ORIGINAL PAPER

GEOMETRIC INTERPRETATION OF MULTIPLICATIVE CALCULUS IN LORENTZIAN SPACE

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Abstract. *In this paper, by using matrix methods we obtain pole points, pole orbits, acceleration center and combinations of these accelerations of multiplicative one-parameter motion on the Lorentzian plane motions (i.e., kinematic). Moreover, some new theorems are given.*
Keywords: *Lorentz space; Multiplicative one-parameter motion; Plane kinematics; pole points; Acceleration center.*

1. INTRODUCTION

Grossman and Katz presented multiplicative calculus which is also named Non-Newtonian calculus. Two operations of multiplicative calculus are multiplicative derivative and multiplicative integral. We refer to [1-7] for different types of Non-Newtonian calculus and its applications. Bashirov et al. [8] gave a complete mathematical description of multiplicative calculus. An extension of multiplicative calculus to functions of complex variables can be found in [9-13]. Çakmak et al. [14] characterized matrix transformations in sequence spaces based on multiplicative calculus. Boruah et al. [15] gave the geometric real number line in the geometric coordinate system. Hazarika et al. [16] gave the trigonometric ratios and the relationship between geometric trigonometry and trigonometric functions as we know them. Georgiev et al. [17], gave some concepts like multiplicative vector spaces, multiplicative vector spaces with inner product, multiplicative matrices, and some basic properties of these concepts using multiplicative arithmetic. Es [18] gave plane kinematics in multiplicative calculus some basic. Aslan et al. [19], gave Geometric 3-space and multiplicative quaternions. Nurkan et al. [20], gave Vector properties of geometric calculus.

A branch on non-Newton calculus is Bigeometric-calculus. This allows operations such as differentiation and integration to be performed. Usually, it is often used in growth related such as price elasticity, numerical approximations problems. We refer to know basics of $\alpha-$ generator and geometric arithmetic $(R(G),\oplus,\ominus,\otimes,\oslash)$ [21]. In multiplicative calculus, the sets of integers, real numbers and complex numbers Z(G),R(G) and C(G) were defined by Türkmen and Başar [12], respectively.

$$\mathbb{Z}(G)=\{e^{x}:\ x\in\mathbb{R}\},$$

$$\mathbb{R}(G)=\{e^{x}:\ x\in\mathbb{R}\}=\mathbb{R}^{+}\setminus\{0\},$$

$$\mathbb{C}(G)=\{e^{z}:\ z\in\mathbb{C}\}=\mathbb{C}\setminus\{0\}.$$

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If we take extended real number line, then R (G) = [0, ∞]. (R(G),⊕,⊗) is a field with geometric zero 1 and geometric identity e. Geometric negative real numbers and positive real numbers are defined as follows, respectively.

$$\mathbb{R}^{-}(G) = \{\, x \in \mathbb{R}(G) \ : x < 1\}$$

and

$$\mathbb{R}^{+}(G) = \{\, x \in \mathbb{R}(G) \ : x > 1\} \text{ [16].}$$

We should know that all concepts in classical arithmetic have natural counterparts in ϕ – arithmetic. Consider any generetor ϕ with range $A \subseteq \mathbb{R}$. By ϕ – arithmetic, we mean the arithmetic whose realm is A and whose operations are defined as follows. Its operations are as follows. Let $x, y \in \mathbb{R}$ be and for any generator ϕ the following operations are defined.

$$\phi - addition: \quad x \oplus y = \phi\{\phi^{-1}(x) + \phi^{-1}(y)\}$$

$$\phi - subtraction: \quad x \ominus y = \phi\{\phi^{-1}(x) - \phi^{-1}(y)\}$$

$$\phi - multiplication: \quad x \otimes y = \phi\{\phi^{-1}(x) \times \phi^{-1}(y)\}$$

$$\phi - division: \quad x \oslash y = \phi\{\phi^{-1}(x) \div \phi^{-1}(y)\}$$

Especially if we choose the ϕ - generator as the identity function then $\phi(x) = x$, for all $x \in \mathbb{R}$ which implies that $\phi^{-1}(x) = x$ that is to say that ϕ - arithmetic is reduced to the classical arithmetic.

$$\phi - addition: \quad x \oplus y = \phi\{\phi^{-1}(x) + \phi^{-1}(y)\} = \phi\{x + y\} = x + y \\ : \ classical\ addition$$

$$\phi - subtraction: \ x \ominus y = \phi\{\phi^{-1}(x) - \phi^{-1}(y)\} \ = \phi\{x - y\} = x - y \\ : \ classical\ subtraction$$

$$\phi - multiplication: \quad x \otimes y = \phi\{\phi^{-1}(x) \times \phi^{-1}(y)\} \ = \phi\{x \times y\} = x \times y \\ : classical\ multiplication$$

$$\phi - division: \quad x \oslash y = \ \phi\{\phi^{-1}(x) \div \phi^{-1}(y)\} = \phi\{x \div y\} = x \div y \\ : \ classical\ division$$

If we choose exp as an ϕ - generator defined $\phi(x) = e^{x}$ for all $x \in \mathbb{R}$ then $\phi^{-1}(x) = lnx$, and ϕ - arithmetic turns out to the Geometric arithmetic.

$$\phi - addition: \quad x \oplus y = \phi\{\phi^{-1}(x) + \phi^{-1}(y)\} = e^{lnx+lny} = x \cdot y \\ : \ geometric\ addition$$

$$\phi - subtraction: \ x \ominus y = \phi\{\phi^{-1}(x) - \phi^{-1}(y)\} = x \div y\,, y \neq 0 \\ : geometric\ subtraction$$

$$\phi - multiplication: \ x \otimes y = \phi\{\phi^{-1}(x) \times \phi^{-1}(y)\} = e^{lnx \times lny} = x^{lny} = y^{lnx} \\ : geometric\ multiplication$$

$$\phi - division:\ x \oslash y = \phi\{\phi^{-1}(x) \div \phi^{-1}(y)\} = e^{lnx \div lny} = x^{\frac{1}{lny}}, y \neq 1$$
$$:\ geometric\ division$$

In [12] defined the geometric real numbers $\mathbb{R}(G)$ and geometric integers numbers as follows:

$$\mathbb{R}(G) = \{e^x :\ x \in \mathbb{R}\} = \mathbb{R}^+ \setminus \{0\},$$

$$\mathbb{Z}(G) = \{e^x :\ x \in \mathbb{R}\} = \mathbb{Z} \setminus \{0\}.$$

Then $(\mathbb{R}(G),\ \oplus,\ \otimes)$ is a field with geometric zero 1 and geometric identity e. Then for all $x, y \in \mathbb{R}(G)$:

1. $x \oplus y = x \cdot y$
2. $x \ominus y = x \div y$
3. $x \otimes y = x^{lny} = y^{lnx}$
4. $x \oslash y = x^{\frac{1}{lny}},$
5. $x^{2_G} = x \otimes x = x^{lnx}$
6. $x^{3_G} = x \otimes x \otimes x = (x^{lnx})^{lnx} = x^{ln^2 x}$
7. $x^{p_G} = x \otimes x \otimes x \ldots \otimes x = x^{ln^{p-1} x}$
8. $\sqrt{x}^G = e^{(lnx)^{\frac{1}{2}}}$
9. $x^{-1_G} = e^{\frac{1}{logx}}$
10. $\sqrt{x^{2_G}}^G = |x|_G$ [12,16,17].

Definition 1. The classical derivate of function f at x is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\ .$$

Replace the difference $f(x+h) - f(x)$ by the ratio $f(x+h)/f(x)$ and the division by h by the raising to the reciprocal power $1/h$ in this formula. Then we get the multiplicative derivative f at x is the following limit

$$f^*\ (x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

In [8] it is shown that if f is a positive function on an open subset A of the real line $\mathbb{R}$ and its derivative $f'(x)$ exists, then its multiplicative derivative $f^*\ (x)$ also exists and they are related as

$$f^*(x) = e^{(lnf(x))'}.$$

Moreover, if nth derivative $f^{(n)}(x)$ exists, then nth multiplicative derivative $f^{*(n)}(x)$ also exists and it can be given as

$$f^{*(n)}(x) = e^{(lnf(x))^{(n)}},\ n = 0,1, \ldots [8\text{-}10,\ 17].$$

Definition 2. A Riemann integral in the multiplicative form given in [8] for positive bounded functions and shown its relation to ordinary Riemann integral infinitesimal version of exponential sum:

$$\int_a^b f(x)^{dx} = e^{\int_a^b \ln(f(x))dx} \quad [8\text{-}10, 17, 22].$$

Definition 3. Relation between geometric and ordinary trigonometry is;

$$sing\,\theta = e^{sin\,\theta}, cosg\,\theta = e^{cos\,\theta}, tang\,\theta = e^{\tan\theta} = \frac{\mathrm{sing}\,\theta}{\mathrm{cosg}\,\theta}\,G \quad [16].$$

Definition 4. A square matrix in which all the main diagonal elements are e's and all the remaining elements are 1's is called an multiplicative Identity Matrix. Multiplicative Identity Matrix is denoted as

$$E_{n\times n}^G = E_n^G = E = \begin{bmatrix} e & 1 & \ldots & 1 \\ 1 & e & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & e \end{bmatrix}_{nxn},$$

where $n\times n$ represents the order of the matrix [17-19].

Defination 5. Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{mxn}$. Multiplicative sum of the matrices is calculated as

$$C = A \oplus B = [c_{ij}]_{mxn} = [a_{ij} \oplus b_{ij}] = [a_{ij}b_{ij}] \quad [17, 18].$$

Definition 6. If $A = [a_{ij}]_{mxn}$ is any matrix and $e^c \in F^G$ then the scalar multiplication $B = e^c \otimes A$ is defined by $a_{ij} = e^c \otimes a_{ij}$ all i, j. Here

$$B = e^c \otimes A = [a_{ij}{}^c]_{mxn} \quad [17, 18].$$

Definition 7. Let $A = [a_{ij}]_{mxn}$ and $B = [b_{ij}]_{nxp}$. The multiplication of the multiplicative matrices is calculated as

$$A \otimes B = C = [c_{ij}]_{mxp} \to c_{ij} = \sum_{k=1}^n (a_{ik} \otimes b_{kj}) = \prod_{k=1}^n (a_{ik})^{\ln(b_{kj})} \quad [17, 18].$$

Definition 10. Let $A = [a_{ij}]_{mxn}$. Define the multiplicative transpose of A , denoted by A^T, to be the nxm matrix with entries $(A^T)_{ij} = a_{ji}$ [17, 18].

Definition 11. Let $A = [a_{ij}]_{nxn}$. Define the multiplicative determinant of A to be the value in F^G

$$|A|^G = det^G(A) = \sum^G{}_{i=1}^{n} \left(\left(a_{pk} \otimes A_{pk}^G \right) \right)$$
$$= \prod_{k=1}^n \left(A_{pk}^G \right)^{\ln(a_{pk})} \quad [17, 18].$$

Definition 12. Let $M = (x_0, y_o)$, as a center, be a point of the geometric 2-space, then a circle with radius r can be written as

$$(\ln x - \ln x_0)^2 + (\ln y - \ln y_0)^2 = (\ln r)^2 \text{, see [20].}$$

Similarly, the multiplicative Lorentz circle can be given as

$$(\ln x - \ln x_0)^2 - (\ln y - \ln y_0)^2 = (\ln r)^2.$$

And unit circle in this motion is

$$(\ln x)^2 - (\ln y)^2 = 1.$$

In this paper, using matrix methods, we obtained multiplicative rotation pole in Lorentzian multipiacative one-parameter motion on multiplicative Lorentzian plane kinematics in multiplicative Lorentzian motions and the relationship between the velocities and accelerations of multiplicative the motion and multiplicative pole orbits, multiplicative Lorentzian accelerations and multiplicative Lorentzian combinations of accelerations. Moreover some new theorems regarding to multiplicative plane are given.

2. GEOMETRIC INTERPRETATION OF MULTIPLICATIVE CALCULUS IN LORENTZIAN SPACE

Definition 2.1. Multiplicative Lorentzian space is a real n-dimensional vector space which is equipped with multiplicative Lorentzian inner product

$$\begin{aligned}\langle x,y\rangle_L^G &= (x_1 \otimes y_1) \oplus (x_2 \otimes y_2) \oplus \dots \oplus (x_{n-1} \otimes y_{n-1}) \ominus (x_n \otimes y_n)\\ &= x_1^{lny_1} \oplus x_2^{lny_2} \oplus \dots \oplus x_{n-1}^{lny_{n-1}} \ominus x_n^{lny_n}\\ &= \frac{x_1^{lny_1}.\ x_2^{lny_2} \ldots.. \ x_{n-1}^{lny_{n-1}}}{x_n^{lny_n}}\\ &= \frac{\prod_{i=1}^{n-1} x_i^{lny_i}}{x_n^{lny_n}},\end{aligned} \tag{1}$$

here

$$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n(G).$$

Definition 2.2. Multiplicative Lorentzian plane is a real two-dimensional vector space which is equipped with multiplicative Lorentzian inner product

$$\begin{aligned}\langle x,y\rangle_L^G &= (x_1 \otimes y_1) \ominus (x_2 \otimes y_2)\\ &= \frac{x_1^{lny_1}}{x_2^{lny_2}}\end{aligned} \tag{2}$$

for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2(G)$.

We denote by T, the proper Multiplicative Lorentzian group $SO^+(1,1)_G$ consisting of all matrices of the form

$$A(\theta) = \begin{bmatrix} \cosh g\theta & \sinh g\theta \\ \sinh g\theta & \cosh g\theta \end{bmatrix}, \theta \in \mathbb{R}(G) \tag{3}$$

here, sinhg $\theta = e^{\sinh\theta}$, coshg $\theta = e^{\cosh\theta}$.

In Lorentzian plane, a multiplicative rigid motion is a combination of a rotation and C multiplicative translation that can be given by a matrix operation as

$$\begin{bmatrix} x' \\ y' \\ e \end{bmatrix} = \begin{bmatrix} \cosh g\theta & \sinh g\theta & a \\ \sinh g\theta & \cosh g\theta & b \\ 1 & 1 & e \end{bmatrix} \otimes \begin{bmatrix} x \\ y \\ e \end{bmatrix}. \tag{4}$$

where $A(\theta)$ is a rotation matrix in mültiplicative Lorentzian plane.

The multiplicative Lorentzian plane is represented by L^2 or for the sake of shortness only by L. Also $B_1 = L/L'$, will be used as the motion of L according to L' where L and L' are moving and fixed multiplicative Lorentzian planes, respectively.

Theorem 2.1. The set of multiplicative Lorentzian rigid motions in In multiplicative Lorentzian plane is a group according to the composition operation.

Definition 2.3. The norm of a vector x in multiplicative Lorentzian plane is

$$\|x\|_L^G = e^{\sqrt{|ln^2(x_1) - ln^2(x_2)|}} \tag{5}$$

here $x = (x_1, x_2),\ x \in \mathbb{R}^2(G)$.

Then

$$ln^2(\|x\|_L^G) = |ln(\langle x, y\rangle_L^G)|. \tag{6}$$

Definition 2.4. In multiplicative Lorentzian plane, a multiplicative Lorentzian planar motion can be given as

$$\begin{aligned} y_1 &= x^{\ln\cosh g\theta} . y^{\ln\sinh g\theta} . a \\ y_2 &= x^{\ln\sinh g\theta} . y^{ln\cosh g\theta} . b \end{aligned} \tag{7}$$

If $\theta, a,$ and b are given by the functions of time parameter t, then this motion is called as multiplicative Lorentzian one parameter motion. Multiplicative Lorentzian one parameter planar motion given by (8) can be written in the form

$$\begin{bmatrix} Y \\ e \end{bmatrix} = \begin{bmatrix} A & C \\ 1 & e \end{bmatrix} \otimes \begin{bmatrix} X \\ e \end{bmatrix} \tag{8}$$

or

$$Y = A \otimes X \oplus C, A = A(t)$$

and

$$C = C(t) \in \mathbb{R}_1^2(G), t \in \mathbb{R}(G) \tag{9}$$

$$\begin{aligned} Y &= (y_1 \quad y_2)^T, \\ X &= (x \quad y)^T, \\ C &= (a \quad b)^T. \end{aligned} \tag{10}$$

where Y and X are the position vectors of the same point B, respectively, for the multiplicative fixed and multiplicative moving systems, and C is the multiplicative translation vector. By taking the derivates with respect to t in (10), we get

$$Y^* = A^* \otimes X \oplus A \otimes X^* \oplus C^* \tag{11}$$

here

$$\begin{aligned} V_a &= Y^*, \\ V_f &= A^* \otimes X \oplus C^* \end{aligned} \tag{12}$$

and

$$V_r = A \otimes X^*$$

are called multiplicative Lorentzian absolute, multiplicative Lorentzian sliding, and multiplicative Lorentzian relative velocites of the motion, respectively. These motions in multiplicative Lorentzian plane are indicated by $B_1 = L/L'$ where L' and L are fixed and moving multiplicative Lorentzian planes, respectively. By taking the derivatives with respect to t in (12), we get

$$Y^{**} = A^{**} \otimes X \oplus e^2 \otimes (A^* \otimes X^*) \oplus A \otimes X^{**} \oplus C^{**}, \tag{13}$$

$$b_a = b_r . b_c . b_f \tag{14}$$

where the acceleration

$$b_a = Y^{**},$$

$$b_f = A^{**} \otimes X \otimes C^{**},$$

$$b_r = A \otimes X^{**}, \tag{15}$$

and

$$b_c = e^2 \otimes A^* \otimes X^*,$$

are called multiplicative Lorentzian absolute acceleration, multiplicative Lorentzian sliding acceleration, multiplicative Lorentzian relative acceleration and multiplicative Lorentzian Coriolis accelerations, respectively.

Definition 2.5. The velocity vector of the point X with respect to the plane L, i.e. the vectorial velocity of X while it is drawing its orbit in L, is called multiplicative Lorentzian relative velocity of the point X and denoted by V_r.

Definition 2.6. The velocity vector of the point X with respect to multiplicative fixed plane L' is called multiplicative Lorentzian absolute velocity of X and is denoted by V_a. Thus (12) we obtain the relation

$$V_a = V_f . V_r \tag{16}$$

If X is a fixed point in multiplicative moving plane L, since $V_r = 1$, then we have $V_a = V_f$. The equality (17) is said to be the velocity law of the motion $B_1 = L/L'$.

Theorem 2.2. Multiplicative Lorentzian absolute velocity of a point is equal to the sum of multiplicative Lorentzian sliding velocity vector and multiplicative Lorentzian relative velocity vector. So it is

$$V_a = V_f . V_r \tag{17}$$

3. POLES OF ROTATING AND ORBIT

The point in which multiplicative Lorentzian sliding velocity V_f at each moment t of a fixed point X in L in the one-parameter motion $B_1 = L/L'$ are fixed points in moving and fixed plane. These points are called multiplicative Lorentzian pole points of the motion.

Theorem 3.1. In a motion $B_1 = L/L'$ whose multiplicative Lorentzian angular velocity is not 1, there exists a unique point which is fixed in both planes at every moment *t*.

Proof: Since the point $X \in L$ is fixed both in the L plane and in the L' plane, multiplicative Lorentzian relative velocity and multiplicative Lorentzian sliding velocity of the same point will be 1, respectively. So if multiplicative Lorentzian siliding velocity for such points is one

$$A^* \otimes X \oplus C^* = 1 \tag{18}$$

and

$$X = e^{-1} \otimes (A^*)^{-1} \otimes C^*. \tag{19}$$

where $(A^*)^{-1}$ is the multiplicative inverse of A^*.

Since

$$A = \begin{bmatrix} \cosh g\theta & \sinh g\theta \\ \sinh g\theta & \cosh g\theta \end{bmatrix},$$

$$C = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$A^* = \begin{bmatrix} (\sinh g\theta)^{\theta.\ln(\theta^*)} & (\cosh g\theta)^{\theta.\ln(\theta^*)} \\ (\cosh g\theta)^{\theta.\ln(\theta^*)} & (\sinh g\theta)^{\theta.\ln(\theta^*)} \end{bmatrix},$$

$$C^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix}$$

we get $det^G A^* = \frac{1}{(e)^{(\theta.\ln\theta^*)^2}}$. Thus A^* is regular and

$$(A^*)^{-1} = \begin{bmatrix} (\sinh g\theta)^{\frac{-1}{\theta.\ln(\theta^*)}} & (\cosh g\theta)^{\frac{1}{\theta.\ln(\theta^*)}} \\ (\cosh g\theta)^{\frac{1}{\theta.\ln(\theta^*)}} & (\sinh g\theta)^{\frac{-1}{\theta.\ln(\theta^*)}} \end{bmatrix}$$

Hence there exists a unique solution X of the equation $V_f = 1$. Point X is called multiplicative Lorentzian pole point in moving plane. For this reason (20) leads to

$$X = \begin{bmatrix} (\sinh g\theta)^{\frac{lna^*}{\theta.\ln(\theta^*)}}.(\cosh g\theta)^{\frac{-lnb^*}{\theta.\ln(\theta^*)}} \\ (\cosh g\theta)^{\frac{-lna^*}{\theta.\ln(\theta^*)}}.(\sinh g\theta)^{\frac{lnb^*}{\theta.\ln(\theta^*)}} \end{bmatrix} \tag{20}$$

the pole point in the multiplicative Lorentzian fixed plane is

$$P' = A \otimes P \oplus C \tag{21}$$

setting these values in their planes and calculating we have

$$P' = \begin{bmatrix} (b^*)^{\frac{-1}{\theta.\ln(\theta^*)}}.a \\ (a^*)^{\frac{1}{\theta.\ln(\theta^*)}}.b \end{bmatrix} \tag{22}$$

or as a vector

$$P' = \left((b^*)^{\frac{-1}{\theta.\ln(\theta^*)}}.a, (a^*)^{\frac{1}{\theta.\ln(\theta^*)}}.b\right). \quad (23)$$

Here we assume that multiplicative Lorentzian $\theta^*(t) \neq 1$ for all t. That is, multiplicative Lorentzian angular velocity is not 1. In this case there exists a unique pole point in each of the moving and fixed planes of each moment t.

Definition 3.1. The point $P = (p_1, p_1)$ is called multiplicative Lorentzian instantaneous rotation center or the pole at moment t of the one parameter motion $B_1 = L/L'$.

Theorem 3.2. The pole ray from the pole P to the point X is multiplicative Lorentzian perpendicular to multiplicative Lorentzian velocity vector V_f at each instant moment.

Proof: The pole point in multiplicative moving plane $Y = A \otimes X \oplus C$ implies that

$$X = (A)^{-1} \otimes \left(Y \oplus ((e)^{-1} \otimes C)\right),$$

$$V_f = A^* \otimes X \oplus C^* \quad (24)$$

and

$$A^* \otimes X \oplus C^* = 1$$

that leads to

$$X = P = e^{-1} \otimes (A^*)^{-1} \otimes C^*. \quad (25)$$

Now let us find pole points in multiplicative fixed plane. Then we have from equation

$$Y = A \otimes X \oplus C$$
$$Y = P' = A \otimes (e^{-1} \otimes (A^*)^{-1} \otimes C^*) \oplus C \;) \quad (26)$$

hence, we get

$$C^* = A^* \otimes (A)^{-1} \otimes \left(C \oplus (e^{-1} \otimes P')\right)$$

If we substitute this values in the equation $V_f = A^* \otimes X \oplus C^*$ we have $V_f = A^* \otimes (A)^{-1} \otimes P'Y$. Now let us calculate the value of $A^* \otimes (A)^{-1} \otimes P'Y$, where $P'Y = \left(\frac{y_1}{p_1}, \frac{y_2}{p_2}\right)$, then

$$V_f = e^{(\theta.\ln(\theta^*))} \otimes \begin{bmatrix} \frac{p_2}{y_2} \\ \frac{y_1}{p_1} \end{bmatrix} \quad (27)$$

or as a vector

$$V_f = \left(\left(\frac{y_2}{p_2}\right)^{(\theta.\ln(\theta^*))}, \left(\frac{y_1}{p_1}\right)^{(\theta.\ln(\theta^*))}\right) \quad (28)$$

thus we obtain

$$\langle V_f, P'Y \rangle_L^G = 1.$$

Corollary 3.1. In a $B_1 = L/L'$ multiplicative Lorentzian motion, the focus of X point of L is an orbit that it’s normals pass through the rotation pole P.

Theorem 3.3. let X be a moving point in L and P be a rotation pole of L/L' motion, then

$$\left\|V_f\right\|_L^G = (\|PX\|_L^G)^{|\theta.\ln(\theta^*)|} \tag{30}$$

Definition 3.2. In motion $B_1 = L/L'$ the geometric place of the pole points P in the moving plane L is called multiplicative moving pole curve of the motion $B_1 = L/L'$and is denoted by (P). The geometric place of the pole points P in multiplicative fixed plane L' is called multiplicative fixed and is denoted by $(P)'$.

Theorem 3.4. The velocity on the curve (P) and $(P)'$ of every moment t of the rotating pole P which draws the pole curves in multiplicative fixed and moving planes are equal to each other. In other words, two curves are always tangent to each other.

Proof: The velocity of the point $X \in L$ while drawing the curve (P) is V_r and the velocity of this point while drawing the curve $(P)'$ is V_a. Since $V_f = 1$ then $V_a = V_r$. And this completes the proof of the theorem.

Corollary 3.2. During the motion $B_1 = L/L'$, (P) and $(P)'$ roll, upon each other without sliding.

Definition 3.3. Let β and β' be two curves. These two curves are tangent to each other at every moment t, and if the lengths ds and ds' of the paths taken by the point drawing these two curves on these curves in dt time are equal, these curves are said to roll over each other without sliding.

Theorem 3.5. In the one parameter planer motion $B_1 = L/L'$ the moving pole curve (P) of the plane L revolves by sliding on the fixed pole curve $(P)'$ of the plane L'.
Proof: From theorem 3.4 , (P) and $(P)'$ curves are tangent to each other at every time t. The arc length of (P) between the points corresponding to t_0, t_1becomes

$$s = \int_{t_0}^{t_1} \|V_r\|_L^{G\,dt} = e^{\int_{t_0}^{t_1} \ln(\|V_r\|_L^G)dt}$$

$$d^*s = (\|V_r\|_L^G)^{lndt}$$

The arc length of $(P)'$ between the points corresponding to t_0, t_1 is

$$s' = \int_{t_0}^{t_1} \|V_a\|_L^{G\,dt} = e^{\int_{t_0}^{t_1} \ln(\|V_a\|_L^G)dt}$$

$$d^*s' = (\|V_a\|_L^G)^{lndt},$$

It was shown from theorem 3.4 that $V_a = V_r$. Hence $d^*s = d^*s'$.

Definition 3.4. The vector V_a is called multiplicative Lorentzian absolute acceleration vector with respect to the L' plane of the point X and is denoted by b_a. Since $V_a = Y^*$ then $b_a = V_a{}^* = Y^{**}$.

Definition 3.5. Let $X \in L$ be a fixed point in motion $B_1 = L/L'$. Multiplicative Lorentzian acceleration vector of X with respect to L' is called multiplicative Lorentzian sliding acceleration vector. This multiplicative Lorentzian sliding acceleration vector is denoted by b_f.

Since acceleration of the multiplicative Lorentzian sliding acceleration X is a fixed point of L, then $b_f = {V_f}^* = A^{**} \otimes C^{**}$.

4. ACCELERATIONS AND UNION OF ACCELERATIONS

Assume that $B_1 = L/L'$ of the moving plane L with respect to the fixed plane L' exists. In this motion, let us consider a point X moving with respect to the plane L, and thus moving respect to the plane L'. We have obtained multiplicative Lorentzian velocity formulas concerning the motion of X, now we will obtain multiplicative Lorentzian acceleration formulas of the point X.

Definition 4.1. We know that point X is multiplicative Lorentzian relative velocity vector V_r to L. The vector b_r obtained by taking the derivative of V_r is called multiplicative Lorentzian relative acceleration vector of X in L. This multiplicative Lorentzian relative acceleration vector is represented by b_r. Considering point X as a moving point in L, matrix A is taken as constant.

Theorem 4.1. Let X be a point moving in the L plane according to a parameter t. The relation between multiplicative Lorentzian acceleration formulas of this point is as follows.

$$b_a = b_r . b_c . b_f$$

here $b_c = A^* \otimes X^*$ is called multiplicative Lorentzian Corilois acceleration.

Corollary 4.1. If point X is a fixed point of multiplicative moving plane, mutiplicative Lorentzian sliding acceleration of point X is equal to multiplicative Lorentzian absolute acceleration of that point.

Proof: Note that

$$V_a = A^* \otimes X \oplus A \otimes X^* \oplus C^*,$$

differentiating the both sides we have

$${V_a}^* = A^{**} \otimes X \oplus e^2 \otimes A^* \otimes X^* \oplus A \otimes X^{**} \oplus C^{**}$$

since the point X is constant its derivative is 1. Hence

$$\begin{aligned} b_a &= {V_a}^* \\ &= A^{**} \otimes X \oplus C^{**} \\ &= b_f . \end{aligned}$$

Theorem 4.2. The multiplicative Lorentzian plane b_c multiplicative Lorentzian coriolis acceleleration vector and V_r multiplicative Lorentzian relative velocity vector are multiplicative Lorentzian perpendicular to each other.

Proof:

$$V_r = A \otimes X^*$$

$$b_c = e^2 \otimes A^* \otimes X^*$$

So it is obvious that

$$\langle b_c, V_r \rangle_L^G = 1$$

Corollary 4.2. Let X be a moving point in L and $b_c = 1$ then $B_1 = L/L'$. motion is only a slide and vice versa.

Proof: Because of $b_c = 1$ then

$$b_c = e^2 \otimes A^* \otimes X^*$$
$$= \begin{bmatrix} \left((\text{sinhg}\,\theta)^{lnx_1{}^*}(\text{coshg}\,\theta)^{lnx_2{}^*}\right)^{2(\theta.\ln\theta^*)} \\ \left((\text{coshg}\,\theta)^{lnx_1{}^*}(\text{sinhg}\,\theta)^{lnx_2{}^*}\right)^{2(\theta.\ln\theta^*)} \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

therfore $e^{2\theta.\ln(\theta^*)} = 1$ and thus $\theta^* = 1$.

So θ is constant. That is $B_1 = l/L'$ must be a slide. The other side of the theorem is obvious.

5. THE ACCELERATION POLES OF THE MOTIONS

The solution of the equation $V_f{}^* = A^{**} \otimes X \oplus C^{**} = 1$ gives us multiplicative Lorentzian acceleration pole of multiplicative motion.

$$V_f{}^* = A^{**} \otimes X \oplus C^{**}$$

implies

$$X = e^{-1} \otimes (A^{**})^{-1} \otimes C^{**}.$$

Now calculating the matrices $e^{-1} \otimes (A^{**})^{-1}$ and C^{**}, and setting these in $X = P_1 = e^{-1} \otimes (A^{**})^{-1} \otimes C^{**}$, we obtain

$$X = P_1 = \begin{bmatrix} \left((\text{sinhg}\,\theta)^K(\text{coshg}\,\theta)^{L^2}\right)^{\frac{\ln(a^{**})}{W}} \left((\text{coshg}\,\theta)^{-K}(\text{sinhg}\,\theta)^{L^2}\right)^{\frac{\ln(b^{**})}{W}} \\ \left((\text{coshg}\,\theta)^K(\text{sinhg}\,\theta)^{-L^2}\right)^{\frac{\ln(a^{**})}{W}} \left((\text{sinhg}\,\theta)^K(\text{coshg}\,\theta)^{L^2}\right)^{\frac{\ln(b^{**})}{W}} \end{bmatrix},$$

where $(A^{**})^{-1}$ is the multiplicative inverse of A^{**}. Here P_1 is called multiplicative Lorentzian pole curve in multiplicative Lorentzian moving plane. If multiplicative Lorentzian pole curve in multiplicative lorentzian fixed plane is denoted by ${P_1}'$we get

$$ {P_1}' = A \otimes P_1 \oplus C \tag{31} $$

Hence

$$ {P_1}' = \begin{pmatrix} \left((b^{**})^{-K}(a^{**})^{L^2}\right)^{\frac{1}{W}}.a \\ \left((a^{**})^{K}(b^{**})^{L^2}\right)^{\frac{1}{W}}.b \end{pmatrix} \tag{32} $$

where

$$ W = (\theta.\ln\theta^*)^4 - \left(\theta.(\ln^2\theta^* + \ln\theta^{**})\right)^2, $$

$$ K = \theta.(\ln^2\theta^* + \ln\theta^{**}) $$

and

$$ L = \theta.\ln\theta^* $$

6. CONCLUSIONS

Rotation and acceleration poles in multiplicative one Lorentzian parameter motion on plane kinematics in multiplicative Lorentzian motions are given. Moreover multiplicative Lorentzian pole orbits, mulltiplicative Lorentzian accelerations and multiplicative Lorentzian combinations of accelerations are obtained.

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