ORIGINAL PAPER

GAUSSIAN FERMAT NUMBERS, POLYNOMIALS AND THEIR ASSOCIATED TRANSFORMS

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Manuscript received: 18.02.2024; Accepted paper: 12.08.2024;
Published online: 30.09.2024.

Abstract. *In this paper, we introduced Gaussian Fermat numbers and polynomials. We provided the Binet formula, generating functions, and the exponential generating function for these numbers and polynomials. Additionally, we derived several identities for these polynomials, including the Cassini identity, Catalan identity, Vajda identity, Halton identity, Gelin-Cesaro identity, and D'Ocagne's identity. We demonstrated that Gaussian Fermat numbers and polynomials can also be obtained through matrix representations and discussed key propositions based on the fact that the determinants of these matrix representations are constant. Furthermore, we explored the relationship between Gaussian Fermat numbers, polynomials, Mersenne numbers, and Jacobsthal numbers. We also presented the Catalan, Binomial, and Binomial of Catalan transformations of the Gaussian Fermat sequence and polynomials. Finally, we introduced the generating function for the Catalan transformation of the Gaussian Fermat numbers and polynomials.*

Keywords: *Gaussian Fermat numbers; Gaussian Fermat polynomials; Catalan transform; Hankel Transform; Binomial transform; Ballot transform.*

1. INTRODUCTION

In this study, we consider positive integer sequence as Gaussian Fermat sequence and polynomials satisfying recurrence relation and we give some well-known identities for these sequence and polynomials [1]. Undouble, the most well-known of integer sequence is Fibonacci (and Lucas) sequence. Many researchers are dedicated to Fibonacci sequence such as Hoggart and Koshy also many researchers investigated the polynomials of integer sequences [2-4]. In this study we will follow closely some of these studies. Fermat numbers and Fermat primes were first studied by Pierre de Fermat, who supposed that all Fermat numbers are prime. Indeed, first five of these numbers are prime but later Leonard

We aim to investigate Gaussian Fermat numbers and their polynomials satisfying second order recurrence relations. We obtain some important identities of these numbers, and we present some important transform of Gaussian Fermat numbers and polynomials. Also, we derive generating functions, Binet formula and summation formula of these numbers and polynomials. Beside these we present determinants of matrix representations of Gaussian Fermat polynomials.

The Hankel transform is an integral transform that was first developed by the mathematician Herman Hankel. It is also known as the Fourier-Bessel transform. The Hankel determinants calculus is arising in combinatorial analysis. Barry [5] investigated the Hankel

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https://doi.org/10.46939/J.Sci.Arts-24.3-a11

transform, Catalan transform, and binomial transform of integer numbers of sequences and polynomials and obtained very remarkable results. Inspired by these studies we present these transforms in Gaussian Fermat sequences and polynomials. This is done by finding the explicit form for the coefficients in the second-term recurrence relations that the corresponding polynomials satisfy.

Fermat numbers investigated by Tsang et al., [6] are defined by $\mathcal{F}_n = 2^n + 1$, named after French mathematician Pierre de Fermat. These numbers are important in various branches of mathematics as especially number theory and combinatorics. Although Fermat numbers are of interest many researchers, there are few studies on these numbers. I aim is to shed light on future studies on Gaussian Fermat numbers and polynomials. Mersenne numbers are given by $\mathcal{M}_n = 2^n - 1$ named after French mathematician Marin Mersenne. Many researchers have investigated the properties of Mersenne numbers [7-9]. These numbers are important as Fermat numbers in various branches of mathematics as especially number theory and combinatorics.

Fermat number attract the attention of many mathematician [10]. Following recurrence relation is about Fermat numbers.

$$\mathcal{F}_n = 3\mathcal{F}_{n-1} - 2\mathcal{F}_{n-2}, n \geq 2 \tag{1}$$

with $\mathcal{F}_0 = 2, \mathcal{F}_1 = 3$.

Mersenne numbers $\mathcal{M}_n$ by the second order recurrence relation [11] that

$$\mathcal{M}_n = 3\mathcal{M}_{n-1} - 2\mathcal{M}_{n-2}, n \geq 2 \tag{2}$$

with $\mathcal{M}_0 = 0, \mathcal{M}_1 = 1$.

Mersenne polynomials $\mathcal{M}(x)$ by the second order recurrence relation that

$$\mathcal{M}_n(x) = 3x\mathcal{M}_{n-1}(x) - 2\mathcal{M}_{n-2}(x)\ n \geq 2 \tag{3}$$

with initial conditions $\mathcal{M}_0(x) = 0, \mathcal{M}_1(x) = 1$

We can obtain Mersenne numbers giving $x = 1$ in (2). We show some terms of $\mathcal{M}_n(x)$ as follows:

$$\mathcal{M}_2(x) = 3x$$
$$\mathcal{M}_3(x) = 9x^2 - 2$$
$$\mathcal{M}_4(x) = 27x^3 - 12x$$
$$\mathcal{M}_5(x) = 81x^4 - 54x^2 + 4$$

The characteristic equation of $\mathcal{M}_n(x)$ is

$$x^2 - 3x + 2 = 0.$$

The characteristic equation has the following roots are

$$p(x) = \frac{3x - \sqrt{9x^2 - 8}}{2}$$
$$q(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}.$$

This paper introduces a novel extension of Fermat numbers and polynomials to the complex domain, termed Gaussian Fermat numbers and polynomials. We provide a comprehensive analysis of these entities, deriving essential properties such as Binet formulas, generating functions, and various identities analogous to those of classical number sequences. Matrix representations are employed to generate these numbers and polynomials, revealing their intrinsic structure and connections to other number sequences like Mersenne and Jacobsthal numbers. Furthermore, we explore transformations of the Gaussian Fermat sequence and polynomials, including Catalan, binomial, and binomial-Catalan transformations. By establishing these foundational properties and interconnections, this research lays the groundwork for further exploration of the arithmetic and algebraic properties of Gaussian Fermat numbers and polynomials, with potential applications in diverse areas of mathematics and beyond.

2. NEW GAUSSIAN FERMAT NUMBERS

In this section, Gaussian Fermat numbers are introduced and some of their basic properties are presented.

Definition 2.1. Gaussian Fermat number $G\mathcal{F}_n$ are defined by

$$G\mathcal{F}_n = 3G\mathcal{F}_{n-1} - 2G\mathcal{F}_{n-2}, n \geq 2 \tag{4}$$

with $G\mathcal{F}_0 = 2i, G\mathcal{F}_1 = 3$. Here Gaussian Fermat numbers are defined with these initial values by adhering to the recurrence relation of Fermat numbers and ensuring that the original Fermat numbers are obtained when i is replaced by 1.
The first terms are as follows:

$$G\mathcal{F}_2 = 9 - 4i$$
$$G\mathcal{F}_3 = 21 - 12i$$
$$G\mathcal{F}_4 = 45 - 28i$$
$$G\mathcal{F}_5 = 93 - 60i$$

Theorem 2.2 For $n \geq 2$ there is a relationship between Gaussian Fermat numbers and Fermat numbers, expressed as follows:

$$G\mathcal{F}_n = 3(\mathcal{F}_n - 2) - 2i(\mathcal{F}_n - 3)$$

Proof: Let's prove the theorem by induction over n: The theorem is true for $n = 1$, that is:

$$G\mathcal{F}_1 = 3 = 3(3-2) - 2i(3-3)$$

We assume that the theorem holds for $n = k$ that is;

$$G\mathcal{F}_k = 3(\mathcal{F}_k - 2) - 2i(\mathcal{F}_k - 3).$$

Then for $n = k + 1$ we have

$$G\mathcal{F}_{k+1} = 3G\mathcal{F}_k - 2G\mathcal{F}_{k-1}$$
$$= 3[3(\mathcal{F}_k - 2) - 2i(\mathcal{F}_k - 3)] - 2[3(\mathcal{F}_{k-1} - 2) - 2i(\mathcal{F}_{k-1} - 3)]$$
$$= 9(\mathcal{F}_k - 2) - 6i(\mathcal{F}_k - 3) - 6(\mathcal{F}_{k-1} - 2) + 4i(\mathcal{F}_{k-1} - 3)$$

$$= 9\mathcal{F}_k - 6\mathcal{F}_{k-1} - 6i\mathcal{F}_k + 4i\mathcal{F}_{k-1} - 6 + 6i$$
$$= 3(3\mathcal{F}_k - 2\mathcal{F}_{k-1}) - 2i(3\mathcal{F}_k - 2\mathcal{F}_{k-1}) - 6 + 6i$$
$$= 3\mathcal{F}_{k+1} - 2i\mathcal{F}_{k+1} - 6 + 6i = 3(\mathcal{F}_{k+1} - 2) - 2i(\mathcal{F}_{k+1} - 3)$$

as desired. □

Based on the relationship between Gaussian Fermat and Fermat numbers, we obtain the Binet formula of Gaussian Fermat numbers as follows:

$$G\mathcal{F}_n = 3(2^n + 1 - 2) - 2i(2^n + 1 - 3)$$
$$3(2^n - 1) - 2i(2^n - 2)$$
$$= 2^n(3 - 2i) + 4i - 3.$$

From here, the relationship between Gaussian Fermat numbers and Mersenne numbers can be easily seen. Namely;

$$G\mathcal{F}_n = 3M_n - 2i(M_n - 1) = M_n(3 - 2i) + 2i.$$

Theorem 2.3 The finite sum of Gaussian Fermat numbers can be formulated as follows:

$$\sum_{n=0}^{k} G\mathcal{F}_n = (3 - 2i)M_{k+1} + k(4i - 3).$$

Proof:

$$\sum_{n=0}^{k} 2^n(3 - 2i) + 4i - 3 = (3 - 2i)\sum_{n=0}^{k} 2^n + k(4i - 3)$$
$$= (3 - 2i)(2^{k+1} - 1) + k(4i - 3) = (3 - 2i)M_{k+1} + k(4i - 3).$$

□

Theorem 2.4. (Catalan identity). For $n \geq r$, we have

$$G\mathcal{F}_{n+r}G\mathcal{F}_{n-r} - G\mathcal{F}_n^2 = 2^{n-r}(18i - 1)(2^r - 1)^2.$$

Proof: From the Binet formula for Gaussian Fermat number we get,

$$(2^{n+r}(3 - 2i) + 4i - 3)(2^{n-r}(3 - 2i) + 4i - 3) - (2^n(3 - 2i) + 4i - 3)^2$$
$$= 2^n(2^r(4i - 3)(3 - 2i) + 2^{-r}(4i - 3)(3 - 2i) - 2(4i - 3)(3 - 2i))$$
$$= 2^n(18i - 1)\left(2^r + \frac{1}{2^r} - 2\right) = \frac{2^n(18i - 1)(2^r - 1)^2}{2^r} = 2^{n-r}(18i - 1)(2^r - 1)^2.$$

□

Theorem 2.5. (Cassini identity)

$$G\mathcal{F}_{n+1}G\mathcal{F}_{n-1} - G\mathcal{F}_n^2 = 2^{n-1}(18i - 1).$$

Proof: For $r = 1$ in Catalan identity we get Cassini identity.

□

Theorem 2.5. (Halton identity). For $n \geq r, s$ and $r \geq s$ we have

$$G\mathcal{F}_{n+r}G\mathcal{F}_{n-r} - G\mathcal{F}_{n+s}G\mathcal{F}_{n-s} = 2^n(18i - 1)(2^s M_{r-s} + 2^{-s}M_{s-r}).$$

Proof: From the Binet formula for Gaussian Fermat number and after basic algebraic operations we get,

$$(2^{n+r}(3-2i)+4i-3)(2^{n-r}(3-2i)+4i-3)-(2^{n+s}(3-2i)+4i-3)(2^{n-s}(3-2i)+4i-3)$$
$$=2^n(18i-1)(2^r+2^r-2^s-2^{-s})=2^n(18i-1)(2^sM_{r-s}+2^{-s}M_{s-r})$$

□

Theorem 2.6. (Gelin-Cesaro identity).

$$GF_{n+2}GF_{n+1}GF_{n-1}GF_{n-2}-GF_n^4$$
$$=2^n(18i-1)[2^{n-3}(18i-1)53+2^{2n-2}(5-12i)+2^{-2}(-7-24i)11]$$

Proof: From the Binet formula for Gaussian Fermat number we get desired.

□

Theorem 2.7. (d'Ocagne's identity). Let n and r be any integers. Then the following identity is true.

$$GF_{n+1}GF_r-GF_{r+1}=(2^n-2^r)(18i-1)$$

Proof: From the Binet formula for Gaussian Fermat number we get desired.

□

Definition 2.8. For $n\geq 1$, Gaussian Fermat number matrix $g\not{f}_n$ is defined by

$$g\not{f}_n=\begin{pmatrix}9-4i & -6\\ 3 & -4i\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}^n=\begin{pmatrix}GF_{n+2} & -2GF_{n+1}\\ GF_{n+1} & -2GF_n\end{pmatrix}$$

For example,

for $n=2$,

$$g\not{f}_2=\begin{pmatrix}9-4i & -6\\ 3 & -4\end{pmatrix}\begin{pmatrix}7 & -6\\ 3 & -2\end{pmatrix}=\begin{pmatrix}45-28i & -42+24i\\ 21-12i & -18-8i\end{pmatrix}.$$

Proposition 2.9. For $n\geq 1$, we have

$$\det(g\not{f}_n)=2^{n-1}(-72i+4).$$

Proof: We can do the proof by induction method on n such that;

For $n=1$, we have

$$\left|\begin{pmatrix}21-12i & -18+8i\\ 9-4i & -6\end{pmatrix}\right|=-72i+4.$$

For $n=k$, assume that the equation holds. That is:

$$\left|\begin{pmatrix}9-4i & -6\\ 3 & -4i\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}^k\right|=2^{k-1}(-72i+4).$$

Then $n=k+1$, we have

$$\left|\begin{pmatrix}9-4i & -6\\ 3 & -4i\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}^k\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}^1\right|=2.2^k(-36i+2)=2^{k+1}(-72i+4).$$

as desired.

□

Proposition 2.10. For $n \geq 1$ we have

$$\det(g\mathcal{f}_{n+1}^{2}) - \det(g\mathcal{f}_{n}^{2}) = 3.\,2^{2n-2}(-72i+4)^2.$$

Proof. From the definition of Gaussian Fermat polynomial's determinant

$$\det(g\mathcal{f}_{n+1}^{2}) - \det(g\mathcal{f}_{n}^{2}) = 2^{2n}(-72i+4)^2 - 2^{2n-2}(-72i+4)^2$$
$$= 3.\,2^{2n-2}(-72i+4)^2.$$

□

Theorem 2.11. Gaussian Jacobsthal number and Gaussian Fermat number have a relationship shown as follows:

$$G\mathcal{F}_n = \frac{(18-12i)\left(GJ_n - \frac{i-1}{6}(-1)^n\right) - 10 - 11i}{2+i}$$

Proof: Binet formula of Gaussian Jacobsthal in [12] is as follows

$$GJ_n = \frac{2i-2}{6}(-1)^n + \frac{2+i}{6}(2)^n$$

Binet formula of Gaussian Fermat is as follows:

$$G\mathcal{F}_n = 2^n(3-2i) + 4i - 3$$

If the two equations are equal to each other over 2^n , the desired result is achieved.

□

Proposition 2.12 For $m, n \geq 2$, we have

$$\det(g\mathcal{f}_{m+1})\det(g\mathcal{f}_{n+1}) + \det(g\mathcal{f}_{m})\det(g\mathcal{f}_{n}) = 2^{m+n-2}(4-72i)5.$$

Proof:

$$\det(g\mathcal{f}_{m+1})\det(g\mathcal{f}_{n+1}) + \det(g\mathcal{f}_{m})\det(g\mathcal{f}_{n})$$
$$= 2^m(4-72i)2^n(4-72i) + 2^{m-1}(4-72i)2^{n-1}(4-72i)$$
$$= 2^{m+n}(4-72i) + 2^{m+n-2}(4-72i) = 2^{m+n-2}(4-72i)5.$$

□

Proposition 2.13.

$$\det(g\mathcal{f}_{n+1}^{n}) - \det(g\mathcal{f}_{n}^{n}) = 2^{n^2-n}(4-72i)M_n$$

Proof:

$$\det(g\mathcal{f}_{n+1}^{n}) - \det(g\mathcal{f}_{n}^{n}) = 2^{n^2}(4-72i)^n - 2^{n^2-n}(4-72i)^n$$
$$= 2^{n^2-n}(4-72i)(2^n-1) = 2^{n^2-n}(4-72i)M_n.$$

If you pay attention here, it can be seen that the result obtained is related to Mersenne numbers.

□

3. NEW GAUSSIAN FERMAT POLYNOMIALS

In this section Gaussian Fermat polynomials $G\mathcal{F}_n(x)$ introduced and presented some of their basic properties.

Definition 3.1. Gaussian Fermat polynomials $G\mathcal{F}_n(x)$ are defined by

$$G\mathcal{F}_n(x) = 3xG\mathcal{F}_{n-1}(x) - 2G\mathcal{F}_{n-2}(x), n \geq 2 \tag{5}$$

with $G\mathcal{F}_0(x) = 2 + 3xi - \frac{3i}{2}, G\mathcal{F}_1(x) = 3 + 2i$.
The first terms are as follows:

$$G\mathcal{F}_2(x) = 9x - 4 + 3i$$
$$G\mathcal{F}_3(x) = 27x^2 - 12x + 9xi - 6 - 4i$$
$$G\mathcal{F}_4(x) = 81x^3 - 36x^2 + 27x^2i - 36x - 12xi - 6i + 8.$$

Theorem 3.2. For $n \geq 2$, we have a relation between Gaussian Fermat polynomials and Fermat polynomials expressed as follows.

$$G\mathcal{F}_n(x) = \mathcal{F}_n(x) + i\mathcal{F}_{n-1}(x)$$

Proof: we can prove the theorem by the induction method on n. for $n = 2$ we have,

$$G\mathcal{F}_2(x) = \mathcal{F}_2(x) + i\mathcal{F}_1(x) = 9x - 4 + 3i.$$

We assume that the theorem holds for $n = s$ that is;

$$G\mathcal{F}_s = \mathcal{F}_s(x) + i\mathcal{F}_{s-1}(x).$$

Then for $n = s + 1$ we have

$$G\mathcal{F}_{s+1}(x) = 3xG\mathcal{F}_s(x) - 2G\mathcal{F}_{s-1}(x)$$
$$= 3x[\mathcal{F}_s(x) + i\mathcal{F}_{s-1}(x)] - 2[\mathcal{F}_{s-1}(x) + i\mathcal{F}_{s-2}(x)]$$
$$= 3x\mathcal{F}_s(x) - 2\mathcal{F}_{s-1}(x) + i\big(3x\mathcal{F}_{s-1}(x) - 2\mathcal{F}_{s-2}(x)\big) = \mathcal{F}_{s+1}(x) + i\mathcal{F}_s(x).$$

as desired.

□

We defined Binet's formula for the Gaussian Fermat polynomials. Let $\alpha_1(x), \alpha_2(x)$ be the solutions for the

$$a^2 - 3xa + 2 = 0$$
$$\alpha_1(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}, \alpha_2(x) = \frac{3x + \sqrt{9x^2 - 8}}{2}.$$

So, we obtain

$$G\mathcal{F}_n(x) = \mathcal{F}_n(x) + i\mathcal{F}_{n-1}(x)$$

$$= \left(\frac{(2\alpha_1(x)-3)\alpha_1^n(x)}{\alpha_2(x)-\alpha_1(x)} + \frac{(-2\alpha_1(x)+3)\alpha_2^n(x)}{\alpha_2(x)-\alpha_1(x)}\right) + i\left(\frac{(2\alpha_2(x)-3)\alpha_1^{n-1}(x)}{\alpha_2(x)-\alpha_1(x)} + \frac{(-2\alpha_1(x)+3)\alpha_2^{n-1}(x)}{\alpha_2(x)-\alpha_1(x)}\right)$$

$$= \frac{\alpha_1^{n-1}(x)\big((2\alpha_2(x)-3)i+4-3\alpha_1(x)\big)+\alpha_2^{n-1}(x)\big((-2\alpha_1(x)+3)i-4+3\alpha_2(x)\big)}{\alpha_2(x)-\alpha_1(x)}.$$

□

Theorem 3.3 It can be seen that there is a relationship between Gaussian Fermat polynomials and Mersenne polynomials as follows:

$$G\mathcal{F}_n(x) = 3M_n(x) + (3i-4)M_{n-1}(x) - 4iM_{n-2}(x).$$

Proof: When the Binet formula for Mersenne polynomials in [13] is substituted, the desired equality is obtained.

□

Definition 3.4. For $n \geq 1$, Gaussian Fermat number matrix $g\mathcal{f}_n(x)$is defined by

$$g\mathcal{f}_n(\mathrm{x}) = \begin{pmatrix} 9x-4+3i & -6-4i \\ 3+2i & -4-6xi+3i \end{pmatrix}\begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G\mathcal{F}_{n+2}(x) & -2G\mathcal{F}_{n+1}(x) \\ G\mathcal{F}_{n+1}(x) & -2G\mathcal{F}_n(x) \end{pmatrix}$$

For example,

for $n = 1$,

$$g\mathcal{f}_1(\mathrm{x}) = \begin{pmatrix} 9x-4+3i & -6-4i \\ 3+2i & -4-6xi+3i \end{pmatrix}\begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 27x^2-12x+9xi-6-4i & -18x+8-6i \\ 9x-4+3i & -6-4i \end{pmatrix}.$$

Proposition 3.5. For $n \geq 1$, we have

$$\det(g\mathcal{f}_n(x)) = 2^n(-54x^2i+51xi-18x+17).$$

Proof: We can do the proof by induction method on n such that;

For $n = 1$, the claim is true. For $n = s$, assume that the equation holds. That is;

$$\left|\begin{pmatrix} 9x-4+3i & -6-4i \\ 3+2i & -4-6xi+3i \end{pmatrix}\begin{pmatrix} 3x & -2 \\ 1 & 0 \end{pmatrix}^s\right| = 2^s(-54x^2i+51xi-18x+17).$$

Then $n = s+1$, we have

$$\left|\begin{pmatrix} 9x-4+3i & -6-4i \\ 3+2i & -4-6xi+3i \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^s\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^1\right| = 2.2^s(-54x^2i+51xi-18x+17) = 2^{s+1}(-54x^2i+51xi-18x+17).$$

as desired.

□

Proposition 3.6.

$$\det(g\mathcal{f}_{n+1}^{n}(x)) - \det(g\mathcal{f}_{n}^{n}(x)) = 2^{n^2}(-54x^2 i + 51xi - 18x + 17)^n (M_n)$$

Proof:

$$\det\left(g\mathcal{f}_{n+1}^{n}(x)\right) - \det\left(g\mathcal{f}_{n}^{n}(x)\right)$$
$$= \left(2^{n+1}(-54x^2 i + 51xi - 18x + 17)\right)^n$$
$$-\left(2^{n}(-54x^2 i + 51xi - 18x + 17)\right)^n$$
$$= 2^{n^2}(-54x^2 i + 51xi - 18x + 17)^n (2^n - 1) = 2^{n^2}(-54x^2 i + 51xi - 18x + 17)^n (M_n)$$

Similar to Gauss Fermat numbers, it is seen that the result obtained in Gaussian Fermat polynomials is related to Mersenne numbers.

□

Theorem 3.7. The sum of the first n terms of the Gaussian Fermat polynomials are as follows:

$$\sum_{s=0}^{n} G\mathcal{F}_s(x) = \frac{-6x + 5 - 9x^2 i + {}^{i}/_{2} + {}^{15xi}/_{2} - 3xG\mathcal{F}_n(x) + 2G\mathcal{F}_{n-1}(x) + 2G\mathcal{F}_n(x)}{1 - 3x + 2}$$

Proof: From the definition of the Gaussian Fermat polynomials, we have

$$G\mathcal{F}_2(x) = 3xG\mathcal{F}_1(x) - 2G\mathcal{F}_0(x)$$
$$G\mathcal{F}_3(x) = 3xG\mathcal{F}_2(x) - 2G\mathcal{F}_1(x)$$
$$\vdots$$
$$G\mathcal{F}_n(x) = 3xG\mathcal{F}_{n-1}(x) - 2G\mathcal{F}_{n-2}(x)$$

so, we have

$$-5 - 3xi - {}^{i}/_{2} + \sum_{s=0}^{n} G\mathcal{F}_s(x)$$
$$= 3x\left[-2 - 3xi + {}^{3i}/_{2} - G\mathcal{F}_n(x) + \sum_{s=0}^{n} G\mathcal{F}_s(x)\right]$$
$$- 2\left[-G\mathcal{F}_{n-1}(x) - G\mathcal{F}_n(x) + \sum_{s=0}^{n} G\mathcal{F}_s(x)\right]$$
$$-5 - 3xi - {}^{i}/_{2} + \sum_{s=0}^{n} G\mathcal{F}_s(x)$$
$$= -6x - 9x^2 i + {}^{9xi}/_{2} - 3xG\mathcal{F}_n(x) + 3x\sum_{s=0}^{n} G\mathcal{F}_s(x) + 2G\mathcal{F}_n(x) + 2G\mathcal{F}_{n-1}(x)$$
$$- 2\sum_{s=0}^{n} G\mathcal{F}_s(x)$$

If we take the total $\sum_{s=0}^{n} G\mathcal{F}_s(x)$ of the right and left sides of the equation in parentheses and rewrite it, we obtain the desired equality.

$$\sum_{s=0}^{n} G\mathcal{F}_s(x)\,(1-3x+2)$$
$$= -6x - 9x^2 i + {}^{9xi}/_{2} - 3xG\mathcal{F}_n(x) + 2G\mathcal{F}_{n-1}(x) + 2G\mathcal{F}_n(x) + 5 + 3xi + {}^{i}/_{2}$$

thus, we obtain:

$$\sum_{s=0}^{n} G\mathcal{F}_s(x) = \frac{-6x + 5 - 9x^2 i + {}^{i}/_{2} + {}^{15xi}/_{2} - 3xG\mathcal{F}_n(x) + 2G\mathcal{F}_{n-1}(x) + 2G\mathcal{F}_n(x)}{1-3x+2}$$

□

Theorem 3.8. The generating functions for Gaussian Fermat numbers and Gaussian Fermat polynomials are given as follows, respectively:

$$u(x) = \sum_{n=0}^{\infty} G\mathcal{F}_n x^n = \frac{2i + 3x - 6xi}{1 - 3x + 2x^2}$$

and

$$v(x) = \sum_{n=0}^{\infty} G\mathcal{F}_n(x) a^n = \frac{4 + 6xi - 3i + 6a + ai(4 - 18x^2 + 9x) - 12xa}{2(1 - 3xa + 2a^2)}.$$

Proof: By the definition of the Gaussian Fermat sequence, we get

$$u(x) = \sum_{n=0}^{\infty} G\mathcal{F}_n x^n = G\mathcal{F}_0 + G\mathcal{F}_1 x + G\mathcal{F}_2 x^2 + G\mathcal{F}_3 x^3 + \cdots$$
$$u(x) = G\mathcal{F}_0 + G\mathcal{F}_1 x + \sum_{n=2}^{\infty} G\mathcal{F}_n x^n$$
$$= G\mathcal{F}_0 + G\mathcal{F}_1 x + \left(\sum_{n=2}^{\infty} 3G\mathcal{F}_{n-1} - 2G\mathcal{F}_{n-2}\right) x^n$$
$$= G\mathcal{F}_0 + G\mathcal{F}_1 x + 3x(G\mathcal{F}_1 x + G\mathcal{F}_2 x^2 + \cdots) - 2x^2(G\mathcal{F}_0 + G\mathcal{F}_1 x + G\mathcal{F}_2 x^2 + \cdots)$$
$$\sum_{n=0}^{\infty} G\mathcal{F}_n x^n (1 - 3x + 2x^2) = G\mathcal{F}_0 + G\mathcal{F}_1 x - 3xG\mathcal{F}_0 = 2i + 3x - 6xi$$
$$\sum_{n=0}^{\infty} G\mathcal{F}_n x^n = \frac{2i + 3x - 6xi}{1 - 3x + 2x^2}.$$

Similarly;

$$v(x) = \sum_{n=0}^{\infty} G\mathcal{F}_n(x) a^n = G\mathcal{F}_0(x) + G\mathcal{F}_1(x) a + G\mathcal{F}_2(x) a^2 + G\mathcal{F}_3(x) a^3 + \cdots$$
$$v(x) = G\mathcal{F}_0(x) + G\mathcal{F}_1(x) a + \sum_{n=2}^{\infty} G\mathcal{F}_n(x) a^n$$

$$= G\mathcal{F}_0(x) + G\mathcal{F}_1(x)a + \left(\sum_{n=2}^{\infty} 3xG\mathcal{F}_{n-1}(x) - 2G\mathcal{F}_{n-2}(x)\right)a^n$$

$$= G\mathcal{F}_0(x) + G\mathcal{F}_1(x)a + 3xa(G\mathcal{F}_1(x)a + G\mathcal{F}_2(x)a^2 + \cdots) - 2a^2(G\mathcal{F}_0(x) + G\mathcal{F}_1(x)a + \cdots)$$

$$\sum_{n=0}^{\infty} G\mathcal{F}_n(x)a^n(1 - 3xa + 2a^2) = G\mathcal{F}_0(x) + G\mathcal{F}_1(x)a - 3xaG\mathcal{F}_0(x)$$

$$\sum_{n=0}^{\infty} G\mathcal{F}_n(x)a^n = \frac{4 + 6xi - 3i + 6a + ai(4 - 18x^2 + 9x) - 12xa}{2(1 - 3xa + 2a^2)}.$$

□

Theorem 3.9. The exponential generating function for $G\mathcal{F}_n$ and $G\mathcal{F}_n(x)$,

$$\sum_{n=0}^{\infty} \frac{G\mathcal{F}_n x^n}{n!} = (3 - 2i)e^{2x} + (4i - 3)e^x$$

and

$$\sum_{n=0}^{\infty} \frac{G\mathcal{F}_n(x)a^n}{n!} = \frac{\big((2\alpha_2(x) - 3)i + 4 - 3\alpha_1(x)\big)e^{\alpha_1(x)a}}{\alpha_1(x)(\alpha_2(x) - \alpha_1(x))} + \frac{\big((-2\alpha_1(x) + 3)i - 4 + 3\alpha_2(x)\big)e^{\alpha_2(x)a}}{\alpha_2(x)(\alpha_2(x) - \alpha_1(x))}.$$

□

Proof: For the proof, we use Binet formula of $G\mathcal{F}_n$ and $G\mathcal{F}_n(x)$,

$$\sum_{n=0}^{\infty} \frac{G\mathcal{F}_n x^n}{n!} = \sum_{n=0}^{\infty} 2^n (3 - 2i) + (4i - 3)$$

$$= (3 - 2i)\sum_{n=0}^{\infty} 2^n \frac{x^n}{n!} + (4i - 3)\sum_{n=0}^{\infty} \frac{x^n}{n!} = (3 - 2i)e^{2x} + (4i - 3)e^x$$

similarly,

$$\sum_{n=0}^{\infty} \frac{G\mathcal{F}_n(x)a^n}{n!} = \sum_{n=0}^{\infty} \frac{\alpha_1^{n-1}(x)\big((2\alpha_2(x) - 3)i + 4 - 3\alpha_1(x)\big)a^n}{n!\,(\alpha_2(x) - \alpha_1(x))} + \sum_{n=0}^{\infty} \frac{\alpha_2^{n-1}(x)\big((-2\alpha_1(x) + 3)i - 4 + 3\alpha_2(x)\big)a^n}{n!\,(\alpha_2(x) - \alpha_1(x))}$$

$$= \sum_{n=0}^{\infty} \frac{\big(a\alpha_1(x)\big)^n\big((2\alpha_2(x) - 3)i + 4 - 3\alpha_1(x)\big)}{\alpha_1(x)n!\,(\alpha_2(x) - \alpha_1(x))} + \sum_{n=0}^{\infty} \frac{\big(a\alpha_2(x)\big)^n\big((2\alpha_2(x) - 3)i + 4 - 3\alpha_1(x)\big)}{\alpha_2(x)n!\,(\alpha_2(x) - \alpha_1(x))}$$

$$\sum_{n=0}^{\infty} \frac{G\mathcal{F}_n(x)a^n}{n!} = \frac{\big((2\alpha_2(x) - 3)i + 4 - 3\alpha_1(x)\big)e^{\alpha_1(x)a}}{\alpha_1(x)(\alpha_2(x) - \alpha_1(x))} + \frac{\big((-2\alpha_1(x) + 3)i - 4 + 3\alpha_2(x)\big)e^{\alpha_2(x)a}}{\alpha_2(x)(\alpha_2(x) - \alpha_1(x))}.$$

Theorem 3.10. Let $n \to \infty, b > a.$ Then the ratio of the bigger to the smallest of the two consecutive terms of the Gaussian Fermat numbers and polynomials are as follows:

$$\lim_{n\to\infty} \frac{G\mathcal{F}_{n+1}}{G\mathcal{F}_n} = 2$$

and

$$\lim_{n\to\infty} \frac{G\mathcal{F}_{n+1}(x)}{G\mathcal{F}_n(x)} = \alpha_2(x).$$

Proof: The Binet Formula of the Gaussian Fermat numbers and polynomials we get,

$$\lim_{n\to\infty} \frac{G\mathcal{F}_{n+1}}{G\mathcal{F}_n} = \frac{2^{n+1}(3-2i+\frac{4i-3}{2^{n+1}})}{2^n(3-2i+\frac{4i-3}{2^n})} = 2$$

Similarly, as per the limit rules

$$\lim_{n\to\infty} \frac{G\mathcal{F}_{n+1}(x)}{G\mathcal{F}_n(x)}$$
$$= \lim_{n\to\infty}\left(\frac{\frac{\alpha_1^n(x)\big((2\alpha_2(x)-3)i+4-3\alpha_1(x)\big)+\alpha_2^n(x)\big((-2\alpha_1(x)+3)i-4+3\alpha_2(x)\big)}{\alpha_2(x)-\alpha_1(x)}}{\frac{\alpha_1^{n-1}(x)\big((2\alpha_2(x)-3)i+4-3\alpha_1(x)\big)+\alpha_2^{n-1}(x)\big((-2\alpha_1(x)+3)i-4+3\alpha_2(x)\big)}{\alpha_2(x)-\alpha_1(x)}}\right)$$
$$= \lim_{n\to\infty}\left(\frac{\alpha_2^n(x)\frac{\big(\alpha_1^n(x)(2\alpha_2(x)-3)i+4-3\alpha_1(x)\big)}{\alpha_2^n(x)}+(-2\alpha_1(x)+3)i-4+3\alpha_2(x)}{\alpha_2^{n-1}(x)\frac{\big(\alpha_1^{n-1}(x)(2\alpha_2(x)-3)i+4-3\alpha_1(x)\big)}{\alpha_2^{n-1}(x)}+(-2\alpha_1(x)+3)i-4+3\alpha_2(x)}\right)$$
$$= \alpha_2(x).$$

□

4. CATALAN AND BINOMIAL TRANSFORMATION OF GAUSSIAN FERMAT NUMBERS AND POLYNOMIALS

The Catalan transform is a sequence transform introduced by Barry [5]. Also, many researcher [14-17] investigated the Catalan transform of integer sequences. Recall the Catalan numbers $C(n)$ are given by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

Later the formula given above can be written as

$$C_n = \frac{(2n)!}{(n+1)!\,n!}$$

First few Catalan numbers are 1,1,2,5,14,42,132,429, …
Also, one can easily see the following recurrence relation for $C(n)$

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}$$

It is given that the ordinary generating function (o.g.f.) $c(x)$ as follows:

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$
$$c(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$$

We define the Catalan transform of Gaussian Fermat sequence $(CG\mathcal{F}_n)$ as

$$CG\mathcal{F}_n = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} G\mathcal{F}_i$$

for $n \geq 1$,

First few terms of $CG\mathcal{F}_n$ as follow:

$$CG\mathcal{F}_1 = \sum_{1}^{1} \frac{i}{2-i}\binom{2-i}{1-i} G\mathcal{F}_i = 3$$
$$CG\mathcal{F}_2 = \sum_{1}^{2} \frac{i}{4-i}\binom{4-i}{2-i} G\mathcal{F}_i = 12 - 4i$$
$$CG\mathcal{F}_3 = \sum_{1}^{3} \frac{i}{6-i}\binom{6-i}{3-i} G\mathcal{F}_i = 45 - 20i$$
$$CG\mathcal{F}_4 = \sum_{1}^{4} \frac{i}{8-i}\binom{8-i}{4-i} G\mathcal{F}_i = 168 - 84i$$

It can be written the following equations as the product of matrix C and $n \times 1$ matrix $G\mathcal{F}_k$

$$\begin{bmatrix} CG\mathcal{F}_1 \\ CG\mathcal{F}_2 \\ CG\mathcal{F}_3 \\ CG\mathcal{F}_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \end{bmatrix} . \begin{bmatrix} G\mathcal{F}_1 \\ G\mathcal{F}_2 \\ G\mathcal{F}_3 \\ G\mathcal{F}_4 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 12-4i \\ 45-20i \\ 168-84i \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 1 & 0 & 0 & \ldots \\ 5 & 5 & 3 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix} . \begin{bmatrix} 3 \\ 9-4i \\ 21-12i \\ 45-28i \\ \vdots \end{bmatrix}$$

Similary, for Gaussian Fermat polynomials we define the Catalan transform of Gaussian Fermat polynomials $(CG\mathcal{F}_n(x))$ as

$$CG\mathcal{F}_n(x)=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}G\mathcal{F}_i(x)$$

for $n\geq 1$

First few terms of $CG\mathcal{F}_n(x)$as follow:

$$CG\mathcal{F}_1(x)=\sum_{1}^{1}\frac{i}{2-i}\binom{2-i}{1-i}G\mathcal{F}_i(x)=3+2i$$

$$CG\mathcal{F}_2(x)=\sum_{1}^{2}\frac{i}{4-i}\binom{4-i}{2-i}G\mathcal{F}_i(x)=9x-1+5i$$

$$CG\mathcal{F}_3(x)=\sum_{1}^{3}\frac{i}{6-i}\binom{6-i}{3-i}G\mathcal{F}_i(x)=27x^2+6x+9xi+6i-8$$

It can be written the following equations as the product of matrix C and $n\times 1$ matrix $G\mathcal{F}_k(x)$

$$\begin{bmatrix}CG\mathcal{F}_1(x)\\CG\mathcal{F}_2(x)\\CG\mathcal{F}_3(x)\\\dots\\\vdots\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\2&2&1&0&0\\5&5&3&1&0\\\vdots&\vdots&\vdots&\vdots&0\end{bmatrix}.\begin{bmatrix}G\mathcal{F}_1(x)\\G\mathcal{F}_2(x)\\G\mathcal{F}_3(x)\\\dots\\\vdots\end{bmatrix}$$

$$\begin{bmatrix}3+2i\\9x-1+5i\\27x^2+6x+9xi+6i-8\\\dots\\\vdots\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&\dots\\1&1&0&0&0&\dots\\2&2&1&0&0&\dots\\5&5&3&1&0&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\dots\end{bmatrix}.\begin{bmatrix}3+2i\\9x-4+3i\\27x^2-12x+9xi-6-4i\\\dots\\\vdots\end{bmatrix}$$

Bernstein and Solane [18] investigated the binomial transformation of integer sequences. We define the Binomial transform of Gaussian Fermat numbers and polynomials $BG\mathcal{F}_n$ an $(BG\mathcal{F}_n)(x)$ respectively.

$$BG\mathcal{F}_n=\sum_{j=0}^{n}\binom{n}{j}G\mathcal{F}_j.$$

for $n\geq 0$ with $BG\mathcal{F}_0=2i, B\mathcal{F}_1=2i+3$.

First few terms of $BG\mathcal{F}_n$ as follow:

$$BG\mathcal{F}_2=\sum_{j=0}^{2}\binom{2}{j}G\mathcal{F}_j=15-2i,$$

$$BG\mathcal{F}_3=\sum_{j=0}^{3}\binom{3}{j}G\mathcal{F}_j=57-22i,$$

$$BG\mathcal{F}_4 = \sum_{j=0}^{4} \binom{4}{j} G\mathcal{F}_j = 195 - 98i,$$

The Binomial transform of Gaussian Fermat sequences can be written as matrix product as follow:

$$\begin{bmatrix} BG\mathcal{F}_0 \\ BG\mathcal{F}_1 \\ BG\mathcal{F}_2 \\ BG\mathcal{F}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} G\mathcal{F}_0 \\ G\mathcal{F}_1 \\ G\mathcal{F}_2 \\ G\mathcal{F}_3 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 2i \\ 15 - 2i \\ 57 - 22i \\ 195 - 98i \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} 2i \\ 3 \\ 9 - 4i \\ 21 - 12i \\ \vdots \end{bmatrix}.$$

Similarly Gaussian Fermat polynomials first few terms of $BG\mathcal{F}_n$(x) as follow:

$$BG\mathcal{F}_n(x) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{F}_j(x).$$

for $n \geq 0$ with $BG\mathcal{F}_0(x) = 2 + 3xi - \frac{3i}{2}, BG\mathcal{F}_1(x) = 5 + \frac{i}{2} + 3xi.$

$$BG\mathcal{F}_2(x) = \sum_{j=0}^{2} \binom{2}{j} G\mathcal{F}_j(x) = 4 + 9x + 3xi + \frac{11i}{2},$$

$$BG\mathcal{F}_3(x) = \sum_{j=0}^{3} \binom{3}{j} G\mathcal{F}_j(x) = 27x^2 - 7 + 12xi + 15x + \frac{19i}{2},$$

The Binomial transform of Gaussian Fermat polynomials can be written as matrix product as follow:

$$\begin{bmatrix} BG\mathcal{F}_0(x) \\ BG\mathcal{F}_1(x) \\ BG\mathcal{F}_2(x) \\ BG\mathcal{F}_3(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \begin{bmatrix} G\mathcal{F}_0(x) \\ G\mathcal{F}_1(x) \\ G\mathcal{F}_2(x) \\ G\mathcal{F}_3(x) \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} 2+3xi-\frac{3i}{2} \\ 5+\frac{i}{2}+3xi \\ 4+9x+3xi+\frac{11i}{2} \\ 27x^2-7+12xi+15x+\frac{19i}{2} \\ \vdots \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 2+3xi-\frac{3i}{2} \\ 3+2i \\ 9x-4+3i \\ 27x^2-12x+9xi-6-4i \\ \vdots \end{bmatrix}$$

Binomial transform [19] enables the given sequence a_n to be obtained as b_n with the help of the transformation given below, namely;

$$a_n=\sum_{k=0}^{n}\binom{n}{k}b_n.$$

The representation of the matrix coefficients of this transformation is as follows:

$$\mathcal{A}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Also Binomial transform of the Catalan transform can obtain as follows:

$$BC(a_n)=\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{i}{2j-i}\binom{n}{j}\binom{2j-i}{j-i}a_n.$$

Here if we also talk about the binomial transform of the Catalan transform, we can also obtain the binomial transforms of the Catalan transforms of Gaussian Fermat polynomials with the help of the transformation given above.

$$BCG\mathcal{F}_n(x)=\sum_{j=1}^{n}\left(\frac{1}{2j-1}\binom{n}{j}\binom{2j-1}{j-1}G\mathcal{F}_n(x)+\frac{2}{2j-2}\binom{n}{j}\binom{2j-2}{j-2}G\mathcal{F}_n(x)\right.$$
$$\left.+\frac{3}{2j-3}\binom{n}{j}\binom{2j-3}{j-3}G\mathcal{F}_n(x)+\cdots+\binom{n}{j}\binom{j}{0}G\mathcal{F}_n(x)\right)$$

First few terms of $BCG\mathcal{F}_n(x)$ as follow:

$$BCG\mathcal{F}_1(x) = 1.G\mathcal{F}_1(x),$$
$$BCG\mathcal{F}_2(x) = 4.G\mathcal{F}_2(x),$$
$$BCG\mathcal{F}_3(x) = 14.G\mathcal{F}_3(x),$$
$$BCG\mathcal{F}_4(x) = 50.G\mathcal{F}_4(x)$$

These coefficients $\{1,4,14,50,\ldots\}$obtained are the self-convolution of the sequence generated from the Binomial transform of the Catalan numbers Ba_n.
The Binomial transform of the Catalan transform is also called the Ballot transform [5] and is denoted as:

$$Bal = Cat \circ Bin\,.$$

4.1. THE GENERATING FUNCTIONS OF THE CATALAN TRANSFORM OF GAUSSIAN FERMAT NUMBERS AND POLYNOMIALS

Let $u(x), v(x)$ and $c(x)$ be the generating functions of Gaussian Fermat numbers, polynomials and Catalan sequences, respectively. $CG\mathcal{F}_n\big(x * c(x)\big) = j(x)$ and $CG\mathcal{F}_n(x)\big(x * c(x)\big) = \mathcal{J}(x)$, shown the generating function of the Catalan Gaussian Fermat numbers and polynomials. The generating function of the Catalan numbers in [3] are

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

The following equations can be written for the generating function of the Catalan Gaussian Fermat numbers and polynomials resperctively. From Theorem 3.8 we get,

$$j(x) = C\mathcal{F}_n\big(x * c(x)\big) = \frac{-2i+3+(6i-3)\sqrt{1-4x}}{\sqrt{1-4x}+1-4x}$$

and

$$\mathcal{J}(x) = C\mathcal{F}_n(x)\big(x * c(x)\big) = \frac{16-2ai+72aix+\sqrt{1-4x}(-12i+24a+18ai)}{4\big(2-3a+3a\sqrt{1-4x}\big)}.$$

5. CONCLUSION

We introduced the new Gaussian Fermat numbers, $G\mathcal{F}_n$, and polynomials, $G\mathcal{F}_n(x)$, and presented several important identities associated with them. We derived the generating functions for both $G\mathcal{F}_n$ and $G\mathcal{F}_n(x)$, as well as identities including Cassini, Catalan, Vajda, Honsberger, Halton, Gelin-Cesaro, and D'Ocagne's, which are significant in the study of number sequences. Furthermore, we demonstrated that Gaussian Fermat polynomials can be represented through matrix representations, discussing key propositions based on the constant nature of the determinants of these matrices. Additionally, we explored the Catalan, Binomial, and Binomial of Catalan transformations of the Gaussian Fermat sequence polynomials. Finally, we introduced the generating function for the Catalan transformation of the Gaussian Fermat numbers and polynomials.

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