ORIGINAL PAPER

AN APPROXIMATION FORMULA FOR THE WALLIS RATIO IN A SIMPLE FORM

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Manuscript received: 07.07.2024; Accepted paper: 11.11.2024;
Published online: 30.12.2024.

Abstract. *We present a different point of view for improving some approximation formulas for the Wallis ratio. In many published papers several such improvements were announced by adding terms from an asymptotic expansion to an already stated approximation formula. This addition can be done to almost every approximation formula involving the gamma function, or ratio of gamma functions. Unfortunately, these new approximation formulas are obtained via a sacrifice of simplicity. One of the approximation formula for the Wallis ratio of a simple form stated here is the following:*

$$\frac{(2n-1)!!}{(2n)!!} \sim \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+3}} , \quad n \to \infty.$$

Then other formulas are given and a general method for obtaining similar formulas is presented.

Keywords: *Gamma function; Wallis ratio; approximations.*

1. INTRODUCTION

In the recent past, a wide method used by authors for improving some approximation formulas is to add new terms from an asymptotic series. It is true that increasingly accurate approximations are obtained, but a sacrifice of simplicity.

In this paper, we present a method for obtaining more accurate simple approximations. This is important for scientists that use these formulas in other branches in science. They need formulas of simple form, as they use them for other expressions or equations, or for numerical approximations.

Roughly speaking, an approximation formula

$$w_n \sim f(n) \;, \quad n \to \infty \tag{1}$$

is valid if $\lim_{n\to\infty}(w_n/f(n)) = 1$. Its accuracy is measured by the following sequence $(q_n)_{n\in\mathbb{N}}$ (also named the error sequence) defined by the relations:

$$w_n = f(n) \exp q_n \;, \quad n = 1,2,3, \ldots .$$

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https://doi.org/10.46939/J.Sci.Arts-24.4-a01

Evidently, $q_n \to 0$. More precisely, the approximation formula (1) is stronger as $(q_n)_{n\in\mathbb{N}}$ faster converges to zero.

A strong tool for measuring the speed of convergence of $(q_n)_{n\in\mathbb{N}}$ is a result stated by Mortici [1] according to which the sequence $(q_n)_{n\in\mathbb{N}}$ converges faster to zero, as the difference $(q_n - q_{n+1})_{n\in\mathbb{N}}$ faster converges to zero.

Recent results using this lemma were obtained for example in [2-11].

2. THE RESULTS

We show these facts on the problem of Wallis ratio.

The Wallis ratio is defined for every positive integer n, by the formula:

$$w_n = \frac{(2n-1)!!}{(2n)!!}.$$

In terms of gamma function, we have:

$$w_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}, \tag{2}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt\ ,\ \ x > 0.$$

In the recent past, the problem of approximation of Wallis ratio has attracted a large interest through the researchers [2-12].

We illustrate our method motivated by the following double inequality:

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < w_n \le \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+6}}\ ,\ \ n \in \mathbb{N}$$

established by Zhang et al. [12]. This gives two approximation formulas for the Wallis ratio:

$$w_n \sim \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} \quad \text{and} \quad w_n \sim \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+6}} \tag{3}$$

and a first question is to say what of these approximations is better. More exactly, for every $a \ge 0$, we introduce the following class of approximations:

$$w_n \sim \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+a}}\ ,\ \ n \to \infty. \tag{4}$$

The above approximations (4) are obtained for $a = 0$ and $a = 6$, respectively.

We prove that the best approximation is obtained for $a = 3$, namely

$$w_n \sim \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+3}}, \quad n \to \infty. \tag{5}$$

In particular, the approximation formula (5) is more accurate than (3).

3. THE PROOFS

As we explained above, we introduce the error sequence $(q_n)_{n\in\mathbb{N}}$ associated to the approximation formula (4) by the relations:

$$w_n = \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+a}} \exp q_n \quad , \quad n \in \mathbb{N},$$

or, using (2), then multiplying by $\sqrt{n}$:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{1}{\sqrt{e}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+a}} \exp q_n , \quad n \in \mathbb{N}.$$

We get:

$$q_n = \ln\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} - \left\{\left(n-\frac{1}{12n+a}\right)\ln\left(1+\frac{1}{2n}\right)-1\right\}. \tag{6}$$

We use also the well-known following asymptotic formula for the ratio of gamma function [13]:

$$\ln\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = -\frac{1}{8n}+\frac{1}{192n^3}-\frac{1}{640n^5}+\frac{17}{14336n^7}-\frac{31}{18432n^9}+O\left(\frac{1}{n^{10}}\right). \tag{7}$$

By using the standard expansion of $\ln(1+x)$ in power series, we get:

$$\left(n-\frac{1}{12n+a}\right)\ln\left(1+\frac{1}{2n}\right)-1 = -\frac{1}{8n}+\left(\frac{1}{288}a-\frac{1}{192}\right)\frac{1}{n^3}-\left(\frac{1}{3456}a^2+\frac{1}{1152}a-\frac{1}{360}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{8}$$

By combining (7)-(8), we get from (6):

$$q_n = \left(\frac{1}{96} - \frac{1}{288}a\right)\frac{1}{n^3} + \left(\frac{1}{3456}a^2 + \frac{1}{1152}a - \frac{1}{360}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{9}$$

Evidently, the fastest convergence of $(q_n)_{n\in\mathbb{N}}$ is obtained when the coefficient of n^{-3} vanishes, that is:

$$\frac{1}{96} - \frac{1}{288}a = 0 \Rightarrow a = 3.$$

As we have explained, the best approximation (4) is obtained. In this case $a = 3$, we can see from (9) that:

$$q_n = \frac{7}{2880n^4} + O\left(\frac{1}{n^5}\right), \quad (a = 3).$$

For the approximation formulas related to Zhang et al. inequality (3), the obtained error sequences of inferior rate of convergence:

$$q_n = \frac{1}{96n^3} + O\left(\frac{1}{n^4}\right), \quad (a = 0),$$

and

$$q_n = -\frac{1}{96n^3} + O\left(\frac{1}{n^4}\right), \quad (a = 6).$$

4. FURTHER STUDY

We exploit the ideas from previous sections to introduce the following approximation formula:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \sim e^{-\frac{n}{2}}\left(1 + \frac{1}{2n^2 - \frac{7}{12}}\right)^{n^3 - \frac{7n}{24}}, \quad n \to \infty, \tag{10}$$

that is more accurate than the equivalent formula (5).

The method for obtaining such a formula is to search for an expression E_n, with the expansion

$$E_n = \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \frac{\alpha_3}{n^3} + \frac{\alpha_4}{n^4} + \frac{\alpha_5}{n^5} + \cdots, \quad n \to \infty, \tag{11}$$

and an approximation formula:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \sim E_n, \quad n \to \infty, \tag{12}$$

such that the the first coefficients in (11) coincide with the corresponding coefficients from (7). In this way, we search for a fastest possible error sequence $(Q_n)_{n\in\mathbb{N}}$:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \sim E_n \cdot \exp Q_n \quad , \quad n \in \mathbb{N}.$$

After a careful analysis of approximation formula (12), and having in mind the previous considerations, we introduce the following family of approximation formulas:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \sim e^{-\frac{n}{2}}\left(1+\frac{1}{2n^2+a}\right)^{n^3+bn} \quad , \qquad n \to \infty,$$

where $a, b \in \mathbb{R}$. The error sequence $(Q_n)_{n\in\mathbb{N}}$ is given by:

$$\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = e^{-\frac{n}{2}}\left(1+\frac{1}{2n^2+a}\right)^{n^3+bn} \cdot \exp Q_n \quad , \qquad n \in \mathbb{N}. \tag{13}$$

By taking the logarithm in (13), we get:

$$Q_n = \ln\frac{\sqrt{n}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} - \rho_n \quad , \quad n \to \infty,$$

where

$$\rho_n = (n^3+bn)\ln\left(1+\frac{1}{2n^2+a}\right) - \frac{n}{2}.$$

By using the classical power series of the logarithm function, we obtain:

$$\begin{aligned}\rho_n = &-\left(\frac{a}{4}-\frac{b}{2}+\frac{1}{8}\right)\frac{1}{n}+\left(\frac{a}{8}-\frac{b}{8}-\frac{ab}{4}+\frac{a^2}{8}+\frac{1}{24}\right)\frac{1}{n^3}\\ &-\left(\frac{a}{16}-\frac{b}{24}-\frac{a^2b}{8}-\frac{ab}{8}+\frac{3a^2}{32}+\frac{a^3}{16}+\frac{1}{64}\right)\frac{1}{n^5}\\ &+\left(\frac{a}{32}-\frac{b}{64}-\frac{3a^2b}{32}-\frac{a^3b}{16}-\frac{ab}{16}+\frac{a^2}{16}+\frac{a^3}{16}+\frac{a^4}{32}+\frac{1}{160}\right)\frac{1}{n^7}+O\left(\frac{1}{n^8}\right).\end{aligned}$$

By comparing with the expansion (7), we impose the condition that the coefficients of n^{-1} and n^{-3} coincide respectively, that is:

$$\begin{cases} \frac{a}{4}-\frac{b}{2}+\frac{1}{8}=\frac{1}{8} \\ \frac{a}{8}-\frac{b}{8}-\frac{ab}{4}+\frac{a^2}{8}+\frac{1}{24}=\frac{1}{192} \end{cases}.$$

The solution of this system is

$$a = -\frac{7}{12} \quad , \quad b = -\frac{7}{24}.$$

For these values, the approximation formula (10) is obtained. The corresponding error sequence is

$$Q_n = -\frac{1}{512n^5} + O\left(\frac{1}{n^7}\right).$$

Note that this sequence $(Q_n)_{n\in\mathbb{N}}$ is faster than the error sequences associated with the approximation formulas stated in the previous section.

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