ORIGINAL PAPER

NUMERICAL INVESTIGATION OF THE ALPHA PARAMETRIZED DTM WITH THE CLASSICAL DTM

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Manuscript received: 11.01.2024; Accepted paper: 23.07.2024; Published online: 30.09.2024.

Abstract. *In this paper, solutions of some differential equations are found using alpha parametrized differential transform method (α–PDTM). The results are compared with the exact solution and the results found from the classical differential transform method (DTM). The optimal values of α are found for which the error at some particular domain point is minimal. It is found that the alpha parametrized differential transform method is more accurate and precise method.*

Keywords: *DTM; alpha parametrized DTM; TSM; Transforms.*

1. INTRODUCTION

There are many situations in various fields of engineering, physics, chemistry and economics which can be modelled to differential equations. These differential equations may be complicated to solve analytically in most of the cases. So, rather than the analytical solutions of the problems, numerical solutions can be found. A number of semi-analytical and numerical methods are available in the literature which can be used to find solutions of differential equations. These methods involve: the Finite Difference Method (FDM), the Shooting Method (SM), the Adomian Decomposition Method (ADM), the Rationalized Haar Function Method (RHFM), the Galerkin Method (GM), the Homotopy Perturbation Method (HPM), Differential Transform Method (DTM), Variational Iteration Method (VIM), the Predictor Correctors Method (PCM) and the like.

Zhou introduced, in [1], a semi-analytical method, called the DTM and the method was used to find solution of differential equations involved in electrical circuits analysis. It was based on the classical Taylor Series Method (TSM) and is one of simplest and accurate methods that can be used to find approximate solution of linear and nonlinear ordinary and partial differential equations and system of differential equations without requiring linearization of the differential equation. It covers wide range of applications and requires less computational work to attain the accuracy and has high convergence rate. Chen and Ho, in [2], applied the DTM to find solutons of a number of eigenvalue problems and the results were compared with some other analytical methods. Chiou and Tzeng, in [3], used the TSM to find solution of some nonlinear variational problems. The Van der Pol's equation and motion behavior of an earthquake isolation system were studied and the results were compared with some other methods. Chen and Ho found solution of a number of partial differential equations using two dimensional DTM in [4].

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https://doi.org/10.46939/J.Sci.Arts-24.3-a13

Jang *et al.* approximated solutions of a number of IVPs using the classical DTM in [5]. Jang and Chen, in [6], demonstrate that the DTM was a feasible tool for obtaining the analytic solutions of linear and nonlinear PDEs. Ayaz applied the DTM to find solution of system of partial differential equations in [7]. Biaza and Eslami used the DTM to find the solutions of some quadratic reccati differential equations in [8]. Soltanalizadeh, in [9], utilized the DTM to solve the hyperbolic Telegraph equation. Some numerical tests were presented to demonstrate the effectiveness and accuracy of the proposed method. Soltanalizadeh and Zarebnia, in [10], used DTM to find the exact solution of the Kuramoto-Sivashinsky equation. Exact solution achieved by the known forms of the series solutions proved that the DTM was one of the powerful method for linear and nonlinear equations. Dogan *et al.*, in [11], uesd DTM initially to solve linear singularly perturbed two-point boundary value problems. Some examples were established to signify that linear singularly perturbed two-point boundary value problems with high accuracy was obtained by DTM. Tabatabaei and Gunerhan used the DTM to solve the Duffing equation in [12]. Opanuga *et al*. used DTM to find the solution of two- point boundary value problems. By comparing the result with exact solution, it was concluded that DTM attained great accuracy in [13].

Mukhtarov *et al.*, in 0, proposed $\alpha-$ PDTM to solve two Sturm–Liouville problems, and the obtained results were compared with those obtained by the classical DTM and by the other methods. The result reveals that $\alpha-$parameterized differential transform method was more accurate and effective numerical algorithm. Mukhtarov *et al.*, in [15], used $\alpha-$ PDTM to approximate the solution of the boundary value problems and then applied the proposed method to two boundary value problems for different values of the parameter α. Afterward, its solutions were compared with those of DTM and exact solutions and it was found that the proposed method attained more accurate method. Yucel and Mukhtarov, in [16], proposed $\alpha-$ PDTM to solve the boundary value transmission problem for the third-order differential equation. A comparable solution for the identical issue was achieved through the classical DTM, and a graphical comparison was conducted for the obtained solutions.

2. MATERIALS AND METHODS

2.1. THE CLASSICAL DTM

Using the Taylor series about $x = x_0$, an analytic function $f(x)$ can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dx^k}(f(x))\right]_{x=x_0}(x-x_0)^k. \tag{1}$$

Let O be an operator defined as $O\{f(x)\} = \frac{1}{k!}\left[\frac{d^k}{dx^k}(f(x))\right]_{x=x_0}$ $(= F(k)$, say). The inverse operator O^{-1}, by definition, follows the property $O^{-1}(F(k)) = f(x)$ and has already been defined in (1). One may write (1) in the form

$$f(x)=\sum_{k=0}^{\infty}F(k)\ (x-x_0)^k. \tag{2}$$

For practical usage, we define $f_n(x)=\sum_{k=0}^{n}F(k)\ (x-x_0)^k$. It is easy to see that

$$H(k)=\alpha F(k)\pm\beta G(k) \text{ as } h(x)=\alpha f(x)\pm\beta g(x),$$

where α and β are constants,

$$H(k)=\prod_{j=1}^{n}(k+j)\ F(k+n) \text{ as } h(x)=\frac{d^n}{dx^n}(f(x)),$$

$$H(k)=\sum_{j=0}^{k}F(k-j)G(j) \text{ as } h(x)=f(x)g(x),$$

$$H(k)=\delta(k-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n\end{cases} \text{ as } h(x)=x^n,$$

$$H(k)=a^k\frac{e^b}{k!} \text{ as } h(x)=e^{ax+b}$$

and

$$H(k)=\frac{1}{k!}\alpha^k\cos(\frac{k}{2}\pi+\beta) \text{ as } h(x)=\cos(\alpha x+\beta),$$ where α and β are constants.

2.2. THE $\alpha-$ PARMETRIZED DTM

Let a and b be two real numbers such that $a<b$. Then the interval $[a,b]$ is equivalent to the unit interval $[0,1]$ due to the bijection $g:[0,1]\to[a,b]$ defined by $g(\alpha)=\alpha a+(1-\alpha)b$. Let $f:[a,b]\to\mathbb{R}$ be infinitely differentiable function. Then instead of using (2), one may consider

$$f(x)=\sum_{k=0}^{\infty}F_\alpha(k)\ (x-x_\alpha)^k, \tag{3}$$

where $x_\alpha=\alpha a+(1-\alpha)b$. For practical usage, we define

$$f_n(x)=\sum_{k=0}^{n}F_\alpha(k)\ (x-x_\alpha)^k. \tag{4}$$

We define an operator O_α as

$$O_\alpha(f(x))=\alpha\frac{1}{k!}\left[\frac{d^k}{dx^k}(f(x))\right]_{x=a}+(1-\alpha)\frac{1}{k!}\left[\frac{d^k}{dx^k}(f(x))\right]_{x=b}\ (=F_\alpha(k)).$$

One may use the notation $\frac{1}{k!}\left[\frac{d^k}{dx^k}(f(x))\right]_{x=a} = F_\alpha(k;a)$ for convenience. The inverse operator O_α^{-1}, by definition, follows the property $O_\alpha^{-1}(F_\alpha(k)) = f(x)$ and has already been defined in (3).

3. RESULTS AND DISCUSSION

3.1. RESULTS

It is easy to see that

$$H_\alpha(k) = pF_\alpha(k) \pm qG_\alpha(k) \text{ as } h(x) = pf(x) \pm qg(x),$$

where p and q are constants. Moreover

$$H_\alpha(k) = \prod_{j=1}^{n}(k+j)\, F_\alpha(k+n) \text{ as } h(x) = \frac{d^n}{dx^n}(f(x)),$$

$$H_\alpha(k) = \sum_{j=0}^{k}\left(\alpha F_\alpha(k;a)G_\alpha(k-j;a) + (1-\alpha)F_\alpha(k;b)G_\alpha(k-j;b)\right) \text{ as } h(x) = f(x)g(x),$$

$$H_\alpha(k) = \begin{cases} \overset{n}{\underset{k}{C}}\left((1-\alpha)a^{n-k} + \alpha\, b^{n-k}\right), & k < n, \\ 1, & k = n, \\ 0, & k > n \end{cases}$$

as $h(x) = x^n$,

$$H_\alpha(k) = \frac{p^k}{k!}\left(\alpha\cos(\frac{k}{2}\pi + pa + q) + (1-\alpha)\cos(\frac{k}{2}\pi + pb + q)\right) \text{ as } h(x) = \cos(px+q),$$

where p and q are constants and

$$H_\alpha(k) = \frac{c^k e^d}{k!}\left[\alpha e^{ca} + (1-\alpha)e^{cb}\right] \text{as } h(x) = e^{cx+d},$$

where a and b are constants.

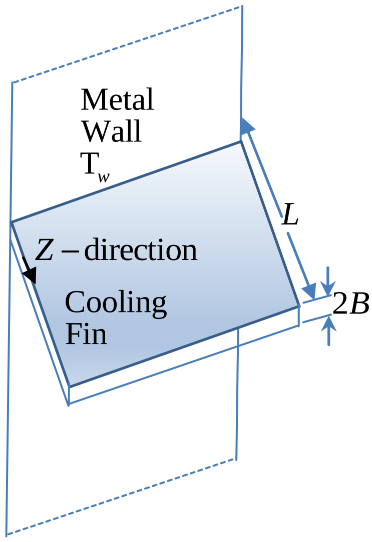

Figure 1. Cooling Fin.

3.2. DISCUSSION

The effectiveness of the method "$\alpha - PDTM$" is demonstrated in the following examples

Problem 1. An application of heat conduction is calculation of efficiency of cooling fin which is used to increase the area available for heat conduction between metal wall and poorly conducting fluid like gases (see the Figure 1). In order to calculate the efficiency, we firstly, find the temperature profile of the fin. For $L >> B$, no heat is lost from the edges so the heat flux is given by $q = \eta(T - T_a)$, where the surrounding fluid temperature T_a and the convective heat transfer coefficient η are constant. Thus, the governing DE becomes $\frac{d^2T}{dz^2} = \frac{\eta}{kB}(T - T_a)$ while k is the thermal conductivity of the fin. Moreover, $T(0) = T_w$ and $\frac{dT}{dz}(L) = 0$. Using $\theta = \frac{T - T_a}{T_w - T_a}$, $x = \frac{z}{L}$ and $H = \sqrt{\frac{\eta L^2}{kB}}$; the boundary value problem is reformulated in the form $\frac{d^2\theta}{dx^2} = H^2\theta$; $\theta(0) = 1$, $\frac{d\theta}{dx}(1) = 0$. We consider the problem for particular value of H, say for $H = 2$. Thus, the boundary value problem

$$\frac{d^2\theta}{dx^2} = 4\theta; \tag{5}$$

$$\theta(0) = 1, \tag{6}$$

$$\frac{d\theta}{dx}(1) = 0. \tag{7}$$

has the exact solution

$$\theta(x)=\frac{e^{-2x}(e^{4}+e^{4x})}{1+e^{4}}. \tag{8}$$

For $(a,b)=(0,1)$; $x_\alpha=1-\alpha$. Applying the operator O_α, (5) leads to the recurrence relation

$$\Theta_\alpha(k+2)=\frac{4\Theta_\alpha(k)}{(k+1)(k+2)}. \tag{9}$$

We use $n=5$ to approximate $\theta(x)$ as

$$\theta_n(x,\alpha)=\sum_{k=0}^{n}\Theta_\alpha(k)\,(x-x_\alpha)^k \tag{10}$$

and find $\Theta_\alpha(2),\Theta_\alpha(3),\Theta_\alpha(4),\Theta_\alpha(5)$ in terms of $\Theta_\alpha(0)$ and $\Theta_\alpha(1)$ from the recurrence relation (9). Then, we use these coefficients $\Theta_\alpha(2),\Theta_\alpha(3),\Theta_\alpha(4),...$ in (10) to write it in terms of $\Theta_\alpha(0)$ and $\Theta_\alpha(1)$ only. After this, the values of $\Theta_\alpha(0)$ and $\Theta_\alpha(1)$ are found using the boundary conditions, (6) and (7), in the mature (the last found) form of $\theta_n(x,\alpha)$. Here, these values $(\Theta_\alpha(0),\Theta_\alpha(1))$ are found in terms of α only. After this, the value of α is found, it becomes $\alpha=0.4256$ in this case, for which the response has minimum error with the exact value at the mid point of the domain, the approximate response becomes

$$\begin{aligned}\theta_n(x,\alpha)&=\theta_n(x,0.4256)=\theta_5(x)\\&=0.3690-0.5087(-0.5744+x)+0.7381(-0.5744+x)^2\\&\quad-0.3391(-0.5744+x)^3+0.24604(-0.5744+x)^4-0.0678(-0.5744+x)^5,\end{aligned}$$

which is an approximation of the exact solution $\theta(x)$ given in (8).

The responses at different domain values are compared with those of the classical DTM and the solutions are shown in Table 1.

Table 1. Error Analysis α -PDTM vs the Classical DTM

x	θ				
	Exact	**DTM**	**Absolute Error**	α **-PDTM**	**Absolute Error**
0	1	1	0	1	8.32667×10^{-17}
0.1	0.8259733	0.8370339	1.10606×10^{-2}	0.8267338	7.60534×10^{-4}
0.2	0.6850958	0.7076558	2.256×10^{-2}	0.6860549	9.59084×10^{-4}
0.3	0.5717136	0.6066291	3.49155×10^{-2}	0.5726795	9.65862×10^{-4}
0.4	0.4812763	0.5297358	4.84595×10^{-2}	0.4822191	9.42769×10^{-4}
0.5	0.4101543	0.4734848	6.33306×10^{-2}	0.4110989	9.44627×10^{-4}
0.6	0.3554932	0.4348218	7.93286×10^{-2}	0.3564766	9.83452×10^{-4}
0.7	0.3150993	0.4108376	9.57383×10^{-2}	0.3161609	1.06161×10^{-3}
0.8	0.2873514	0.3984776	1.11126×10^{-1}	0.2885299	1.17849×10^{-3}
0.9	0.2711360	0.3942509	1.23115×10^{-1},	0.2724501	1.31412×10^{-3}
1	0.2658022	0.3939394	1.28137×10^{-1}	0.2671948	1.39254×10^{-3}

One may obseve that the $\alpha-$parametrized method is more reliable due to minimum errors as shown in the table. The approximate solution by the classical DTM and α-PDTM and the exact solution, for this problem, are drawn and compared in the following figures:

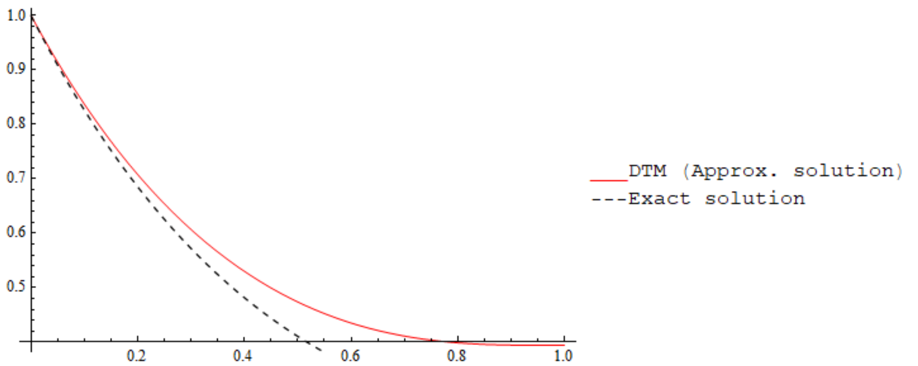

Figure 2. Error Analysis DTM vs the Exact Solution for the problem-1.

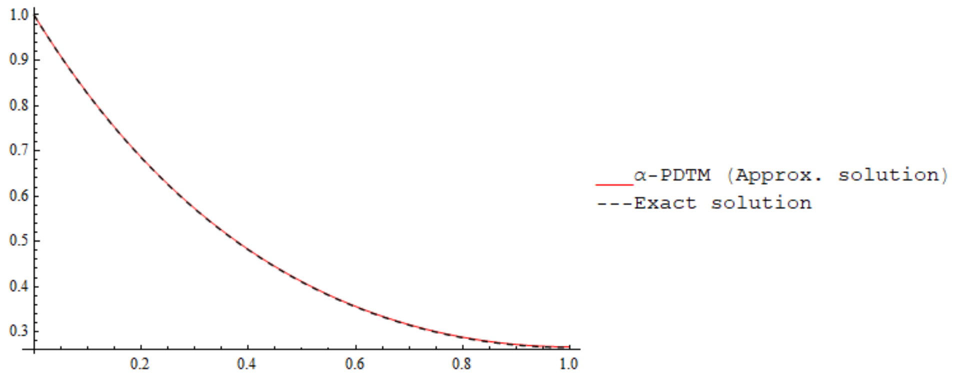

Figure 3. Error Analysis α-PDTM vs Exact Solution for the problem-1.

Problem 2. Consider the following nonhomogeneous second order boundary value problem (Cauchy Euler equation)

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=\frac{1}{x};\ y(\frac{1}{e})=0,\ y(1)=1. \tag{11}$$

For $x=e^t$; the above boundary value problem can be written in the form

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=e^{-t}; \tag{12}$$

$$y(-1)=0, \tag{13}$$

$$y(0)=1. \tag{14}$$

The BVP has the exact solution

$$y(t)=\frac{8}{9}e^{2t}+\frac{8+e^3}{9}te^{2t}+\frac{1}{9}e^{-t}. \tag{15}$$

Here $(a,b)=(-1,0)$. Thus $t_\alpha=-\alpha$. We use $n=8$ to approximate $y(t)$ using the same algorithm of that of the previous example, α becomes $\alpha=0.9944$ in this case, and find the approximate response

$$\begin{aligned} y_n(t,\alpha) &= y_n(t,0.9944)=y_8(t) \\ &= 0.9999999+4.7733398\,t+7.9855836\,t^2+7.1581610\,t^3 \\ &\quad +4.3383184\,t^4+1.8460344\,t^5+0.54616792\,t^6+0.1014857\,t^7+0.0089510\,t^8, \end{aligned}$$

which is an approximation of the exact solution $y(t)$ given in (15).

The responses at different values of the domain are compared with those of the classical DTM and with the exact solution in the following (Table 2):

Table 2. Error Analysis α -PDTM vs the Classical DTM

t	y				
	Exact	DTM	Absolute Error	α -PDTM	Absolute Error
-1	0	0	0	3.2454666×10^{-18}	3.24547×10^{-18}
-0.9	-0.0440292	-0.0566253	1.2596×10^{-2}	-0.0441078	7.85852×10^{-5}
-0.8	-0.0772872	-0.0961335	1.88463×10^{-2}	-0.0774168	1.29672×10^{-4}
-0.7	-0.0957264	-0.1177678	2.20414×10^{-2}	-0.0958696	1.43199×10^{-4}
-0.6	-0.0937609	-0.1174418	2.36809×10^{-2}	-0.0938667	1.05836×10^{-4}
-0.5	-0.0638099	-0.0880728	2.42629×10^{-2}	-0.0638118	1.94421×10^{-6}
-0.4	0.0042887	-0.0194514	2.37401×10^{-2}	0.0044694	1.80691×10^{-4}
-0.3	0.1240279	0.1022763	2.17516×10^{-2}	0.1244613	4.33396×10^{-4}
-0.2	0.3131893	0.2954773	1.7712×10^{-2}	0.3138743	6.85075×10^{-4}
-0.1	0.5950631	0.5842463	1.08167×10^{-2}	0.5957796	7.16526×10^{-4}
0	1	1	0	0.9999999	2.22045×10^{-16}

One may obseve that the $\alpha-$parametrized method is more reliable due to minimum errors as shown in the table. The approximate solution by the classical DTM and α -PDTM and the exact solution, for this problem, are drawn and compared in the following figures:

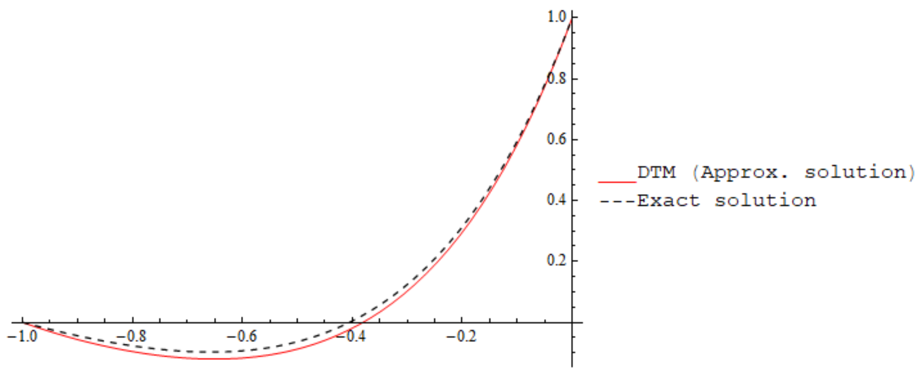

Figure 4. Error Analysis DTM vs the Exact Solution for the problem-2.

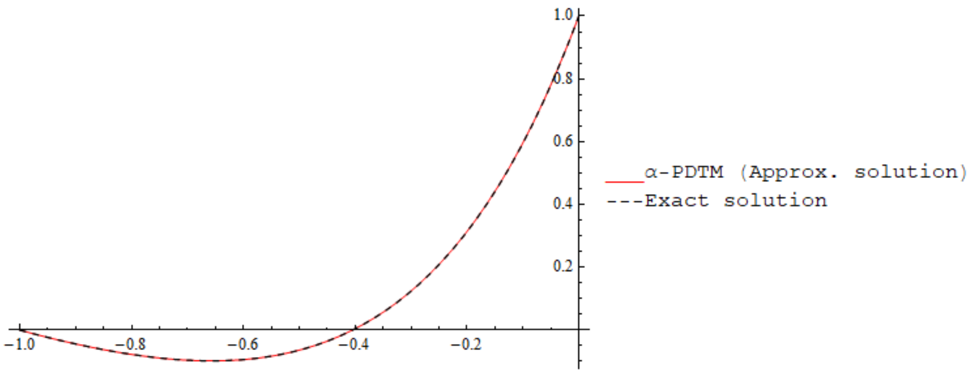

Figure 5. Error Analysis α -PDTM vs Exact Solution for the problem-2.

4. CONCLUSIONS

The alpha parametrized method ($\alpha -$PDTM) is applied on some boundary values problems. The exact solutions and the solution found by the classical DTM are compared with those of the alpha parametrized method. It is found that the $\alpha -$PDTM is more precise and accurate method as the errors found by this method are minimum. The parameter α is infact dependent on the number n of terms of the corresponding Taylor series and on the point of the domain. The optimized values of α for which the error at the mid point of the domain is minimum, are also found. However, the pointwise optimial values of α can also be found and discussed.

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