii)The following formulas: $\langle\vec{\bar{q}}(u), \vec{\bar{\bar{\vartheta}}}^*(u)\rangle = 0$ and $\left|\vec{\bar{q}}(u)\right| = 1$ have to be satisfied.

(iii) $\vec{\bar{W}}$ requires to be directed in a fixed non-zero direction and a constant angle. Namely,

$$\langle\vec{\bar{\bar{W}}}(u), \vec{u}\rangle = \cos\gamma.$$

Consider $\hat{\Gamma}(u) = \mathcal{E}\vec{\bar{\bar{\vartheta}}}^*(u) + \vec{\bar{q}}(u)$ represent the striction curve of the natural lift curve on DS^2 with paramerter u, under the previously stated conditions. The $\vec{\bar{W}}$-slant ruled surface obtained from $\hat{\Gamma}(u)$ is written in $\mathbb{R}^3$, as follows:

$$\beta(u) = \vec{\bar{q}}(u) \times \vec{\bar{\vartheta}}(u) \tag{19}$$

where

$$\bar{\beta}(u) = \vec{\bar{q}}(u) \times \vec{\bar{\bar{\vartheta}}}^*(u) \tag{20}$$

refers the base curve of $\hat{\phi}$. Now, we will give some substantial theorems for $\vec{\overline{W}}$-slant ruled surface.

Theorem 4.2. Let $\bar{\phi}(u,v)$ be the ruled surface and $\hat{\Gamma}(u) = \left(\bar{q}(u), \bar{\vartheta}^*(u)\right) \in T\bar{M}$ be its striction curve. As $\hat{\phi}(u,v)$ is a Darboux slant ruled surface produced by $\hat{\Gamma}(u)$, the canonical curvature κ is constant.

Proof: Let $\bar{\phi}(u,v)$ be a Darboux slant ruled surface. From the definition of $\vec{\overline{W}}$-slant ruled surface, there exists a non-zero fixed direction $\vec{u}$, which satisfies the following condition:

$$\langle \overline{W}, \vec{u} \rangle = constant \tag{21}$$

Taking the derivative of Eq. (21), we have

$$\langle \dot{\vec{W}}, \vec{u} \rangle = 0 \tag{22}$$

Exploiting Frenet formulas, we write

$$\dot{\kappa} \langle \vec{\bar{q}}, d \rangle = 0 \tag{23}$$

Thus, it is evident that we have two options:

$$\kappa = constant \tag{24}$$

or

$$\langle \vec{\bar{q}}, \vec{u} \rangle = 0. \tag{25}$$

As $\vec{u}$ is orthogonal to $\vec{\bar{q}}$ if $\langle \vec{\bar{q}}, \vec{u} \rangle = 0$. Thus, we can write

$$\vec{u} = a_1 \vec{\bar{h}} + a_2 \vec{\bar{a}} \tag{26}$$

Here a_1, a_2 are smooth functions. Taking the derivative of $\vec{u}$ and using the Frenet formulas, we have

$$-a_1 \vec{\bar{q}} + (\dot{a_1} - \kappa a_2) \vec{\bar{h}} + (\dot{a_2} + \kappa a_1) \vec{\bar{a}} = 0.$$

Considering the linear independence of $\vec{\bar{q}}, \vec{\bar{h}}, \vec{\bar{a}}$, we obtain

$$a_1 = 0,$$
$$\dot{a_1} - \kappa a_2 = 0,$$
$$\dot{a_2} + \kappa a_1 = 0.$$

So, we get $a_1 = a_2 = 0$ which gives that $\vec{u} = 0$. That is a contradiction. Consequently, $\langle \vec{\bar{q}}, \vec{u} \rangle \neq 0$ is inferred. Then, κ is constant.

Remark 4.3. $\det\left(\vec{\overline{W}}, \dot{\vec{\overline{W}}}, \ddot{\vec{\overline{W}}}\right) = 0$ if $\hat{\phi}(u,v)$ is a Darboux slant ruled surface acquired by $\hat{\Gamma}(u)$.

Proof: Using the derivatives of the Darboux vector, we obtain

$$\vec{\overline{W}} = \kappa\vec{\overline{q}} + \vec{\overline{a}},$$
$$\dot{\vec{\overline{W}}} = \dot{\kappa}\vec{\overline{q}},$$
$$\ddot{\vec{\overline{W}}} = \ddot{\kappa}\vec{\overline{q}} + \dot{\kappa}\vec{\overline{h}}.$$

The value of determinant is calculated by

$$det\left(\vec{\overline{W}}, \dot{\vec{\overline{W}}}, \ddot{\vec{\overline{W}}}\right) = (\dot{\kappa})^2.$$

From Theorem 4.2, we deduce that κ is constant if $\hat{\phi}(u,v)$ is Darboux slant ruled surface. So, the desired result is obtained.

Theorem 4.4. A $\vec{\overline{\mathrm{W}}}$-slant ruled surface is equivalent to any $\vec{\overline{\mathrm{h}}}$ slant ruled surface.

Proof: Assume that $\hat{\phi}(u,v)$ is an $\overline{h}$-slant ruled surface. Hence, it satisfies

$$\langle\vec{\overline{h}}, \vec{u}\rangle = \cos\theta = constant. \tag{27}$$

where θ represents the constant angle formed by $\vec{\overline{h}}$ and $\vec{u}$. In Theorem 2.1, the axis of the $\overline{h}$-slant ruled surface is

$$\vec{u} = \frac{\kappa}{\sqrt{1+\kappa^2}}\vec{\overline{q}} + \cos\theta\,\vec{\overline{h}} + \frac{1}{\sqrt{1+\kappa^2}}\vec{\overline{a}}. \tag{28}$$

Using Eq. (28) and Darboux vector, it is possible to write

$$\vec{u} = \frac{\vec{\overline{W}}}{\left|\vec{\overline{W}}\right|} + \cos\theta\,\vec{\overline{h}}\,. \tag{29}$$

This indicates that $|\vec{u}| = \sqrt{1+cos^2\theta}$ is constant. We derive from Eq. (29)

$$\langle\vec{\overline{W}}, \vec{u}\rangle = \left|\vec{\overline{W}}\right| + \cos\theta\,\langle\vec{\overline{W}}, \vec{\overline{h}}\rangle. \tag{30}$$

Moreover, we understand that $\langle\vec{\overline{W}}, \vec{u}\rangle = \left|\vec{\overline{W}}\right||\vec{u}|\cos\lambda$, where the angle between $\vec{\overline{W}}$ and $\vec{u}$ is denoted by $\cos\lambda$. Taking into account Eq. (30), we have

$$|\vec{u}|\cos\lambda = 1. \tag{31}$$

Using Eq. (31), we obtain

$$\cos\lambda = \frac{1}{\sqrt{1+cos^2\theta}} = constant,$$

which means that $\langle\vec{\overline{W}}, \vec{u}\rangle = constant$. That, $\hat{\phi}(u,v)$ is $\vec{\overline{W}}$ $-$slant ruled surface.

Theorem4.5. Consider $\widehat{\phi}(u,v)$ as a Darboux slant ruled surface generated by $\widehat{\Gamma}(u)$. Assuming a $\vec{\vec{h}}$-slant ruled surface $\widehat{\phi}(u,v)$, it can also be a $\vec{\vec{q}}$- or $\vec{\vec{a}}$-slant ruled surface.

Proof: As a Darboux slant ruled surface, $\widehat{\phi}(u,v)$, the conical curvature κ, is constant.

According to Theorem 2.1 in [3], if $\hat{\phi}(u,v)$ is a $\vec{\vec{h}}$-slant ruled surface, then the surface is axis is

$$\vec{u} = \frac{\kappa}{\sqrt{1+\kappa^2}}\vec{\vec{q}} + d\vec{\vec{h}} + \frac{1}{\sqrt{1+\kappa^2}}\vec{\vec{a}}\,. \tag{32}$$

Since κ is constant, the last equlity gives that $\langle\vec{\vec{q}},\vec{u}\rangle = constant$, $\left(or\ \langle\vec{\vec{a}},\vec{u}\rangle = constant\right)$, i.e., $\hat{\phi}(u,v)$ is also $\vec{\vec{q}}$-slant $\left(or\ \vec{\vec{a}} - slant\ \right)$ ruled surface. The Theorem's converse is only true for a certain value of coefficient a_3. As a result, we provide the subsequent unique requirement.

Theorem 4.6. Define $\widehat{\phi}(u,v)$ to be a Darboux slant ruled surface acquired by $\widehat{\Gamma}(u)$. If $\widehat{\phi}(u,v)$ is also a $\vec{\vec{a}}$-slant ruled surface with constant angle α, as shown by $\langle\vec{\vec{a}},\vec{u}\rangle = \cos\alpha = \frac{\cos\gamma}{1+\kappa^2}$, then $\widehat{\phi}(u,v)$ is a $\vec{\vec{h}}$-slant ruled surface. Here the fixed vector $\vec{u}$ is non-zero.

Proof: Given that $\hat{\phi}(u,v)$ is a surface determined by the Darboux vector, we obtain

$$\langle\vec{\vec{W}},\vec{u}\rangle = \cos\varphi = constant, \tag{33}$$

where the constant angle φ between the vectors $\vec{\vec{W}}$and $\vec{u}$ is expressed and $\vec{u}$ is a non-zero fixed vector. We can write $\vec{u}$ as

$$\vec{u} = a_1\vec{\vec{q}} + a_2\vec{\vec{h}} + a_3\vec{\vec{a}}\,, \tag{34}$$

where $a_i = a_i(s_1)$ are smooth functions of s_1. By calculating the derivative of Eq. (34), we derive

$$(\dot{a_1} - a_2)\vec{\vec{q}} + (a_1 + \dot{a_2} - a_3\kappa)\vec{\vec{h}} + (a_2\kappa + \dot{a_3})\vec{\vec{a}} = 0. \tag{35}$$

Since the linear independence of $\vec{\vec{q}}, \vec{\vec{h}}, \vec{\vec{a}}$, we have the following system

$$\dot{a_1} - a_2 = 0,$$
$$a_1 + \dot{a_2} - a_3\kappa = 0,$$
$$a_2\kappa + \dot{a_3} = 0.$$

The solution gives the following relations:

$$\kappa a_1 + a_3 = \cos\varphi$$

and

$$\dot{a_2} = \frac{(1+\kappa^2)a_3 - \cos\varphi}{\kappa}.$$

κ is constant as $\hat{\phi}(u,v)$ is a Darboux slant ruled surface. Thus, a_2 is constant if and only if a_3, which can be found using the formula $a_3 = \frac{\cos\varphi}{1+\kappa^2}$, is also constant. Specifically, when

$\hat{\phi}(u,v)$ is similarly $\vec{\vec{a}}$, then $\hat{\phi}(u,v)$ is a $\vec{\vec{h}}$-slant ruled surface. The $\langle \vec{\vec{a}}, \vec{\vec{u}} \rangle = \cos\lambda = \frac{\cos\varphi}{1+\kappa^2}$ defines $\vec{\vec{a}}$-rules surface with constant angle λ.

Example 4.7. Let us consider the curve $\vec{\vec{q}}(u) = (\cos u, \sin u, 0)$ and the vector $\overrightarrow{\overline{\vartheta^*}}(u) = (v \sin u, -v\cos u, 0)$ in $\mathbb{R}^3$. The slant ruled surface generated by $\hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u))$ is given by $\hat{\phi}(u,v) = (v\cos u, v\sin u, -u)$, where the case curve is

$$\bar{\beta}(u) = (0,0,-t).$$

Moreover, $\langle \vec{\vec{q}}(u), \overline{\vartheta^*}(u) \rangle = 0$ and $\left|\vec{\vec{q}}(u)\right| = 1$. So, $\hat{\Gamma}(u) = \left(\bar{q}(u), \overline{\vartheta^*}(u)\right) \in T\bar{M}$. The Frenet operators of $\hat{\phi}(u,v)$ are, respectively.

$$\vec{\vec{q}}(u) = (\cos u, \sin u, 0),$$
$$\vec{\vec{h}}(u) = (-\sin u, \cos u, 0),$$
$$\vec{\vec{a}}(u) = (0,0,1),$$

where the first and second curvatures are calculated by $\overrightarrow{\overline{k_1}} = 1$ and $\overrightarrow{\overline{k_2}} = 0$. Hence, the canonical curvature κ is founded by zero. From Theorem 4.1. in [12], we have

$$\frac{\overline{k_1^2}}{\left(\overline{k_1^2}+\overline{k_2^2}\right)^{\frac{3}{2}}}\left(\frac{\overline{k_1}}{\overline{k_2}}\right)' = 0 = constant.$$

As a result, $\vec{\vec{h}}$- is the slant ruled surface. We know that every $\vec{\vec{h}}$-slant ruled surface is Darboux slant surface and Darboux vector is also

$$\overrightarrow{\overline{W}} = (0,0,1).$$

Fig. 1 is about the Darboux slant ruled surface obtained by the striction curve $\hat{\Gamma}(u)$, given as follows:

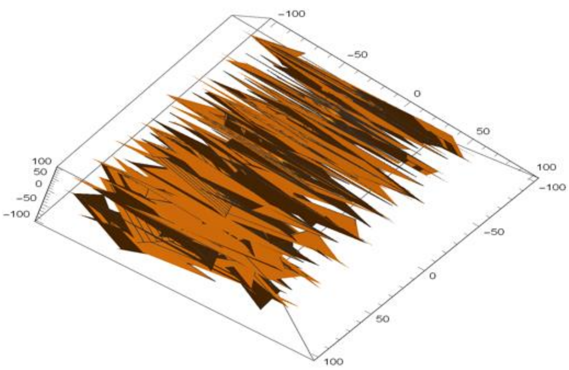

Figure 1. Darboux slant ruled surface acquired by the striction curve $\hat{\Gamma}(u)$.

Now let's compute the shape operator, Gauss and mean curvatures of $\hat{\phi}(u,v)$:

$$\widehat{\phi_u}(u,v) = \frac{1}{\sqrt{1+v^2}}(\cos u, \sin u, 0),$$

$$\widehat{\phi_v}(u,v) = \frac{1}{(\sqrt{1+v^2})^{\frac{3}{2}}}(\sin u, \cos u, -1).$$

The normal vector is calculated by

$$N = \frac{1}{\sqrt{1+v^2}}(\sin u, \cos u, -v).$$

Then, the shape operator is given as

$$\begin{bmatrix} 0 & \frac{1}{(\sqrt{1+v^2})^{\frac{3}{2}}} \\ \frac{1}{\sqrt{1+v^2}} & 0 \end{bmatrix}.$$

Gauss and mean curvatures are, respectively,

$$K = \frac{-1}{(1+v^2)^2},$$
$$H=0.$$

5. QUATERNIONIC EXPRESSION OF $\overrightarrow{\overrightarrow{W}}$-SLANT RULED SURFACES

The quaternionic expression for $\overrightarrow{\overrightarrow{W}}$-slant ruled surface is provided in this section. Moreover, this expression is used to denote an example. Let u be a real variable and $\overrightarrow{\overrightarrow{\vartheta^*}}(u)$ a curve in $\mathbb{R}^3$. A quaternion $H = (u,v)$ is shown by the sum of $\overrightarrow{\overrightarrow{\vartheta^*}}(u)$ and u.

Furthermore, the parametric version is

$$H(u,v) = u + \overrightarrow{\overrightarrow{\vartheta^*}}(u), u \in I \subseteq \mathbb{R}, u \in \mathbb{R}. \tag{36}$$

Here, the scalar and vector parts of the quaternion $H(u,v)$ is indicated by $S\big(H(u,v)\big) = u$ and $V\big(H(u,v)\big) = \overrightarrow{\overrightarrow{\vartheta^*}}(u)$. Additionally, the modulus of $H(u,v)$ is

$$|H(u,v)| = \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}(u)\right|^2}. \tag{37}$$

As a result, the normalized vector of $H(u,v)$ can be expressed as

$$q_0(u,v) = \frac{H(u,v)}{|H(u,v)|},$$
$$= \cos\theta\,(u,v) + \sin\theta\,(u,v)\vec{y}(u),$$

where

$$\cos\theta\,(u,v) = \frac{u}{\sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}(u)\right|^2}}$$

$$\sin\theta\,(u,v) = \frac{\left|\overrightarrow{\overline{\vartheta^*}}(u)\right|}{\sqrt{u^2 + \left|\overrightarrow{\overline{\vartheta^*}}(u)\right|^2}}$$

$$\vec{y}(u) = \frac{\overrightarrow{\overline{\vartheta^*}}(u)}{\left|\overrightarrow{\overline{\vartheta^*}}(u)\right|}.$$

Hence, $H(u,v)$ is defined as quaternionic operatör denoted by

$$H(u,v) = \sqrt{u^2 + \left|\overrightarrow{\overline{\vartheta^*}}(u)\right|^2}\, q_0(u,v) \tag{38}$$

in order to construct the slant ruled surface.

Theorem 5.1. The position vector of $\hat{\Gamma}(u) = (\bar{q}(u), \bar{\vartheta}^*(u)) \in T\bar{M}$ should be perpendicular to the position vector of the curve. Therefore, the quaternion product of the pure quarternion $\vec{\bar{q}}(u)$ and the quaternion operator $H(u,v) = u + \overrightarrow{\overline{\vartheta}^*}(u)$,

$$H(u,v) * \vec{\bar{q}}(u) = \hat{\phi}(u,v) = \bar{\beta}(u) + v\vec{\bar{q}}(u), u \in \mathbb{R}, v \in \mathbb{R}. \tag{39}$$

for $\{\vec{\bar{q}}, \vec{\bar{h}}, \vec{\bar{a}}, \overline{k_1}, \overline{k_2}\}$. Furthermore,

$$\langle \overrightarrow{\overline{W}}, \vec{u}\rangle = \cos\gamma = constant;\ \gamma \neq \frac{\pi}{2}$$

exist with a fixed non-zero direction $\vec{u}$ in the space, the Darboux vector makes a constant angle θ. Thus, the $\hat{\phi}$ slant ruled surface is $\overrightarrow{\overline{W}}$-slant ruled surface.

Proof: As $S(H(u,v)) = u, V(H(u,v)) = \overrightarrow{\overline{\vartheta^*}}(u), S(\bar{q}(u)) = 0, V(\bar{q}(u)) = \vec{\bar{q}}(u)$, we get

$$H(u,v) * \vec{\bar{q}}(u) = (u + \overrightarrow{\overline{\vartheta^*}}(u)) * \vec{\bar{q}}(u)$$
$$= -\langle \overrightarrow{\overline{\vartheta^*}}(u), \vec{\bar{q}}(u)\rangle + v\vec{\bar{q}}(u) + \vec{\bar{q}}(u) \times \overrightarrow{\overline{\vartheta^*}}(u).$$

Exploiting $\langle \overrightarrow{\overline{\vartheta^*}}(u), \vec{\bar{q}}(u)\rangle = 0$, we say that

$$H(u,v) * \vec{\bar{q}}(u) = \vec{\bar{q}}(u) \times \overrightarrow{\overline{\vartheta^*}}(u) + v\bar{q}(u).$$

Thus, we have

$$H(u,v) * \vec{\bar{q}}(u) = \bar{\beta}(u) + v\vec{\bar{q}}(u) = \hat{\phi}(u,v),$$

where $\bar{\beta}(u) = \vec{\bar{q}}(u) \times \overrightarrow{\overline{\vartheta^*}}(u)$ and $\vec{\bar{q}}(u)$ denote curves in $\mathbb{R}^3$. From Theorem 2.1 in [3], we know that if

$$\frac{\dot{\kappa}}{(1+\kappa^2)^{\frac{3}{2}}} \tag{40}$$

is constant if and only if $\hat{\phi}$ is $\vec{\vec{h}}$ -slant ruled surface. Additionally, from Theorem 4.4, it is also $\vec{\vec{W}}$-slant ruled surface.

Theorem5.2. Assume that $\widehat{\Phi}(u,v)$ is the slant ruled surface obtained by quaternion product of a pure unit quaternion $\vec{\vec{q}}(u)$, where $\langle\vec{\vec{q}}(u), \overrightarrow{\overrightarrow{\vartheta^*}}(u)\rangle = 0$ and a quaternionic operator $H(u,v) = u + \overrightarrow{\overrightarrow{\vartheta^*}}(u)$. Then, 2-parameter homothetic motion may be employed to express it as

$$\hat{\phi}(u,v) = H(u,v) * \vec{\vec{q}}(u),$$
$$= h(u,v)M(u,v)\vec{\vec{q}}(u).$$

$M(u,v)$ represents the orthogonal matrix that fulfills

$$H(u,v)\vec{\omega}(u) = q_0 * \vec{\vec{q}}(u), q_0(u,v) = \frac{H(u,v)}{|H(u,v)|},$$

$h(u,v) = \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}\right|^2}$ is a homothetic sclar and s, u are homothetic parameter.

Proof: Let $\hat{\phi}(u,v)$ satisfy the criteria stated in Definition 4.1. We obtain the slant ruled surface $\hat{\phi}(u,v)$ using Eq. (38) as

$$\hat{\phi}(u,v) = \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}\right|^2}\, q_0(u,v) * \vec{\vec{q}}(u) \tag{41}$$

where $q_0(u,v) = \cos\theta\,(u,v) + \sin\theta\,(u,v)\vec{y}(u)$. If $p(u,v) = \cos\theta\,(u,v) + \sin\theta\,(u,v)\vec{y}(u)$ is the unit quaternion in the formula $p_0(u,v) = \cos\frac{\theta(u,v)}{2} + \sin\frac{\theta(u,v)}{2}\vec{y}(u)$, we acquire the orthogonal matrix that matches the linear mapping $\varphi(\vec{\vec{q}}) = p_0 * \vec{\vec{q}} * p_0^{-1}$ as

$$M(u,v) = \begin{pmatrix} cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2}(y_1^2 - y_2^2 - y_3^2) & 2sin^2\frac{\theta}{2}y_1y_2 - 2cos\frac{\theta}{2}sin\frac{\theta}{2}y_3 & 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}y_2 + 2sin^2\frac{\theta}{2}y_1y_3 \\ 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}y_3 + 2sin^2\frac{\theta}{2}y_1y_2 & cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2}(y_2^2 - y_1^2 - y_3^2) & 2sin^2\frac{\theta}{2}y_2y_3 - 2cos\frac{\theta}{2}sin\frac{\theta}{2}y_1 \\ 2sin^2\frac{\theta}{2}y_1y_3 - 2cos\frac{\theta}{2}sin\frac{\theta}{2}y_2 & 2cos\frac{\theta}{2}sin\frac{\theta}{2}y_1 + 2sin^2\frac{\theta}{2}y_2y_3 & cos^2\frac{\theta}{2} + sin^2\frac{\theta}{2}(y_3^2 - y_2^2 - y_1^2) \end{pmatrix},$$

where $\vec{y}(u) = (y_1(u), y_2(u), y_3(u)) \in \mathbb{R}^3$. The vector $\vec{\vec{q}}(u)$ in $\mathbb{R}^3$ is rotated by $M(u,v)$ around the axis $y(u)$ by an angle $\theta(u,v) \in \mathbb{R}$. Hence, we give

$$q_0(u,v) * \vec{\vec{q}}(u) = p_0(u,v) * \vec{\vec{q}}(u) * p_0^{-1}(u,v) = M(u,v)\vec{\vec{q}}(u) \tag{42}$$

Considering these equalities and $h(u,v) = \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}(u)\right|^2}$, $\hat{\phi}(u,v)$ is represented by

$$\hat{\phi}(u,v) = \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}(u)\right|^2} * \vec{\vec{q}}(u),$$
$$= \sqrt{u^2 + \left|\overrightarrow{\overrightarrow{\vartheta^*}}(u)\right|^2}\, M(u,v)\vec{\vec{q}}(u),$$
$$= h(u,v)M(u,v)\vec{\vec{q}}(u).$$

$\hat{\phi}(u,v)$ can therefore be written as a 2-parameter homothetic motion.

Example 5.3. Let us consider Example 4.7 again. The Darboux slant ruled surface generated by $\hat{\Gamma}(u) = (\overline{q}(u), \overline{\vartheta}^*(u))$ is

$$\hat{\phi}(u,v) = (v\cos u, v\sin u, -u),$$

where $\vec{\overline{q}}(u) = (\cos u, \sin u, 0)$ is the curve and the vector $\overrightarrow{\overline{\vartheta^*}}(u) = (v\sin u, -v\cos u, 0)$ in $\mathbb{R}^3$. Then, we write

$$\hat{\phi}(u,v) = H(u,v) * \vec{\overline{q}}(u),$$
$$= \vec{\overline{q}}(u) \times \overrightarrow{\overline{\vartheta^*}}(u) + v\vec{\overline{q}}(u),$$
$$= (v\cos u, v\sin u, -u).$$

The modulus of $H(u,v)$ is given by

$$H(u,v) = \sqrt{u^2+v^2}.$$

After normalizing $H(u,v)$, we get

$$q_0(u,v) = \frac{H(u,v)}{|H(u,v)|},$$
$$= \frac{u + (v\sin u, -v\cos u, 0)}{\sqrt{u^2+v^2}},$$
$$= \cos\theta\,(u,v) + \sin\theta\,(u,v)\vec{y}(u).$$

Here

$$\cos\theta\,(u,v) = \frac{u}{\sqrt{u^2+v^2}},$$
$$\sin\theta\,(u,v) = \frac{v}{\sqrt{u^2+v^2}},$$
$$\overrightarrow{y}(u) = \frac{(v\sin u, -v\cos u, 0)}{v}.$$

Moreover, the Darboux slant ruled surface can be expressed as follows:

$$\hat{\phi}(u,v) = h(u,v)q_0(u,v)\vec{\overline{q}}(u),$$

where $h(u,v) = \sqrt{u^2+v^2}$ and from Theorem 5.2, the 2-parameter homothetic motion for $\hat{\phi}(u,v)$ is

$$h(u,v)M(u,v)\vec{\overline{q}}(u).$$

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